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Feedback Linear Equivalence for nonlinear time delay systems

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***Abstract*— In the present paper the linear feedback equivalence problem is addressed for time delay systems. It is shown that thanks to the use of new mathematical tools recently introduced in the literature for dealing with time delay systems, it is possible to define necessary and sufficient conditions for the solvability of the problem.**

I. INTRODUCTION

Geometric tools for addressing control problems have been extensively used both in the linear and nonlinear context (recall the pioneering works [24] and [13]). In the nonlinear context one of the first topics addressed was the definition of the conditions under which a given nonlinear accessible single input system was diffeomorphic eventually up to a regular static state feedback to a linear system. It was shown that the solution is linked to the involutivity of a specific distribution defined by the vector fields which characterize the dynamics of the given system ($(g, ad_f g, \cdots ad_f^{n-2} g)$ for continuous–time systems and $(G^0, Ad_{F_0} G^0, \cdots Ad_{F_0}^{n-2} G^0)$ for discrete–time systems). This property in fact implies the existence of a function with relative degree equal to n, thus defining both the change of coordinates and the regular static state feedback (see for example [2], [14], [11], [20], [16], [15], [6]).

In [22] a first attempt was pursued to introduce geometric tools to deal with time–delay systems which are gaining more and more attention due to their importance in several applications such as those concerning the delay in the signal transmission over communication networks (see for example [1], [9], [17], [22], [21], [23]).

In the present paper we consider the *accessibility submodules* [3] linked to the accessibility property of the system as well as an operation over their elements, the extended Lie bracket operation recently introduced in [5] to deal with time-delay systems. This operation generalizes the delayed state bracket introduced in [22]. We will show that the tools introduced can be efficacely used to characterize if a NLTDS is equivalent or not, to a Linear Time Delay System (LTDS) by bicausal change of coordinates and bicausal static state feedback. With respect to ([9], [22]) we will consider a more general class of systems where there is no assumption on the delay of the input. For notational simplicity we will assume that the maximal delays on the state and input variables do coincide.

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The paper is organized as follows. Section II concerns recalls and notations about time–delay systems and the geometric framework. In Section III some geometric tools for dealing with time–delay systems are introduced and discussed. In Section IV the proposed approach is used to address the problem of the equivalence under bicausal coordinates change and static state feedback to a linear time delay system.

II. RECALLS AND NOTATIONS

The following notation and definitions, taken from [19], [25], will be used: $\mathcal{K}$ denotes the field of meromorphic functions of a *finite* number of variables in $\{x(t-i), u(t-i), \dot{u}(t-i), \ldots, u^{(k)}(t-i), i, k \in I\!N\}$; d is the standard differential operator; δ represents the backward time-shift operator, that is, given $a(\cdot), f(\cdot) \in \mathcal{K}$:

$$\delta(a(t)\,\mathrm{d}f(t)) = a(t-1)\delta\mathrm{d}f(t) = a(t-1)\mathrm{d}f(t-1);$$

$\deg(\cdot)$ is the polynomial degree in δ of its argument; $\mathcal{K}(\delta]$ is the (left) ring of polynomials in δ with coefficients in $\mathcal{K}$. Every element of $\mathcal{K}(\delta]$ may be written as $\alpha(\delta] = \alpha_0(t) + \alpha_1(t)\,\delta + \cdots + \alpha_{r_\alpha}(t)\,\delta^{r_\alpha}, \quad \alpha_i \in \mathcal{K}$, where $r_\alpha = \deg(\alpha(\delta])$. Addition and multiplication on this ring are defined by $\alpha(\delta] + \beta(\delta] = \sum_{i=0}^{\max\{r_\alpha, r_\beta\}} (\alpha_i(t) + \beta_i(t))\delta^i$ and $\alpha(\delta]\beta(\delta] = \sum_{i=0}^{r_\alpha} \sum_{j=0}^{r_\beta} \alpha_i(t)\,\beta_j(t-i)\delta^{i+j}$. Although this ring is non-commutative, it is an Euclidean ring ([25], [19], [10]); $\mathcal{R}(\delta] = \mathrm{span}_{\mathcal{K}(\delta]}\{r_1, \ldots, r_s\}$, is the right module spanned over $\mathcal{K}(\delta]$ by the column elements $r_1, \ldots, r_s \in \mathcal{K}^n(\delta]$; a polynomial matrix $A \in \mathcal{K}^{n\times n}(\delta]$ is unimodular if it has a polynomial inverse. If $\deg(A) = s$, then $\deg(A^{-1}) \le (n-1)\,s$.

Example 1: Let $f(t) = x(t)\,x(t-2) \in \mathcal{K}$. Then

$$\begin{aligned} \delta f(t) &= x(t-1)\,x(t-3)\delta \in \mathcal{K}(\delta], \\ \mathrm{d}f(t) &= x(t)\mathrm{d}x(t-2) + x(t-2)\mathrm{d}x(t) \\ &= x(t)\delta^2\mathrm{d}x + x(t-2)\mathrm{d}x, \\ \delta\mathrm{d}f(t) &= x(t-1)\delta^3\mathrm{d}x + x(t-3)\delta\mathrm{d}x. \end{aligned}$$

Let us consider the nonlinear dynamics with delays

$$\Sigma: \;\; \dot{x}(t) \;=\; F(\mathbf{x}_{[s]}) + \sum_{j=0}^{s} G_j(\mathbf{x}_{[s]})u(t-j) \qquad (1)$$

where $\mathbf{x}_{[s]}^T = (x^T(t), \cdots x^T(t-s))$ with $x \in I\!R^n$, $u \in I\!R$.

In the following, $\mathbf{x}_{[s]}(-p) := (x^T(t-p), \cdots x^T(t-s-p))^T$. $\mathbf{u}_{[s]}, \mathbf{u}_{[s]}(-p), \mathbf{z}_{[s]}$, and $\mathbf{z}_{[s]}(-p)$ are defined in a similar vein. When no confusion is possible the subindex will be omitted so that $\mathbf{x}$ will stand for $\mathbf{x}_{[s]}$ and $\mathbf{x}(-p)$ will stand for $\mathbf{x}_{[s]}(-p)$.

