2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

# Active mode and switching time estimation for switched linear systems

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*Abstract*— **This paper is concerned with the determination of the switching times for linear time-invariant switched systems. Using a distribution framework, an implementable computational method is proposed in order to obtain a finite time estimation of the switching times by explicit and non asymptotic formulae, avoiding derivative measurements. The above procedure also determines the sequence of active modes when the initial mode is known. Numerical simulations illustrating our method are provided.**

## I. INTRODUCTION

Hybrid systems provide an appropriate mathematical framework to describe real phenomena that exhibit discontinuous behaviors. Roughly speaking, hybrid systems are systems in which continuous dynamics and discrete events coexist and interact between each other. Switched linear systems are an important subclass of hybrid systems that can be used to model a large number of physical processes, and to approximate nonlinear dynamics, e.g., via multiple linearizations at different operating points. They are widely used because the well-mastered tools for analysis and control of linear systems can be extended, under some assumptions, to this class of systems. Research on switched systems is mainly focused on the fields of stability [3], [4], [5], [6], [7], stabilization [8], [9], [10], or controllability [11], [12].

The knowledge of the mode describing the evolution of the system at any moment is a crucial piece of information that simplifies the application of the various results coming from the fields of identification, control, stability analysis, and state estimation. Ackerson and Fu [13] were the first to consider the question of the determination of the active mode by stating the problem in the form of a state estimation problem in a noisy environment. A residual-based methodology for the design of a location observer for hybrid systems is proposed in [14]. In [15], [16], the recognition of the active mode is carried out by the means of model-based diagnosis techniques. In [17], [18], sliding mode based techniques are employed to estimate both continuous and discrete states related to the system active dynamic. Several observability concepts for switched systems were introduced in [19], [20], [21], [22].

The aim of this paper is to estimate in real time the switching times sequence of some class of switched linear systems using the methodology proposed in [23] and relaxing the assumptions given in [24]. The approach considered here takes root in the work developed in [25] for parameter identification of continuous linear systems, using a distribution framework, and results in finite time estimates given by explicit algebraic formulae. The main advantage in using the distribution theory is that by differentiation we do not miss essential information such as discontinuities. Recently, those results have been also extended to the problems of state, parameter and unknown input (such as faults) for other classes of continuous time systems in [26], [27], [28], [29], [30], [31]. The efficiency of the proposed estimation algorithm mainly lies in its non asymptotic nature.

The paper is organized as follows. Section II introduces the class of switched systems considered and the distribution framework in which the method will be developed. The main results are derived in section III. First, the switching times identification between two modes is analyzed. Then, the result is extended to the case of commutations among an arbitrary number of modes. An illustrative example is presented in section IV. Finally, some concluding remarks are made in section V.

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## II. PRELIMINARIES

### A. Problem Statement

Consider the system represented by equation:

$$\dot{y} = A_q y \tag{1}$$

where $y \in \mathbb{R}^p$ are measured signals and $q \in \{1, 2, \dots, N\}$. Equation (1) describes an autonomous switched system with $N$ modes, represented by the system state matrix which takes its value in a finite set $A \in \{A_1, A_2, \dots, A_N\}$. The variable $q$ denotes the active mode at each time $t \in \mathbb{R}$. For such a system, we consider the unknown switching instants sequence $\{t_j\}_{j=1}^{\infty}$ assuming that the switchings are arbitrary and independent of the systems state variable. It is assumed that Zeno phenomena cannot occur, i.e., only one discrete event can act on the system at each time $t \in \mathbb{R}$.

The objective of this paper is to estimate on-line the switching times $\{t_j\}_{j=1}^{\infty}$ and to determine the active modes sequence using only the dynamics of $y$.

Due to the occurrence of non-smooth dynamics, all derivatives considered in this paper has to be understood in the distribution sense. Next subsection introduces the distribution framework in which our method will be developed.

