A sensitivity trade-off arising in small-gain design for nonlinear systems: an iISS framework

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Abstract— This note investigates the trade-off arising in disturbance attenuation for nonlinear feedback systems in the framework of integral input-to-state stability. Similarly to the linear case, we show that if a gain tuning on one subsystem is used to drastically reduce the effect of its exogenous disturbances, then the other subsystem's disturbance attenuation is qualitatively the same as in open loop.

I. INTRODUCTION

The objective of the present paper is to provide some insights on how the well-known sensitivity / co-sensitivity trade-off arising for feedback linear time-invariant (LTI) systems extends to nonlinear plants. More precisely, given two feedback nonlinear subsystems, assume that the nonlinear gain of one subsystem can be made smaller by a convenient control design. Then the nonlinear loop gain becomes smaller and the small-gain stability criterion is satisfied with a larger margin. A natural question is then whether this induces stronger robustness to disturbances for the overall feedback system. We give an answer to this question in the dissipative formulation for input-to-state stability (ISS, [19]) and integral ISS ([21]) systems.

The results presented along this paper rely on small gain arguments. More precisely, we make use of recent results on Lyapunov-based small gain theorems for iISS [11], which include ISS as a special case. Compared to other nonlinear small gains existing in the literature such as [12], [13], [22], [1], [5], this result allows both to deal with not necessarily ISS systems, and to provide an explicit construction of a Lyapunov function for the overall interconnection in the presence of exogenous inputs, which are two helpful features for this work.

Instead of relying on the exact knowledge of differential equation models, we employ iISS dissipation inequalities to describe nonlinear systems in feedback loop. Compared to the frequency analysis for LTI systems (cf. classical textbooks such as [6]), iISS dissipation inequalities do not provide an equality between the input and its response, but rather an *inequality* that provides only a "worst-case" estimate (sometimes not very tight) of the input influence on the overall system: no distinction can be made between

systems that are strongly sensitive to inputs, and those for which the dissipation inequality is simply too loose.

In order to overpass this difficulty, we proceed in two different manners. The first one (Section IV) consists in building, for a given pair (α, γ) of iISS supply rates, an iISS system $\dot{x} = f(x, d)$ for which these estimates are tight, in the sense that *all* disturbances that may act on that system have a negative impact on the system's performance and that this effect is not compensated by a dissipation rate stronger than the prescribed one. Roughly speaking, this is done by imposing (at least in some relevant state regions)

$$\frac{\partial V}{\partial x}(x)f(x,d) = -\alpha(|x|) + \gamma(|d|),$$

where V denotes a given Lyapunov function candidate. The equality sign in this equation guarantees the sought tightness of the estimates. We show that, given an iISS supply pair (α, γ), Lyapunov-based small gain arguments always authorize the existence of such a system and consequently the non-rejection of some disturbances. Of course, this first approach is of purely theoretical interest, as the constructed system has typically no practical relevance. The second approach (Section V) demonstrates this trade-off without introducing such fictitious subsystems. Assuming that a disturbance does have a negative effect on one subsystem's performance, we show that, in feedback, this effect cannot be attenuated by the gain tuning of the other subsystem.

Notation. Given $x \in \mathbb{R}^n$, $|x|$ denotes its Euclidean norm. Given a set $\mathcal{A} \subset \mathbb{R}^n$, $|x|_{\mathcal{A}} := \inf_{z \in \mathcal{A}} |x - z|$. Given a constant $\delta > 0$, $\mathcal{B}_\delta := \{x \in \mathbb{R}^n : |x| \le \delta\}$. Given a set $A \subset \mathbb{R}$ and a constant $a \in \mathbb{R}$, $A_{\ge a} := \{s \in A : s \ge a\}$. $\text{sat}_\delta : \mathbb{R}^n \to \mathbb{R}^n$ is defined, for all $x \in \mathbb{R}^n$, by $\text{sat}_\delta(x) := (\delta \text{sat}(x_1/\delta), \ldots, \delta \text{sat}(x_n/\delta))^T$, where $\text{sat}(s) := \min(|s|, 1)\text{sign}(s)$ for all $s \in \mathbb{R}$. Given a function $\sigma : \mathbb{R}^m \to \mathbb{R}^n$, $\ker(\sigma) := \{x \in \mathbb{R}^m : \sigma(x) = 0\}$. A continuous function $\alpha : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is said to be of class $\mathcal{PD}$ if it is positive definite. It is said to be in class $\mathcal{K}$ if, in addition, it is increasing. It is said to be of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and $\alpha(s) \to \infty$ as $s \to \infty$. A function $\beta : \mathbb{R}_{\ge 0} \times \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is said to be of class $\mathcal{KL}$ if $\beta(\cdot, t) \in \mathcal{K}$ for any fixed $t \ge 0$ and $\beta(s, \cdot)$ is continuous non-increasing and tends to zero at infinity for any fixed $s \ge 0$. Given $\alpha \in \mathcal{K}$, $\alpha(\infty) \in \mathbb{R}_{\ge 0} \cup \{\infty\}$ is defined as $\lim_{s \to \infty} \alpha(s)$. Given $\alpha, \gamma \in \mathcal{K}$, $\alpha(\infty) > \gamma(\infty)$ means that either $\alpha \in \mathcal{K}_\infty$, or $\alpha(\infty) = c_\alpha \in \mathbb{R}_{\ge 0}$ and $\gamma(\infty) = c_\gamma \in \mathbb{R}_{\ge 0}$ with $c_\alpha > c_\gamma$. $\mathcal{U}^m$ is the set of measurable locally essentially bounded signals $d : \mathbb{R}_{\ge 0} \to \mathbb{R}^m$. Given $u \in \mathcal{U}^m$, $\|u\| := \text{ess sup}_{t \ge 0} |u(t)|$. Given $\Delta \ge 0$,