With such notation, Σ_L, the differential form representation of Σ, is given by

$$\Sigma_L: \quad d\dot{x} \quad = \quad f(\mathbf{x}_{[s]}, \mathbf{u}_{[s]}, \delta)dx + g(\mathbf{x}_{[s]}, \delta)du \tag{2}$$

with

$$\begin{aligned} f(\mathbf{x}_{[s]}, \mathbf{u}_{[s]}, \delta) &= \sum_{i=0}^{s} \frac{\partial F(\mathbf{x}_{[s]}, \delta)}{\partial x(t-i)} \delta^i \\ &\quad + \sum_{j=0}^{s} u(t-j) \sum_{i=0}^{s} \frac{\partial G_j(\mathbf{x}_{[s]})}{\partial x(t-i)} \delta^i \\ g(\mathbf{x}_{[s]}, \delta) &= \sum_{j=0}^{s} G_j(\mathbf{x}_{[s]}) \delta^j \end{aligned}$$

Let us end this section by recalling the definition of a bicausal change of coordinates given in [19].

Definition 1 (Bicausal change of coordinates): Consider the dynamics Σ. $\mathbf{z}_{[0]} = \phi(\mathbf{x}_{[s]}), \phi \in \mathcal{K}^n$ is a *bicausal* change of coordinates for Σ if there exist an integer $\ell \in I\!N$ and a function $\phi^{-1}(\mathbf{z}_{[\ell]}) \in \mathcal{K}^n$ such that $\mathbf{x}_{[0]} = \phi^{-1}(\mathbf{z}_{[\ell]})$.

By definition a bicausal change of coordinates satisfies the following properties:

P1) $T[\mathbf{x}, \delta] = \sum\limits_{i=0}^{s} \dfrac{\partial \phi(\mathbf{x}_{[s]})}{\partial x(t-i)} \delta^i = \sum\limits_{i=0}^{s} T^i(\mathbf{x}) \delta^i$ is unimodular

P2) The inverse $T^{-1}[\mathbf{z}, \delta]$ of $T[\mathbf{x}, \delta]$ is unimodular and given by

$$T^{-1}[\mathbf{z}, \delta] = \sum_{i=0}^{\ell} \frac{\partial \phi^{-1}(\mathbf{z}_{[\ell]})}{\partial z(t-i)} \delta^i = \sum_{i=0}^{\ell} \bar{T}^i(\mathbf{z}) \delta^i.$$

It follows that under the bicausal change of coordinates $\mathbf{z}_{[0]} = \phi(\mathbf{x}_{[\alpha]})$ the differential form (2) is transformed into

$$d\dot{z}(t) \quad = \quad \tilde{f}(\mathbf{z}, \mathbf{u}, \delta)dz + \tilde{g}(\mathbf{z}, \delta)du \tag{3}$$

with

$$\begin{aligned} \tilde{f}(\mathbf{z}, \mathbf{u}, \delta) &= \left[\left(T(\mathbf{x}, \delta) f(\mathbf{x}, \mathbf{u}, \delta) + \dot{T}(\mathbf{x}, \delta)\right) T^{-1}(\mathbf{x}, \delta\right]_{\phi^{-1}(\mathbf{z})} \\ \tilde{g}(\mathbf{z}, \delta) &= \left(T(\mathbf{x}, \delta) g(\mathbf{x}, \delta)\right)_{\phi^{-1}(\mathbf{z})} \end{aligned}$$

III. THE GEOMETRY OF TIME–DELAY SYSTEMS

In this Section we recall the Extended Lie derivative and Extended Lie bracket operators recently introduced in [5], [4] to deal with time–delay systems. Their usefulness has already been tested with respect to some basic control problems for time delay systems such as the linear equivalence problem [3] or the equivalence to the observer canonical form [4]. In fact it has been shown in [5] that the Extended Lie bracket operator characterizes the integrability conditions of one forms depending not only on the state variables but also on their repeated delay.

Definition 2: Let $r_\beta(\mathbf{x}, \mathbf{u}, \delta) = \sum\limits_{j=0}^{s} r_\beta^j(\mathbf{x}, \mathbf{u}) \delta^j$, with $\beta = 1, 2$. Then the Extended Lie bracket $[r_1^k(\cdot, \mathbf{u}), r_2^l(\cdot, \mathbf{u})]_{E_i}$, on $I\!R^{(i+1)n}$, $i \geq 0$, is defined as

$$\begin{aligned} &\left[r_1^k(\cdot, \mathbf{u}), r_2^l(\cdot, \mathbf{u})\right]_{E_i} = \\ &\sum_{j=0}^{\bar{k}} \left([r_1^{k-j}(\cdot, \mathbf{u}), r_2^{l-j}(\cdot, \mathbf{u})]_{E_0}\right)^T_{|(\mathbf{x}(-j), \mathbf{u}(-j))} \tfrac{\partial}{\partial x(t-j)}, \end{aligned} \tag{4}$$

with $\bar{k} = min(k, l, i)$, and

$$\begin{aligned} &[r_1^k(\cdot), r_2^l(\cdot, \mathbf{u})]_{E_0} = \\ &\sum_{i=0}^{k} \frac{\partial r_2^l(\mathbf{x}, \mathbf{u})}{\partial x(t-i)} r_1^{k-i}(\mathbf{x}(-i), \mathbf{u}(-i)) + \\ &- \sum_{i=0}^{l} \frac{\partial r_1^k(\mathbf{x}, \mathbf{u})}{\partial x(t-i)} r_2^{l-i}(\mathbf{x}(-i), \mathbf{u}(-i)). \end{aligned} \tag{5}$$

As in the delay–free case it is convenient to introduce an Extended Lie derivative whose definition is given below and is slightly different from the one given in [21].

Definition 3: Given the function $\lambda(\mathbf{x}_{[s]})$ and the submodule element $r_i(\mathbf{x}, \delta) = \sum\limits_{j=0}^{\bar{s}} r_i^j(\mathbf{x}) \delta^j$, the Extended Lie derivative $L_{r_i^j(\mathbf{x})} \lambda(\mathbf{x}_{[s]})$

$$L_{r_i^j(\mathbf{x})} \lambda(\mathbf{x}_{[s]}) = \sum_{l=0}^{j} \frac{\partial \lambda(\mathbf{x}_{[s]})}{\partial x(t-l)} r_i^{j-l}(\mathbf{x}(-l)) \tag{6}$$

Accordingly setting $\bar{k} = min(k, l, i)$

$$\begin{aligned} &\left[r_1^k(\cdot), r_2^l(\cdot)\right]_{Ei} = \\ &\sum_{j=0}^{\bar{k}} \left(L_{r_1^{k-j}(\mathbf{x})} r_2^{l-j}(\mathbf{x}) - L_{r_2^{l-j}(\mathbf{x})} r_1^{k-j}(\mathbf{x})\right)^T_{(\mathbf{x}(-j))} \tfrac{\partial}{\partial x(t-j)}. \end{aligned}$$

thus recovering the definitions of Lie derivative and Lie bracket in the delay free case.

From (1), consider now the module element

$$\mathbf{F}(\mathbf{x}, \delta) = \sum_{j=0}^{ns} F^j(\mathbf{x}) \delta^j = \sum_{j=0}^{ns} F(\mathbf{x}) \delta^j. \tag{7}$$

Thus, the i–th derivative of $\lambda(\mathbf{x})$ computed for $\mathbf{u} = 0$, is given by $\lambda^{(i)}(\mathbf{x}, 0) = L^i_{F^{ns}(\mathbf{x})} \lambda(\mathbf{x})$, for any i. The definition of relative degree can be then formulated as follows.

Definition 4: The function $\lambda(\mathbf{x})$ has relative degree $k > 0$ if

$$L_{g^j} L^i_{F^{ns}} \lambda(\mathbf{x}) \quad = \quad 0, \ \forall j \geq 0, \forall 0 \leq i < k-1$$

and there exists an index $j \geq 0$ such that

$$L_{g^j} L^{k-1}_{F^{ns}} \lambda(\mathbf{x}) \quad \neq \quad 0.$$

It will have strong relative degree if (8) is satisfied for $j = 0$.

To deal with the integrability of one-forms, we need now to recall the following definitions of an integrable submodule $\Delta = \text{span}_{\mathcal{K}(\delta]}\{r_1(\mathbf{x}, \delta), \cdots, r_j(\mathbf{x}, \delta)\}$, with $r_l(\mathbf{x}, \delta) = \sum_{t=0}^{s} r_l^t(\mathbf{x}) \delta^t$, $l \in [1, j]$. To this end, let $\mathbf{x}^0 = (x^0(t)^T, \cdots, x^0(t-\gamma)^T)^T$.

Definition 5: Δ is nonsingular locally around $\mathbf{x}^0$ if $\text{rank}(\Delta(x)) = j$, $\forall \mathbf{x} \in \mathcal{U}_0$ an open and dense subset of $\mathbf{x}^0$.

Definition 6: Δ nonsingular locally around $\mathbf{x}^0$, is integrable if there exist $n-j$ independent functions $\lambda_l(x(t), \cdots, x(t-\gamma))$, $l \in [1, n-j]$ such that $\text{rank}\frac{\partial \lambda(\mathbf{x})}{\partial x(t)} = n-j$ and

$$\sum_{p=0}^{\gamma} \frac{\partial \lambda_l(\mathbf{x})}{\partial x(t-p)} \delta^p \sum_{k=0}^{s} r_i^k(\mathbf{x}) \delta^k = 0, \quad \forall l \in [1, n-j],\ \forall i \in [1, j].$$

Lemma 1: Let $r_\beta(\mathbf{x}, \mathbf{u}, \delta) = \sum_{j=0}^{s} r_\beta^j(\mathbf{x}, \mathbf{u}) \delta^j$, $\beta = 1, 2$ and assume under the bicausal change of coordinates $\mathbf{z}_{[0]} = \phi(\mathbf{x}_{[\alpha]})$, with $dz = T(\mathbf{x}, \delta) dx$, the submodule element $r_\beta(\mathbf{x}, \mathbf{u}, \delta)$ is transformed as

$$\tilde{r}_\beta(\mathbf{z}, \mathbf{u}, \delta) = [T(\mathbf{x}, \delta) r_\beta(\mathbf{x}, \mathbf{u}, \delta)]_{|\mathbf{x}=\phi^{-1}(\mathbf{z})}, \ \beta = 1, 2. \quad (8)$$

Then

$$[\tilde{r}_1^k(\mathbf{z}, \mathbf{u}), \tilde{r}_2^l(\mathbf{z}, \mathbf{u})]_{E_i} = \left(\Gamma^{l,i}(\mathbf{x}) [r_1^k(\mathbf{x}, \mathbf{u}), r_2^l(\mathbf{x}, \mathbf{u})]_{E_l}\right)_{|\mathbf{x}=\phi^{-1}(\mathbf{z})}$$

where, setting $T^j = 0$ for $j > \alpha$,

$$\Gamma^{l,i}(\mathbf{x}) = \begin{pmatrix} T^0(\mathbf{x}) & \cdots & \cdots & \cdots & T^l(\mathbf{x}) \\ 0 & \ddots & & & \vdots \\ 0 & 0 & T^0(\mathbf{x}(-i)) & \cdots & T^{l-i}(\mathbf{x}(-i)) \end{pmatrix}.$$

A. Integrability Conditions of one-forms

We will now recall the necessary and sufficient conditions given in [5] under which some given one-forms are integrable. The obtained results are based on the consideration that though when dealing with delay systems one ends up on an infinite dimensional system, the elements that one considers are characterized by a finite number of components.