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$\mathcal{U}_{\leq\Delta}^m := \{u \in \mathcal{U}^m : \|u\| \leq \Delta\}$. $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is called a Lyapunov function candidate if it is C^1, positive definite and radially unbounded.

II. PROBLEM STATEMENT

We consider two dynamical systems Σ_1 and Σ_2 interconnected in a feedback configuration through their outputs y_1 and y_2, and subject to exogenous disturbances d_1 and d_2, cf. Fig. 1.
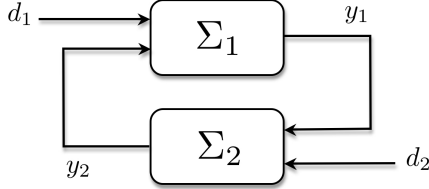

Fig. 1.   Feedback interconnection.

It is well known that, when Σ_1 and Σ_2 are LTI, the sensitivity / co-sensitivity tradeoff impedes the disturbance rejection of both d_1 and d_2 at the same frequency. To sketch out this tradeoff, consider single input - single output systems and let H_i denote the transfer function of Σ_i, $i \in \{1, 2\}$. If H_1 is tuned in such a way that $H_2 H_1 (1 - H_2 H_1)^{-1} \to 0$ at a given frequency, then one cannnot avoid $(1 - H_1 H_2)^{-1} \to 1$. This results in $H_2(1 - H_1 H_2)^{-1} \to H_2$, meaning that the d_1-rejection imposes that the effect of d_2 is the same as in open-loop. This fundamental obstruction to control design was first studied in [3]. It imposes, in particular, a compromise between precision / output disturbance rejection and sensor noise attenuation. See [9], [16], [18], [8] for an in-depth analysis. The aim of this paper is to analyze to what extent this result can be adapted to nonlinear plants. The feedback interconnection considered in this note is

$$\dot{x}_1 = f_1(x_1, x_2, d_1, \theta) \tag{1a}$$
$$\dot{x}_2 = f_2(x_2, x_1, d_2), \tag{1b}$$

where $(x_1^T, x_2^T)^T =: x \in \mathbb{R}^{n_1 + n_2}$ denote the state of each subsystem, $(d_1^T, d_2^T)^T =: d \in \mathcal{U}^{m_1 + m_2}$ are exogenous disturbances, and $\theta \in \Theta \subset \mathbb{R}^p$ is a free parameter as, for instance, a vector of tuning gains. The functions f_1 and f_2 are assumed to be continuous. We stress that this structure does not necessarily require that the subsystems (1a) and (1b) be connected through their whole states, but rather authorizes output feedback interconnection as f_1 (resp. f_2) may involve only part of x_2 (resp. x_1) or a function of its entries.

While the above LTI reasoning does not require any stability assumption on Σ_1 and Σ_2 when considered individually, the small-gain approach we follow in this note imposes that each subsystem be iISS with a class $\mathcal{K}$ dissipiation rate [21].

Assumption 1 *There exist $\alpha_1, \gamma_1, \varphi_1 \in \mathcal{K}$, $\underline{\alpha}_1, \overline{\alpha}_1 \in \mathcal{K}_\infty$, and a C^1 function $V_1 : \mathbb{R}^{n_1} \to \mathbb{R}_{\geq 0}$ satisfying $\underline{\alpha}_1(|x_1|) \leq V_1(x_1) \leq \overline{\alpha}_1(|x_1|)$ such that, given any $\lambda > 1$, there exist $\theta \in \Theta$ such that, for all $(x_1, x_2) \in \mathbb{R}^{n_1 + n_2}$ and all $d_1 \in \mathbb{R}^{m_1}$,*

$$\frac{\partial V_1}{\partial x_1} f_1 \leq -\alpha_1(|x_1|) + \frac{1}{\lambda} \left[\gamma_1(|x_2|) + \varphi_1(|d_1|) \right]. \tag{2}$$