Consider $P_j(\mathbf{x}, \delta) = [r_1(\mathbf{x}, \delta), \cdots, r_j(\mathbf{x}, \delta)] = \sum_{l=0}^{s} P_{jl}(\mathbf{x}) \delta^l$ with $P_{j0}(\mathbf{x})$ of dimension j and $r_k(\mathbf{x}, \delta) = \sum_{l=0}^{s} r_k^s(\mathbf{x}) \delta^s$, $k \in [1, j]$. Consider the distributions Δ_i and Δ_i', $i \geq 0$ defined on $I\!R^{(i+1)n}$ and with vector fields parameterized by $x(t-i-1), \cdots, x(t-i-s)$, for $i \geq 0$

$$\Delta_i = \text{span}_{\mathcal{K}} \left\{ \sum_{l=0}^{\gamma} \left(r_k^{\gamma-l}(\mathbf{x}(-l)) \right)^T \frac{\partial}{\partial x(t-l)} \begin{array}{l} k \in [1, j] \\ \gamma \in [0, i] \end{array} \right\}, \quad (9)$$

$$\Delta_i' = \text{span}_{\mathcal{K}} \left\{ \sum_{l=0}^{min(\gamma, i)} \left(r_k^{\gamma-l}(\mathbf{x}(-l)) \right)^T \frac{\partial}{\partial x(t-l)} \begin{array}{l} k \in [1, j] \\ \gamma \in [0, i+s] \end{array} \right\}.$$

By construction $\Delta_i \subseteq \Delta_i'$. Let locally around $\mathbf{x}^0$, $\rho_i = \text{rank}(\Delta_i')$, then $\Delta_i' = \text{span}\{\tau_l(\mathbf{x}), l \in [1, \rho_i]\} \subset I\!R^{(i+1)n}$ while its elements depend on the variables $\mathbf{x}_{[i+s]}$. Let us thus consider the series development of τ_l with respect to the parameters $\mathbf{x}(-i-1)$ locally around $\mathbf{x}^0(-i-1)$ which without loss of generality can be assumed to be the origin, that is

$$\begin{aligned} \tau_l(\mathbf{x}) &= \tau_{l0}(\mathbf{x}_{[i]}) + \sum_{j=1}^{s} \sum_{\alpha=1}^{n} {}_{\alpha_j}\tau_{l1}(\mathbf{x}_{[i]}) x_\alpha(-i-j) \\ &+ \frac{1}{2} \sum_{\alpha,\beta=1}^{n} \sum_{j,k=1}^{s} {}_{\alpha_j \beta_k}\tau_{l2}(\mathbf{x}_{[i]}) x_\alpha(-i-j) x_\beta(-i-k) + \cdots \end{aligned}$$

and consider the possibly infinite set of distributions

$$\begin{aligned} \Delta_{i0}' &= \text{span}\{\tau_{l0}, \ l \in [1, k]\} \\ \Delta_{i1}' &= \text{span}\{{}_{\alpha_j}\tau_{l1}, \ l \in [1, k], \ j \in [1, s], \alpha \in [1, n]\} \\ &\vdots \end{aligned} \quad (10)$$

Set $\rho_{i0} = rank \ \overline{(\sum_{k \geq 0} \Delta_{ik}')}$. We can now recall the main result concerning integrability.

Theorem 1: [5] Consider the submodule $\Delta = \text{span}_{\mathcal{K}(\delta]} \{r_1(\mathbf{x}, \delta), \cdots, r_j(\mathbf{x}, \delta)\}$ with $r_i(\mathbf{x}, \delta) = \sum_{l=0}^{s} r_i^l(\mathbf{x}) \delta^l$, and such that the matrices $P_j(\mathbf{x}, \delta) = (r_1(\mathbf{x}, \delta), \cdots, r_j(\mathbf{x}, \delta)) = \sum_{l=0}^{s} P_{jl}(\mathbf{x}) \delta^l$ and $P_{j0}(\mathbf{x})$ are of dimension j. Let Δ_i' and $\overline{(\sum_{k \geq 0} \Delta_{ik}')}$ be the associated set of distributions defined respectively by (9) and (10) which are assumed to be locally non singular on $\mathbf{x}^0 = (x^0(t)^T, \cdots x^0(t-i)^T)^T$ with $\rho_i = rank \ \Delta_i'$ and $\rho_{i0} = rank \ \overline{(\sum_{k \geq 0} \Delta_{ik}')}$ (with $\rho_{-1} = \rho_{-1,0} = 0$). Then Δ is integrable iff there exists an index γ such that the following conditions are satisfied

a) $\forall l, k \in [1, j]$ and $t \leq p \leq i+s$, $[r_l^t(\cdot), r_k^p(\cdot)]_{E_i} \in \Delta_i'$, $\quad i \in [0, \gamma]$
b) $\rho_\gamma - \rho_{\gamma-1} = j$
c) $\rho_{\gamma 0} - \rho_{\gamma-1,0} = j$

Example 2: Consider the submodule $\Delta = \text{span}_{\mathcal{K}(\delta]} \left\{ x_2(t-1) x_1(t-1) \delta \frac{\partial}{\partial x_1(t)} + x_1(t) \frac{\partial}{\partial x_2(t)} \right\}$. According to Theorem 1, to check if there exists a one-form which lies in the left kernel of the given submodule, we must consider

$$\begin{aligned} \Delta_0' &= \text{span}_{\mathcal{K}} \left\{ x_1(t) \frac{\partial}{\partial x_2(t)}, x_2(t-1) x_1(t-1) \frac{\partial}{\partial x_1(t)} \right\}, \\ \Delta_1' &= \text{span}_{\mathcal{K}} \left\{ x_1(t) \frac{\partial}{\partial x_2(t)} \right\} \\ &+ \text{span}_{\mathcal{K}} \left\{ x_1(t-1) x_2(t-1) \frac{\partial}{\partial x_1(t)} + x_1(t-1) \frac{\partial}{\partial x_2(t-1)} \right\} \\ &+ \text{span}_{\mathcal{K}} \left\{ x_2(t-2) x_1(t-2) \frac{\partial}{\partial x_1(t-1)} \right\}, \\ &\vdots \end{aligned}$$

Since $\rho_0 = \rho_{00} = rank \ (\Delta_0') = 2$ locally around $x^0 \neq 0$, then condition a) of Theorem 1 holds true for Δ_0'. As for Δ_1', it is readily seen that it is independent of $x(t-i)$ $i \geq 2$ and is involutive, so that condition a) is satisfied for Δ_1'; moreover $\rho_1 = \rho_{1,0} = rank \ (\Delta_1') = 3$ so that also condition b) and c) are satisfied being $\rho_1 - \rho_0 = \rho_{10} - \rho_{00} = 1$. Thus there exists one integrable one–form in the left kernel of Δ given by $\omega_2(\mathbf{x},\delta)dx = dx_1 - x_2(t-1)\delta dx_2 = x_1(t) - \frac{1}{2}[x_2(t-1)]^2$.

IV. Linear feedback equivalence of Accessible Time Delay Systems

We will now show how the results proposed in the previous section can be efficacely used to address the problem of linear feedback equivalence for accessible time delays. To this end, some preliminary definitions related to the accessibility properties of a NLTDS are in order.

A. Accessibility submodules

Let $g_1(\mathbf{x}_{[s]},\delta) := g(\mathbf{x}_{[s]},\delta)$. The module generators g_k are recursively defined for $k \geq 2$ as

$$g_k(\mathbf{x}_{[s]},\mathbf{u}_{[s]},\delta) = \\ f(\mathbf{x}_{[s]},\mathbf{u}_{[s]},\delta)g_{k-1}(\mathbf{x}_{[s]},\mathbf{u}_{[s]},\delta) - \dot{g}_{k-1}(\mathbf{x}_{[s]},\mathbf{u}_{[s]},\delta).$$

Definition 7: The accessibility submodules $\mathcal{R}_i$ of system Σ, are defined as

$$\mathcal{R}_i = \mathrm{span}_{\mathcal{K}(\delta]}\{g_1(\mathbf{x},\delta)\cdots g_i(\mathbf{x},\mathbf{u},\delta)\},\ i \geq 1.$$

The following results hold true [3].

Proposition 1: If $g_{i+1}(\mathbf{x},\mathbf{u},\delta) \in \mathcal{R}_i$ then $\forall j \geq 1$, $g_{i+j}(\mathbf{x},\mathbf{u},\delta) \in \mathcal{R}_i$.

Proposition 2: Under the change of coordinates $\mathbf{z}_{[0]} = \phi(\mathbf{x}_{[\alpha]})$, with $dz = T(\mathbf{x},\delta)dx$ the accessibility submodules elements $g_j(\cdot)$ are transformed as

$$\tilde{g}_j(\mathbf{z},\mathbf{u},\delta) = [T(\mathbf{x},\delta)g_j(\mathbf{x},\mathbf{u},\delta)]_{\mathbf{x}=\phi^{-1}(\mathbf{z})}. \qquad (11)$$

An immediate consequence is the following.

Corollary 1: Under a bicausal change of coordinates $\mathbf{z}_{[0]} = \phi(\mathbf{x}_{[\alpha]})$

$$\begin{aligned}\mathcal{R}_i &= \mathrm{span}_{\mathcal{K}(\delta]}\{g_1(\mathbf{x},\delta)\cdots g_i(\mathbf{x},\mathbf{u},\delta)\} \equiv \tilde{\mathcal{R}}_i \\ &= \mathrm{span}_{\mathcal{K}(\delta]}\{\tilde{g}_1(\mathbf{z},\delta)\cdots \tilde{g}_i(\mathbf{z},\mathbf{u},\delta)\}.\end{aligned}$$

Denote now for $i \geq 1$ and $k \geq 0$, and setting $\bar{\gamma} = min(\gamma,k)$

$$\mathcal{R}_i(\mathbf{x},0,\delta) = \mathrm{span}\{g_1(\mathbf{x},\delta),\cdots,g_i(\mathbf{x},0,\delta)\}$$

$$\mathcal{R}_i^k(\mathbf{x}) = \mathrm{span}\left\{\sum_{l=0}^{\gamma} g_j^{\gamma-l}(\mathbf{x}(-l))\frac{\partial}{\partial x(t-l)},\ \begin{matrix} j\in[1,i], \\ \gamma\in[0,k]\end{matrix}\right\}$$

$$\mathcal{R}_i^{k'}(\mathbf{x}) = \mathrm{span}\left\{\sum_{l=0}^{\bar{\gamma}} g_j^{\gamma-l}(\mathbf{x}(-l))\frac{\partial}{\partial x(t-l)},\ \begin{matrix} j\in[1,i], \\ \gamma\in[0,k+\bar{s}]\end{matrix}\right\}.$$

Following the notation of (9),(10) and according to Theorem 1, set $\mathcal{R}_{n-1,k}^{\gamma,0'} = \sum_{k\geq 0}\mathcal{R}_{n-1,k}^{\gamma'}(\mathbf{x})$. Then the following result holds true

Lemma 2: There exists a function $\lambda(\mathbf{x})$ with relative degree n if and only if

i) $\mathrm{rank}_{\mathcal{K}(\delta]}\mathcal{R}_n(\mathbf{x},0,\delta) = n$

and there exists an index γ such that denoting by $\rho_\gamma = \mathrm{rank}\mathcal{R}_{n-1}^{\gamma'}(\mathbf{x},0)$, $\rho_{\gamma,0} = \mathrm{rank}\mathcal{R}_{n-1}^{\gamma,0'}(\mathbf{x},0)$, one has that

ii) for $l,p\in[1,n-1]$, $\forall j \leq t \leq k+\bar{s}$, $[g_l^j(\mathbf{x},0),g_p^t(\mathbf{x},0)]_{E_k} \in \mathcal{R}_{n-1}^{k'}(\mathbf{x})$, for $k\in[0,\gamma]$.

iii) $\rho_\gamma - \rho_{\gamma-1} = \rho_{\gamma,0} - \rho_{\gamma-1,0} = 1$

The proof, omitted for space reasons, can be easily carried out using the definition of relative degree.