This first assumption not only guarantees iISS for the x_1-subsystem (1a), but also that the disturbance rejection for this subsystem can be tuned at will by a convenient choice of the parameter θ. More precisely, considering $u_1 := (x_2^T, d_1^T)^T$ as the exogenous input of (1a) and relying on classical reasonings for iISS systems (cf. [2, Corollary IV.3]), Assumption 1 naturally yields the following trajectory estimate for (1a):

$$|x_1(t)| \leq \beta(|x_{10}|, t) + \eta\left(\frac{1}{\lambda}\int_0^t \tilde{\gamma}_1(|u_1(\tau)|)d\tau\right) \tag{3}$$

where $x_1(\cdot) := x_1(\cdot; x_{10}, x_2, d_1, \theta)$, $x_2(\cdot) := x_2(\cdot; x_{20}, x_1, d_2)$, $\tilde{\gamma}_1(\cdot) := 2\max\{\gamma_1(\cdot), \varphi_1(\cdot)\}$ and η and β denote respectively class $\mathcal{K}$ and $\mathcal{KL}$ functions. Thus, once the exogenous signals x_2 and d_1 are given, the above estimate illustrates the possibility to arbitrarily reject their effect on the behavior of the x_1-subsystem by conveniently tuning θ (*i.e.* by increasing λ). Note that the dissipation rate α_1 is assumed to belong to class $\mathcal{K}$ rather than simply $\mathcal{PD}$ as in [2]. This is motivated by the small gain argument [11] we invoke in the sequel. Hence, Assumption 1 actually imposes iISS plus ISS with respect to small inputs[1] with an assignable supply rate. We stress that, under specific matching conditions, Assumption 1 can be ensured by control designs available in the literature such as [17, Lemma 3]. The results in [23], [15] may also be inspiring. See [4, Proposition 1] for a relevant example. Anyway, even though Assumption 1 may be hard to achieve in practice, this note aims precisely at showing that, despite such a strong stabilizability assumption, disturbance rejection cannot be expected to be arbitrary in a feedback interconnection.

On the other hand, the x_2-subsystem is assumed to be iISS, with a fixed supply rate. Again, the dissipation rate is assumed to be in class $\mathcal{K}$, thus guaranteeing ISS with respect to small inputs.

Assumption 2 *There exist $\alpha_2, \gamma_2, \varphi_2 \in \mathcal{K}$, $\underline{\alpha}_2, \overline{\alpha}_2 \in \mathcal{K}_\infty$, and a C^1 function $V_2 : \mathbb{R}^{n_2} \to \mathbb{R}_{\geq 0}$ such that, for all $(x_1, x_2) \in \mathbb{R}^{n_1 + n_2}$ and all $d_2 \in \mathbb{R}^{m_2}$,*

$$\underline{\alpha}_2(|x_2|) \leq V_2(x_2) \leq \overline{\alpha}_2(|x_2|) \tag{4}$$

$$\frac{\partial V_2}{\partial x_2} f_2(x_2, x_1, d_2) \leq -\alpha_2(|x_2|) + \gamma_2(|x_1|) + \varphi_2(|d_2|). \tag{5}$$

III. TUNING FOR d_1−REJECTION

The following result formally shows that, as expected, the tuning of θ allows for arbitrary attenuation of d_1.

Proposition 1 *Let Assumptions 1 and 2 hold and assume that the following implication holds true for each $i \in \{1, 2\}$:*

$$\gamma_{3-i} \in \mathcal{K}_\infty \quad \Rightarrow \quad \alpha_i \in \mathcal{K}_\infty. \tag{6}$$

[1]This combination is sometimes referred to as Strong iISS.

Assume also that the small-gain condition[2]

$$c_2\gamma_2 \circ \underline{\alpha}_1^{-1} \circ \overline{\alpha}_1 \circ \alpha_1^{-1} \circ c_1\gamma_1(s) \leq \alpha_2 \circ \overline{\alpha}_2^{-1} \circ \underline{\alpha}_2(s) \quad (7)$$

holds for all $s \geq 0$, where $c_1 > 0$ and $c_2 > 1$ denote some constants. Then, there exist $\beta \in \mathcal{KL}$, $\underline{\alpha}, \gamma, \zeta \in \mathcal{K}_\infty$, and $\Delta > 0$ and, given any $\ell > 1$, there exist $\theta \in \Theta$ such that, for all $x_0 \in \mathbb{R}^{n_1+n_2}$, all $d_1 \in \mathcal{U}^{m_1}$ and all $d_2 \in \mathcal{U}^{m_2}$, the feedback interconnection (1) is iISS and ISS with respect to small inputs, and its solution satisfies

$$\begin{aligned} \underline{\alpha}(|x(t)|) &\leq \beta(|x_0|, t) + \int_0^t \gamma(|d_1(\tau)|/\ell)\, d\tau \\ &+ \int_0^t \gamma(|d_2(\tau)|)d\tau, \quad \forall t \geq 0, \end{aligned} \quad (8)$$

and, for all $d_1 \in \mathcal{U}^{m_1}_{\leq \ell\Delta}$ and all $d_2 \in \mathcal{U}^{m_2}_{\leq \Delta}$,

$$|x(t)| \leq \beta(|x_0|, t) + \zeta(\|d_1\|/\ell) + \zeta(\|d_2\|). \quad (9)$$