Proposition 3: Assume that the conditions of Lemma 2 are satisfied, and let $\bar{\gamma}$ be the smallest index which verifies condition iii). Let $\lambda(\mathbf{x})$ be a function whose differential lies in the kernel of $\mathcal{R}_{n-1}(\mathbf{x},\delta)$ computed starting from $\mathcal{R}_{n-1}^{\bar{\gamma},0'}$, Then

i) there does not exist any function $\bar{\lambda}(\mathbf{x}_{[\bar{\gamma}-1]})$ whose differential lies in the kernel of $\mathcal{R}_{n-1}(\mathbf{x},\delta)$

ii) $\bar{\gamma}$ is the maximum delay in $\lambda(\mathbf{x})$.

iii) given a solution $d\lambda(\mathbf{x}) \in (\mathcal{R}_{n-1}^{\bar{\gamma},0'})^\perp$, any other solution $d\bar{\lambda}(\mathbf{x}) \in (\mathcal{R}_{n-1}^{\bar{\gamma},0'})^\perp$, is given by $\bar{\lambda}(\mathbf{x}) = \varphi(\lambda(\mathbf{x}))$, with $\frac{\partial\varphi}{\partial\lambda} \neq 0$.

It is now possible to state the necessary and sufficient conditions for linear feedback equivalence of time delay systems under bicausal change of coordinates and bicausal static state feedback. In fact while in the delay–free case the existence of a function with relative degree equal to n is necessary and sufficient, for delay systems, the existence of such a function is necessary but not sufficient. This is due to two main problems: first while the differentials of the functions $\lambda(\mathbf{x}),\cdots,\lambda^{(n-1)}(\mathbf{x})$, are ensured to be independent, they may not define a bicausal change of coordinates; secondly there may not exist a bicausal static state which linearizes the input output behavior of the chosen function. These two problems require some additional conditions which are enlightened below.

Theorem 2: System (1) is equivalent, under bicausal static state feedback and bicausal change of coordinates, to a linear weakly accessible delay system if and only if the conditions of Lemma 2 are satisfied and additionally there exist matrices $Q_1(\delta)$ unimodular and lower triangular, $Q_2(\delta) = diag(1,c_2(\delta),\cdots,c_2(\delta)\cdots c_n(\delta))$, $T(\mathbf{x},\delta)$ unimodular and $\Phi_1(\mathbf{x})$ lower triangular, such that denoting by $\lambda(\mathbf{x})$ a function with relative degree n and closed[1] the following conditions are satisfied

$$a)\ \begin{pmatrix} d\lambda(\mathbf{x}) \\ d\dot{\lambda}(\mathbf{x}) \\ \vdots \\ d\lambda^{(n-1)}(\mathbf{x})\end{pmatrix} = \Phi_1(\mathbf{x})^{-1}Q_1(\delta)Q_2(\delta)T(\mathbf{x},\delta)dx$$

[1] that is it is characterized by the minimum possible delay[7],[18] and any other function in the kernel of $\mathcal{R}_{n-1}(\mathbf{x},\delta)$ can be generated from it

where

$$\Phi_1(\mathbf{x}) = \begin{pmatrix} \varphi_1(\lambda) & 0 & \cdots & 0 \\ \dot{\varphi}_1(\lambda) & \varphi_1(\lambda) & \cdots & 0 \\ \vdots & & & \vdots \\ \varphi_1^{(n-1)}(\lambda) & \binom{n-1}{1}\varphi_1^{(n-2)}(\lambda) & \cdots & \varphi_1(\lambda) \end{pmatrix}$$

with $\varphi_1(\lambda) = \frac{\partial\varphi(\lambda)}{\partial\lambda}$.

b) Setting $d\lambda^{(n)} = a(\mathbf{x}, \mathbf{u}, \delta)dx + b(\mathbf{x}, \mathbf{u}, \delta)du$, one must have

$$b(\mathbf{x}, \mathbf{u}, \delta) = b(\mathbf{x}, 0, \delta) = \varphi_1^{-1}(\lambda)\tilde{b}(\delta)\beta(\mathbf{x})$$

and

$$\begin{aligned}\varphi_1(\lambda)a(\mathbf{x}, 0, \delta) + \Phi_{n+1}(\mathbf{x})\Phi_1(\mathbf{x})^{-1}Q_1(\delta)Q_2(\delta)T(\mathbf{x}, \delta) = \\ = (\tilde{a}(\delta) + \tilde{b}(\delta)\Gamma(\mathbf{x}, 0, \delta))T(\mathbf{x}, \delta)\end{aligned}$$

with

$$\Phi_{n+1}(\mathbf{x}) = \begin{pmatrix} \varphi_1^{(n)}(\lambda) & \binom{n}{1}\varphi_1^{(n-1)}(\lambda) & \cdots & \binom{n}{n-1}\dot{\varphi}_1(\lambda) \end{pmatrix}$$

Proof: Necessity. Assume that the system is weakly accessible and equivalent through bicausal change of coordinates $z = \phi(\mathbf{x})$, and bicausal static state feedback $u(t) = \alpha(\mathbf{x}) + \beta(\mathbf{x})v(t)$ to a linear time delay system. Then in the new coordinates and after the bicausal feedback law the differential of the system reads

$$d\dot{z} \quad = \quad A(\delta)dz + B(\delta)dv$$

Furthermore, the change of coordinates can be chosen in order to get

$$A(\delta) = \begin{pmatrix} a_{11}(\delta) & a_{12}(\delta) & 0 & \cdots & 0 \\ \vdots & & & & \\ a_{n-1,1}(\delta) & & & & a_{n-1,n}(\delta) \\ a_{n,1}(\delta) & & & & a_{n,n}(\delta) \end{pmatrix}$$

$$B(\delta) = \begin{pmatrix} 0 & \cdots & 0 & b_1(\delta) \end{pmatrix}^T$$

It follows that the function $\tilde{\lambda}(\mathbf{x}) = z_1$ satisfies

$$\begin{pmatrix} d\tilde{\lambda}(\mathbf{x}) \\ \vdots \\ d\tilde{\lambda}^{(n-1)}(\mathbf{x}) \end{pmatrix} = Q_1(\delta)Q_2(\delta)T(\mathbf{x}, \delta)dx$$

As enlightened in Proposition 3, starting from $\mathcal{R}_{n-1}^{\gamma,0'}(\mathbf{x})$ (γ being the smallest index satisfying the conditions of Lemma 2), any other possible solution $\lambda(\mathbf{x}) = \varphi^{-1}(\tilde{\lambda}(\mathbf{x}))$ so that the generic function $\lambda(\mathbf{x})$ satisfies a) which is invariant under bicausal change of coordinates and bicausal static state feedback. Finally standard computations show that $d\tilde{\lambda}^n(\mathbf{x}, \mathbf{u}) = \tilde{a}(\delta)T(\mathbf{x}, \delta)dx + \tilde{b}(\delta)dv$, so that for any bicausal state feedback $v(t) = \tilde{\alpha}(\mathbf{x}) + \beta(\mathbf{x})u(t)$, to which corresponds the differential $dv = \tilde{\Gamma}(\mathbf{x}, \mathbf{u}, \delta)dx + \beta(\mathbf{x})du = \Gamma(\mathbf{x}, \mathbf{u}, \delta)T(\mathbf{x}, \delta)dx + \beta(\mathbf{x})du$, one gets

$$d\tilde{\lambda}^n(\mathbf{x}, \mathbf{u}) = (\tilde{a}(\delta) + \tilde{b}(\delta)\Gamma(\mathbf{x}, \mathbf{u}, \delta))T(\mathbf{x}, \delta)dx + \tilde{b}(\delta)\beta(\mathbf{x})du.$$

Accordingly for the generic output $\lambda(\mathbf{x}) = \varphi^{-1}(\tilde{\lambda}(\mathbf{x}))$ we get that

$$d\tilde{\lambda}^{(n)}(\mathbf{x}, \mathbf{u}) = \sum_{i=0}^{n} \binom{n}{i} (\frac{\partial\varphi}{\partial\lambda})^{(n-i)} d\lambda^{(i)}(\mathbf{x}, \mathbf{u})$$

from which we immediately recover b).

Sufficiency. Assume that there exists a function $\lambda(\mathbf{x})$ with relative degree n which satisfies a) and b). Then consider $dz = T(\mathbf{x}, \delta)dx$ with $T(\mathbf{x}, \delta)$ defined by a). Note that due to the relations between the exact differentials $d\lambda(\mathbf{x}), \cdots, d\lambda^{n-1}(\mathbf{x})$ and $T(\mathbf{x}, \delta)$, this last matrix certainly represents the differential representation of $\mathbf{z}_{[0]} = \phi(\mathbf{x})$ which is bicausal being $T(\mathbf{x}, \delta)$ unimodular. Under such a change of coordinates and with respect to the function $\tilde{\lambda}(\mathbf{x}) = \varphi(\lambda)$ defined by a), one gets that

$$\begin{pmatrix} d\tilde{\lambda}(\mathbf{z}) \\ d\dot{\tilde{\lambda}}(\mathbf{z}) \\ \vdots \\ d\tilde{\lambda}^{(n-1)}(\mathbf{z}) \end{pmatrix} = Q_1(\delta)Q_2(\delta)dz \tag{12}$$

Furthermore according to b)

$$d\tilde{\lambda}^{(n)}(\mathbf{z}, \mathbf{u}) = (\tilde{a}(\delta) + \tilde{b}(\delta)\Gamma(\mathbf{z}, \mathbf{u}, \delta))dz + \tilde{b}(\delta)\beta(\mathbf{z})du. \tag{13}$$

Again due to the structure of $d\tilde{\lambda}^n(\mathbf{z}, \mathbf{u})$, the existence of a function $\alpha(\mathbf{z})$ such that $\Gamma(\mathbf{z}, \mathbf{u}, \delta)) = \sum_{i=0}^{s} \frac{\partial\alpha(\mathbf{z})}{\partial z(t-i)}\delta^i + v(t)\sum_{i=0}^{s} \frac{\partial\beta(\mathbf{z})}{\partial z(t-i)}\delta^i$ is guaranteed. As a consequence there exists a bicausal static state feedback $u = \beta^{-1}(\mathbf{z})[-\alpha(\mathbf{z}) + v(t)]$ such that $d\tilde{\lambda}^n(\mathbf{z}, \mathbf{u}) = \bar{a}(\delta)dz + \tilde{b}(\delta)dv$. ■