It is worth noting that the upper and lower bounds on V_i (namely, $\underline{\alpha}_i$ and $\overline{\alpha}_i$), $i \in \{1, 2\}$, involved in (7) could be removed if (2) and (5) were replaced by dissipation inequalities involving only V_i rather than x_i. We keep the original small-gain condition (7) of [11] as the bounds (2) and (5) are usually easier to establish in practice.

We also stress that small-gain condition in [11] requires both c_1 and c_2 to be greater than 1. Relaxing to only $c_1 > 0$ in (7) is made possible by the fact that, in the context of this article, the constant λ multiplying the supply rate γ_1 is tunable at will through the parameter θ (cf. Assumption 1).

Apart from these details, the iISS and ISS with respect to small inputs of the feedback interconnection (1) under (6)-(7) directly follows from previous results of the second author [11]. See [4] for the complete proof. Let us recall that the small-gain condition (7) is not symmetric. We have chosen to assume (7) rather than its counterpart:

$$c_1\gamma_1 \circ \underline{\alpha}_2^{-1} \circ \overline{\alpha}_2 \circ \alpha_2^{-1} \circ c_2\gamma_2(s) \leq \alpha_1 \circ \overline{\alpha}_1^{-1} \circ \underline{\alpha}_1(s),$$

in order to allow for the interconnection of not necessarily ISS subsystems. See [11] for details.

Compared to [11], the novelty of Proposition 1 stands in the explicit estimate of the disturbance attenuation allowed by the tuning gain θ. Indeed, since the functions $\underline{\alpha}, \beta, \gamma$ and ζ in (8)-(9) are independent of ℓ, Proposition 1 guarantees that the effect of the exogenous disturbance d_1 over the solutions' behavior can be made arbitrarily small provided a convenient tuning of θ (*i.e.*, corresponding to sufficiently large λ and ℓ). In addition, since (9) ensures ISS with respect to all d_1 of amplitude smaller than $\ell\Delta$, with Δ independent of ℓ, the class of ISS-tolerated disturbances can be enlarged at will. These constitute two interesting features for the rejection of the d_1 disturbance.

However, no such d_2-disturbance attenuation appears in the trajectory estimates (8)-(9). This fact could either be

due to an intrinsic property of feedback interconnections, or simply to the looseness of the upper bounds (8)-(9). The rest of the paper shows that this property is indeed intrinsic and that no such d_2-attenuation can be expected in general.

The proof of Proposition 1 is omitted due to lack of space, but can be found on the on-line preprint [4]. It relies on the following lemma, whose proof can be found along the lines of [11].

Lemma 1 *For each $i \in \{1, 2\}$, let $V_i : \mathbb{R}^{n_i} \to \mathbb{R}_{\geq 0}$ be a C^1 function satisfying, for all $x_i \in \mathbb{R}^{n_i}$, $\underline{\alpha}_i(|x_i|) \leq V_i(x_i) \leq \overline{\alpha}_i(|x_i|)$ with $\underline{\alpha}_i, \overline{\alpha}_i \in \mathcal{K}_\infty$, and assume that there exist $\alpha_i, \gamma_i, \varphi_i \in \mathcal{K}$ such that (6) holds and, for all $s \geq 0$,*

$$c_2\gamma_2 \circ \underline{\alpha}_1^{-1} \circ \overline{\alpha}_1 \circ \alpha_1^{-1} \circ c_1\gamma_1(s) \leq \alpha_2 \circ \overline{\alpha}_2^{-1} \circ \underline{\alpha}_2(s) \quad (10)$$

with $c_1, c_2 > 1$. Then there exist $\rho_1, \rho_2, \alpha, \gamma \in \mathcal{K}$ such that, for all $(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$,

$$\begin{aligned} \sum_{i=1}^2 \rho_i(V_i(x_i)) \Big[-\alpha_i(|x_i|) + \gamma_i(|x_{3-i}|) + \varphi_i(|d_i|) \Big] \\ \leq -\alpha(|x|) + \gamma(|d|). \end{aligned}$$

IV. SENSITIVITY TO d_2: A "WORST CASE" SYSTEM

In contrast to the previous section, we now show that the increase of λ, by a convenient tuning of the gain θ, is in general of no help in reducing the influence of d_2 over x_2. The proof of this result is provided in Section VII-A.

Theorem 1 *Let Assumption 1 hold, let $d_2^{min} < d_2^{max}$ be two positive constants, and let $\alpha_2, \gamma_2, \varphi_2$ denote some given $\mathcal{K}$ functions. Let $V_2 : \mathbb{R}^{n_2} \to \mathbb{R}_{\geq 0}$ be any Lyapunov function candidate satisfying*

$$\frac{\partial V_2}{\partial x_2}(x_2) \neq 0, \quad \forall x_2 \neq 0. \quad (11)$$

Then one can always find class $\mathcal{K}$ functions ν_2 and η_2, and a vector field $f_2 : \mathbb{R}^{n_2} \times \mathbb{R}^{n_1} \times \mathbb{R}^{m_2} \to \mathbb{R}^{n_2}$, continuous on $\mathbb{R}^{n_2} \times \mathbb{R}^{n_1} \times (\mathbb{R}^{m_2} \setminus \{0\})$, satisfying Assumption 2 with these prescribed functions α_2, γ_2 and φ_2, and such that, given any $\theta \in \Theta$, any initial state $x_{20} \in \mathbb{R}^{n_2}$ and any disturbance $d_1 \in \mathcal{U}^{m_1}$ and $d_2 \in \mathcal{U}^{m_2}$ satisfying

$$d_2^{min} \leq \|d_2(t)\| \leq d_2^{max}, \quad (12)$$

all forward complete solutions of (1) starting with $|x_{20}| \geq \eta_2(\|d_2\|)$ satisfy

$$|x_2(t)| \geq \nu_2(\text{ess inf}_{\tau \geq 0} |d_2(\tau)|), \quad \forall t \geq 0. \quad (13)$$

Theorem 1 shows that the only knowledge of the dissipation inequality associated to each subsystem cannot guarantee, in general, an arbitrary d_2-disturbance attenuation even when control gains can be tuned in order to decrease the sensitivity of the x_1-subsystem with respect to its inputs. Indeed, it guarantees that such an interconnection may always yield, for some particular systems, the existence of an

incompressible lower bound (13) whose size is somewhat "proportional" to the minimal value of $|d_2|$, for solutions starting sufficiently far from the origin. The crucial point is that this lower bound holds regardless of the chosen gain θ. It is therefore hopeless to expect arbitrary d_2-disturbance rejection for this system by relying only on the associated dissipation inequalities.

Remark 1 *If in addition to the assumptions of Theorem 1, the small gain condition (7) holds, then the assumptions of Proposition 1 are satisfied and consequently the feedback interconnection (1) is iISS and ISS with respect to small inputs (cf. (8)-(9)) if λ is made small enough by a convenient choice of θ. In particular, (1) results forward complete and the lower bound (13) holds at all times.*

The property stated as Theorem 1 is quite intuitive once the inequality (5) is sufficiently tight. The contribution of this result is, in fact, to show that such a dissipation inequality is always tight for some particular systems. More precisely, the proof of Theorem 1 relies on the following lemma, that may have interest on its own. It is similar in spirit to [11, Lemma 1], but applies to any given Lyapunov function candidate. Its proof omitted due to lack of space, but can be found in [4].

Lemma 2 *Given $m, n \in \mathbb{N}_{\geq 1}$, let $\varphi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be any continuous function satisfying*

$$|x| \leq \sigma(u) \quad \Rightarrow \quad \varphi(x, u) \geq 0, \tag{14}$$

for some continuous function $\sigma : \mathbb{R}^m \to \mathbb{R}_{\geq 0}$. Consider any Lyapunov function candidate $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfying

$$\frac{\partial V}{\partial x}(x) \neq 0, \quad \forall x \neq 0. \tag{15}$$

Then, there exists a vector field $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, continuous on $\mathbb{R}^n \times (\mathbb{R}^m \setminus ker(\sigma))$, such that, for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$,

$$\frac{\partial V}{\partial x}(x) f(x, u) \leq \varphi(x, u) \tag{16}$$

$$|x| \geq \sigma(u) \Rightarrow \frac{\partial V}{\partial x}(x) f(x, u) = \varphi(x, u). \tag{17}$$

This lemma shows that, under mild assumptions, the dissipation inequality (5) is always "tight" for what we refer to as a *worst-case* system. In other words, any Lyapunov function candidate constitutes a tight iISS/ISS estimate of the behavior of these systems. This can be seen by taking φ as an iISS or ISS supply pair for this system. Here we refer to a "worst case" situation as, for this system, the application of *any* input signal works against the convergence of the associated Lyapunov function, and that it can be compensated by no greater dissipation rate than $\alpha(|x|)$.