The last step consists in showing that in the new coordinates and with the computed feedback the system is linear. In fact

$$\begin{pmatrix} d\dot{\tilde{\lambda}}(\mathbf{z}) \\ d\tilde{\lambda}^{(2)}(\mathbf{z}) \\ \vdots \\ d\tilde{\lambda}^{(n)}(\mathbf{z}) \end{pmatrix} = Q_1(\delta)Q_2(\delta)d\dot{z}$$

which due to (12-13) can be equivalently written as

$$Q_2(\delta)d\dot{z} = \bar{A}(\delta)dz + \bar{B}(\delta)dv$$

that is

$$d\dot{z} = A(\delta)dz + B(\delta)dv$$

Example 3: Consider the following nonlinear system

$$\begin{aligned} \dot{x}_1 &= x_1(t)x_2(t-1) + (x_2(t)+1)v(t) \\ &+ x_1(t-1)x_2(t-2) + (x_2(t-1)+1)v(t-1) \\ &- 4x_1(t-1)x_2(t-1) - 2x_2(t-1)x_2^2(t-3) \\ &- 4x_2(t-1)x_2^2(t-2) - 2x_1(t-2)x_2(t-1) \\ \dot{x}_2(t) &= 2x_1(t) + x_1(t-1) + 2x_2^2(t-1) + x_2^2(t-2) \end{aligned}$$

The associated differential representation is characterized by

$$\begin{aligned} f(\mathbf{x}, \mathbf{u}, \delta) &= \begin{pmatrix} f_{11}(\mathbf{x}, \mathbf{u}, \delta) & f_{12}(\mathbf{x}, \mathbf{u}, \delta) \\ 2+\delta & 4x_2(t-1)\delta + 2x_2(t-2)\delta^2 \end{pmatrix} \\ g(\mathbf{x}, \delta) &= \begin{pmatrix} (x_2(t)+1) + (x_2(t-1)+1)\delta \\ 0 \end{pmatrix} \end{aligned}$$

with $f_{11}(\mathbf{x},\mathbf{u},\delta) = x_2(t-1)(1-4\delta-2\delta^2)+x_2(t-2)\delta$, $f_{12}(\mathbf{x},\mathbf{u},\delta) = x_1(t)\delta+v(t)+x_1(t-1)\delta^2-4x_1(t-1)\delta+v(t-1)\delta-4x_2^2(t-2)\delta-8x_2(t-1)x_2(t-2)\delta^2-2x_1(t-2)\delta-2x_2^2(t-3)\delta-4x_2(t-1)x_2(t-3)\delta^3$

The system is weakly accessible and $\mathcal{R}_{n-1}(\mathbf{x}_{[0]},\delta) = g(\mathbf{x},\delta)$. One thus computes

$$R^{0'}_{n-1} = \left\{ \begin{pmatrix} x_2(t)+1 \\ 0 \end{pmatrix} \begin{pmatrix} x_2(t-1)+1 \\ 0 \end{pmatrix} \right\}$$

from which we easily get that $\lambda(\mathbf{x}) = x_2(t)$ has relative degree $r = 2$. The first part of Theorem 2 is thus satisfied. We must then verify conditions a) and b). To this end let us now compute

$$\begin{aligned} \begin{pmatrix} d\lambda(\mathbf{x}) \\ d\dot{\lambda}(\mathbf{x}) \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 2+\delta & 4x_2(t-1)\delta+2x_2(t-2)\delta^2 \end{pmatrix} dx \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 2+\delta \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 2x_2(t-1)\delta \end{pmatrix} dx \end{aligned}$$

from which $\Phi^{-1}(\mathbf{x}) = Id$, $Q_1(\delta) = Id$, $Q_2(\delta) = diag(1, 2+\delta)$

$$T(\mathbf{x},\delta) = \begin{pmatrix} 0 & 1 \\ 1 & 2x_2(t-1)\delta \end{pmatrix} \tag{14}$$

which corresponds to the change of coordinates $z_1(t) = x_2(t)$, $z_2(t) = x_1(t)+x_2^2(t-1)$.

$$\begin{aligned} d\lambda^{(2)}(\mathbf{x}) &= (2+\delta)[d\dot{x}_1+2\dot{x}_2(t-1)\delta dx_2+2x_2(t-1)\delta d\dot{x}_2] \\ &= (2+\delta)(\delta+1)x_2(t-1)dx_1 \\ &+ (2+\delta)((1+\delta)x_1(t)\delta+2x_2(t-1)\delta^2)dx_2 \\ &+ (2+\delta)[(x_2(t)+1)+(x_2(t-1)+1)\delta]dv \end{aligned}$$

that is

$$b(\mathbf{x},\mathbf{u},\delta) = (2+\delta)(1+\delta)(x_2(t)+1)$$

while

$$\begin{aligned} a(\mathbf{x},0,\delta) &= (2+\delta)(f_{11}(\mathbf{x},0,\delta)dx_1+f_{12}(\mathbf{x},0,\delta)dx_2) \\ &+(2+\delta)2\dot{x}_2(t-1)\delta(2+\delta)(dx_1+2x_2(t-1)\delta dx_2) \\ &= (2+\delta)(1+\delta)[x_2(t-1)dx_1(t)+x_1(t)\delta dx_2(t)] \end{aligned}$$

so that also b) is satisfied. Comparing the previous relations one gets that the feedback law $u(t) = \beta^{-1}(\mathbf{x})[-\alpha(\mathbf{x})+v(t)]$ which linearizes the input output behavior is characterized by $\beta^{-1}(\mathbf{x}) = \frac{1}{x_2(t)+1}$ and $\alpha(\mathbf{x}) = x_1(t)x_2(t-1)$. In fact with such a bicausal static state feedback and in the considered change of coordinates the system reads

$$\begin{aligned} \dot{z}_1(t) &= 2z_1(t)+z_1(t-1) \\ \dot{z}_2(t) &= v(t)+v(t-1) \end{aligned}$$

Conclusions

In the present paper we have applied the geometric framework introduced in [3],[5] for dealing with nonlinear time–delay systems in order to solve the linear feedback equivalence problem. It is shown that the existence of a function with relative degree equal to n is a necessary condition but not anymore sufficient to guarantee the existence of a bicausal change of coordinates and a bicausal state feedback which linearizes the given dynamics. Some additional conditions must thus be considered to deal with the general case of delay systems.

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