Remark 2 *The right-hand side f of the constructed system may not be locally Lipschitz. However, depending on the choice of the function φ, the existence of solutions may be guaranteed at all time. For instance, the application of the comparison lemma guarantees forward completeness for any function φ satisfying, at least for large $|x|$,*

$$\varphi(x, u) \leq cV(x) + \eta(|u|),$$

where $c \in \mathbb{R}$ and $\eta : \mathbb{R}^m \to \mathbb{R}$ denotes a continuous function. See [10] for further discussions on how forward completeness of feedback systems can be guaranteed. Also, the fact that f is not necessarily continuous in $u = 0$ is not a crucial issue as Lemma 2 will typically be used for inputs lower-bounded away from zero.

V. SENSITIVITY TO d_2: IMPEDING DISTURBANCES

In most situations, exogenous inputs do not systematically work against the convergence of the associated Lyapunov function, as opposed to the worst-case systems developed in Section IV. For instance, for the scalar system $\dot{x} = -x + d$, any positive signal d tends to slowing down the convergence of x to zero for positive values of the initial state x_0, but it actually speeds it up if $x_0 \leq 0$. This observation suggests that no tight Lyapunov function, in the sense of Lemma 2, exists for most dynamical systems of practical relevance, nor can a Lyapunov function candidate W satisfying

$$\dot{W} \geq -\alpha(|x|) + \gamma(|u|), \quad \forall x \in \mathbb{R}^n, \forall u \in \mathbb{R}^m,$$

with $\alpha, \gamma \in \mathcal{K}$, be expected in general. On the other hand, in many cases, disturbances do induce an increase of the associated Lyapunov function at least in some regions of the state space. It is also reasonable to assume that their size are bounded for bounded states. This motivates the following assumption, which can be seen as a destabilizing counterpart of the small control property, cf. *e.g.* [20], [7].

Assumption 3 *There exists a Lyapunov function candidate $W_2 : \mathbb{R}^{n_2} \to \mathbb{R}_{\geq 0}$, a $\mathcal{K}$ function Υ_2 and a continuous[3] function $d_2 : \mathbb{R}^{n_1+n_2} \to \mathbb{R}^{m_2}$ such that, given any $x = (x_1^T, x_2^T)^T \in \mathbb{R}^{n_1+n_2}$,*

$$|d_2(x)| \leq \Upsilon_2(|x|) \tag{18}$$

$$\frac{\partial W_2}{\partial x_2}(x_2) f_2(x_2, x_1, d_2(x)) > 0.$$

This assumption ensures that at least one disturbance, whose size is somewhat "proportional" to the state norm, tends to destabilize the x_2-subsystem with x_1 as an input. For feedback systems satisfying Assumption 3, the following result shows that the tuning of the gain θ cannot be expected to induce arbitrary d_2-disturbance rejection. Due to space constraints, its proof cannot be included here, but can be found in the on-line preprint [4].

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[3]The continuity requirement on d_2 may probably be relaxed by relying on Arstein-type constructions [20] to get a continuous destablizing feedback. Since such a construction is of limited interest in the context of this note, we assume continuity of d_2 for simplicity.

Theorem 2 *Let Assumption 3 hold. Then there exists $\Upsilon \in \mathcal{K}$ such that, given any $\delta > 0$, there exists a signal $d_2^\star \in \mathcal{U}^{m_2}$ satisfying*

$$\|d_2^\star\| \le \Upsilon(\delta) \qquad (19)$$

such that, given any $\theta \in \Theta$ and any $d_1 \in \mathcal{U}^{m_1}$, the set $\mathbb{R}^n \backslash \mathcal{B}_\delta$ is globally attractive for the feedback interconnection (1) (i.e., $\liminf_{t\to\infty} |x(t; x_0, d)| \ge \delta$ for all $x_0 \in \mathbb{R}^n$) if the latter is forward complete.

The above result establishes that, for all systems satisfying Assumption 3, either the resulting interconnection is not forward complete (in which case disturbance rejection is obviously not achieved), or any ball centered at the origin can be made repellent for the overall interconnection, regardless of the choice of the tuning gain θ, by a bounded disturbance $d_2^\star$ whose amplitude is "proportional" to the size of the chosen ball. This means that the maximum disturbance rejection is purely a function of the applied disturbance $d_2^\star$ and that the tuning of θ has no effect on it. We stress that, in the above result, the larger the upper bound in (19) is, the further away from origin solutions will asymptotically go to (as $\mathcal{B}_\delta$ grows larger).

Remark 3 *If, in addition, the vector fields f_1 and f_2 are chosen according to Assumptions 1 and 2 and the small gain condition (7) holds, then Proposition 1 ensures that the overall system is iISS (hence, forward complete).*

VI. Conclusion

Motivated by the observation that the smaller the loop gain is, the larger the internal stability margin is for a feedback system, this paper has investigated the effect of decreasing the loop gain on external stability, and established a natural trade-off between rejection of disturbances entering in different places in the feedback loop. If one subsystem's parameters are tuned to reduce the effects of its disturbances, then the other subsystem eventually has been shown to behave as if it were in open-loop. While this trade-off is quite natural, the dissipation formulation of this paper enables to confirm the property for nonlinear systems, thus without relying on transfer functions. This iISS framework employed in this paper also allows to encompass subsystems whose solutions are not necessarily bounded for bounded inputs. The extension to the interconnection of more than two subsystems cas be envisioned based on large-scale small gain theorems such as [5].

VII. Proofs

A. Proof of Theorem 1

First of all, notice that, since V_2 is a Lyapunov function candidate, (4) holds for some $\underline{\alpha}_2, \overline{\alpha}_2 \in \mathcal{K}_\infty$. Let $u_2 := (x_1^T, d_2^T)^T$ and consider

$$\varphi(x_2, u_2) = -\alpha_2(|x_2|) + \gamma_2(|x_1|) + \tilde{\varphi}_2(|d_2|),$$

where $\tilde{\varphi}_2$ is the class $\mathcal{K}$ function defined as

$$\tilde{\varphi}_2(s) := \frac{1}{2}\min\{\varphi_2(s); \alpha_2(s)\}, \quad \forall s \ge 0.$$

This construction of $\tilde{\varphi}_2$ ensures that the function $\alpha_2^{-1} \circ \tilde{\varphi}_2$ is well defined over $\mathbb{R}_{\ge 0}$. Also, this function satisfies (14) for any continuous nonnegative function σ such that, for all $u_2 \in \mathbb{R}^{n_1+m_2}$, $\sigma(u_2) \le \alpha_2^{-1} \circ \tilde{\varphi}_2(|d_2|)$. In particular, this condition is fulfilled with $\sigma(u_2) = \sigma_2(|d_2|)$, if σ_2 is the $\mathcal{K}$ function defined as

$$\sigma_2(s) := \alpha_2^{-1} \circ \tilde{\varphi}_2\left(\frac{d_2^{min}s}{2d_2^{max}}\right), \forall s \ge 0. \qquad (20)$$

Applying Lemma 2 to V_2 with the above functions φ and σ ensures the existence of a vector field f_2 such that $\dot{V}_2 := \frac{\partial V_2}{\partial x_2}(x_2)f_2(x_2, x_1, d_2) \le -\alpha_2(|x_2|) + \gamma_2(|x_1|) + \tilde{\varphi}_2(|d_2|)$, for all $x \in \mathbb{R}^n$ and all $d_2 \in \mathbb{R}^{n_2}$. This makes Assumption 2 fulfilled by noticing that $\tilde{\varphi}_2(s) \le \varphi_2(s)$ for all $s \in \mathbb{R}_{\ge 0}$. Lemma 2 also guarantees that, for all x and d_2 satisfying $|x_2| \ge \sigma_2(|d_2|)$,

$$\begin{aligned} \dot{V}_2 &= -\alpha_2(|x_2|) + \gamma_2(|x_1|) + \tilde{\varphi}_2(|d_2|) \\ &\ge -\alpha_2(|x_2|) + \tilde{\varphi}_2(|d_2|). \end{aligned} \qquad (21)$$

Note that, since $\sigma(u_2) = \sigma_2(|d_2|)$ and $\sigma_2 \in \mathcal{K}$, $\ker(\sigma) = \mathbb{R}^{n_1} \times (\mathbb{R}^{m_2} \backslash \{0\})$. Lemma 2 thus ensures that f_2 is continuous over $\mathbb{R}^{n_2} \times \mathbb{R}^{n_1} \times (\mathbb{R}^{m_2} \backslash \{0\})$. Now, consider any disturbance $d_2 \in \mathcal{U}^{m_2}$ satisfying (12) and let $\underline{d}_2 := \text{ess inf}_{\tau \ge 0} |d_2(\tau)|$. Note that it holds that

$$\underline{d}_2 \ge d_2^{min}, \quad \|d_2\| \le d_2^{max}. \qquad (22)$$

Let $\theta \in \Theta$ be any arbitrary tuning gain, let $d_1 \in \mathcal{U}^{m_1}$, and consider any forward complete solution of (1) starting with an initial condition $x_0 = (x_{10}^T, x_{20}^T)^T \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ satisfying

$$|x_{20}| \ge \alpha_2^{-1} \circ \tilde{\varphi}_2(\overline{d}_2). \qquad (23)$$

In view of (20), this ensures in particular that

$$|x_{20}| > \sigma_2(\|d_2\|). \qquad (24)$$

Let $t_1 \in \mathbb{R}_{\ge 0} \cup \{\infty\}$ be defined as

$$t_1 := \sup\{t \ge 0 : |x_2(\tau)| > \sigma_2(\|d_2\|) \, \forall \tau \in [0, t)\}. \quad (25)$$

In view of (24) and invoking the continuity of solutions, it holds that $t_1 \in \mathbb{R}_{>0} \cup \{\infty\}$ and, for all $t \in [0, t_1)$, it holds from (4) and (21) that

$$\begin{aligned} \dot{v}_2(t) &\ge -\alpha_2(|x_2(t)|) + \tilde{\varphi}_2(|d_2(t)|) \\ &\ge -\alpha_2 \circ \underline{\alpha}_2^{-1}(v_2(t)) + \tilde{\varphi}_2(\underline{d}_2), \end{aligned} \qquad (26)$$

where $v_2(\cdot) := V_2(x_2(\cdot))$. We then rely on the following lemma, whose proof is given in Section VII-B.

Lemma 3 *Let α be a class $\mathcal{K}$ locally Lipschitz function and let $a \in \mathbb{R}_{\ge 0}$. Let $[0, \overline{t}) \subset \mathbb{R}_{\ge 0}$ be the maximum interval of existence of a diffentiable function v whose derivative satisfies $\dot{v}(t) \ge -\alpha(v(t)) + a$ for all $t \in [0, \overline{t})$. Then the following implication holds:*

$$\alpha(v(0)) \ge a \Rightarrow \alpha(v(t)) \ge a, \quad \forall t \in [0, \overline{t}).$$

Recalling that the function $\alpha_2 \circ \underline{\alpha}_2^{-1}$ is invertible over $[0, \tilde{\varphi}_2(\underline{d}_2)]$ by construction of $\tilde{\varphi}_2$, Equation (26) together with Lemma 3 ensure that

$$v_2(0) \geq \underline{\alpha}_2 \circ \alpha_2^{-1} \circ \tilde{\varphi}_2(\underline{d}_2) \Rightarrow$$
$$v_2(t) \geq \underline{\alpha}_2 \circ \alpha_2^{-1} \circ \tilde{\varphi}_2(\underline{d}_2), \quad \forall t \in [0, t_1),$$

which yields, in view of (4),

$$|x_{20}| \geq \eta_2(\underline{d}_2) \Rightarrow |x_2(t)| \geq \nu_2(\underline{d}_2),$$

where the functions $\eta_2, \nu_2 \in \mathcal{K}$ are defined as

$$\begin{aligned} \eta_2 &:= \alpha_2^{-1} \circ \tilde{\varphi}_2 \\ \nu_2 &:= \overline{\alpha}_2^{-1} \circ \underline{\alpha}_2 \circ \alpha_2^{-1} \circ \tilde{\varphi}_2. \end{aligned} \quad (27)$$

Equation (23) guarantees that the left-hand side of the above implication holds true. Hence

$$|x_2(t)| \geq \nu_2(\underline{d}_2), \quad \forall t \in [0, t_1). \quad (28)$$

In other words, Theorem 1 is proved if we show that $t_1 = +\infty$. If it were not the case, then it would mean, in view of (25), that

$$|x_2(t_1)| = \sigma_2(\|d_2\|). \quad (29)$$

Consider the greatest time $t_2 \geq 0$ for which

$$|x_2(t)| \geq \nu_2(\underline{d}_2), \quad \forall t \in [0, t_2]. \quad (30)$$

In view of (28), we necessarily have that $t_2 \geq t_1$. But (20) and (27) ensure that $\sigma_2(\overline{d}_2) < \nu_2(\underline{d}_2)$. The continuity of solutions together with (25), (29) and (30) then impose that $t_2 < t_1$, which induces a contradiction. Thus, t_1 is infinite, which makes (28) valid for all $t \geq 0$ and concludes the proof.

B. Proof of Lemma 3

We distintiguish between two cases.

Case 1: $a < \alpha(\infty)$. Consider the differential equation $\dot{y} = -\alpha(y) + a$. Then letting $z := y - \alpha^{-1}(a)$ yields $\dot{z} = -\tilde{\alpha}(z)$ where $\tilde{\alpha}$ is the locally Lipschitz class $\mathcal{K}$ function defined as $\tilde{\alpha}(s) := \alpha(s + \alpha^{-1}(a)) - a$. By [14, Lemma 4.4], $z(\cdot)$ exists over $\mathbb{R}_{\geq 0}$ and satisfies $z(t) = \beta(z(0), t)$, where $\beta \in \mathcal{KL}$, for all $z(0) \geq 0$, and all $t \geq 0$. In terms of y, this reads $y(t) = \beta(y(0) - \alpha^{-1}(a), t) + \alpha^{-1}(a)$ for all $y(0) \geq \alpha^{-1}(a)$. But [14, Lemma 3.3] guarantees that, if $v(0) \geq y(0)$, then $v(t) \geq y(t)$ for all $t \in [0, \bar{t})$. It follows that, for all $v(0) \geq \alpha^{-1}(a)$, $v(t) \geq \alpha^{-1}(a)$ for all $t \in [0, \bar{t})$.

Case 2: $a \geq \alpha(\infty)$. In this case, $\dot{v}(t) \geq 0$ for all $t \in [0, \bar{t})$, which makes the claim trivial.

VIII. Acknowledgement

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