### B. Mathematical tools

We recall here some standard definitions and results from distribution theory developed in [32], and fix the notation to be used in what follows. The space of $C^{\infty}$-functions having compact support in an open subset $\Omega$ of $\mathbb{R}$ is denoted by $\mathcal{D}(\Omega)$, and $\mathcal{D}'(\Omega)$ is the space of distributions on $\Omega$, i.e., the space of continuous linear functionals on $\mathcal{D}(\Omega)$. For $u \in \mathcal{D}'$,$\langle u, \varphi\rangle$ denotes the real number which linearly and continuously depends on $\varphi \in \mathcal{D}$. This number is defined as $\langle u, \varphi\rangle = \int_{-\infty}^{+\infty} f\varphi$ for a locally Lebesgue integrable function $u = f$. The support of a distribution $u$, denoted as supp $u$, is defined as the complement of the largest open subset of $\Omega$ in which the distribution $u$ vanishes.

For the Dirac distribution $u = \delta$ and its derivative $\dot{u} = \dot{\delta}$, the functional is defined as $\langle u, \varphi\rangle = \varphi(0)$ and $\langle \dot{u}, \varphi\rangle = -\dot{\varphi}(0)$, respectively. More generally, every distribution is indefinitely differentiable, by virtue of its definition:

$$\langle \dot{u}, \varphi\rangle = -\langle u, \dot{\varphi}\rangle, \quad \forall \varphi \in \mathcal{D}(\Omega). \tag{2}$$

For notational convenience, we shall denote $\dot{u}$ or $u^{(1)}$ the distributional derivative of $u$, while for a function $u$, $\frac{du}{dt}$ will stand for the distribution stemming from the usual derivative (function) of $u$ defined almost everywhere. Throughout this paper, functions (locally Lebesgue integrable) are considered through the distributions they define. Hence, if $u$ is a continuous function except at a point $a$ with a finite jump $\sigma_a$, one can easily show that its distributional derivative writes as $\dot{u} = \frac{du}{dt} + \sigma_a \delta_a$ where $\delta_a$ is the Dirac distribution located at $\{a\}$. This result can be generalized to arbitrary derivation orders and discontinuity points as follows: let $\{t_\nu\}$ be an increasing sequence of points that are finite in number in every finite time interval. Assume also that both left-hand and right hand derivatives $\frac{d^p u}{dt^p}(t_\nu)$ exist and denote the corresponding jump $\sigma_\nu^p = \frac{d^p u}{dt^p}(t_\nu^+) - \frac{d^p u}{dt^p}(t_\nu^-)$. Then one has:

$$u^{(p)} = \frac{d^p u}{dt^p} + \sum_\nu \sum_{k+0}^{p-1} \sigma_\nu^{p-1-k} \delta_{t_\nu}^{(k)}. \tag{3}$$

When rewritten in a distributional sense, the class of differential equations we shall encounter in the sequel will always exhibit such singular terms. The singularities, stemming from the origin $t_0 = 0$, will be gathered into a single distribution denoted $\psi_0$ with support $\{0\}$. With a slight abuse of language, the latter distribution $\psi_0$ will be referred to as the initial condition term.

Another useful result we shall exploit in the sequel is based on properties derived from the multiplication of distributions. Although this operation is not always defined for arbitrary distributions, it turns out that multiplication of two distributions (say $\alpha$ and $u$) is always well-defined when at least one of the two terms (say $\alpha$) is a smooth function. By definition:

$$\langle \alpha u, \varphi\rangle = \langle u, \alpha\varphi\rangle. \tag{4}$$

The previous definitions of derivation (2) and multiplication (4) also allow to transform terms of the form $\alpha u^{(n)}$ into linear combinations of derivatives of products $\alpha^{(k)} u$, using the "reversed" Leibniz rule:

$$\alpha u^{(n)} = \sum_{k=0}^{n} (-1)^{2n-k} C_n^k w_k^{(n-k)}, \quad w_k := \alpha^{(k)} u. \tag{5}$$

One of the nice feature of distribution theory lies in the properties of multiplication, since for every smooth function $\alpha$, one has:

$$\alpha\delta_\tau = \alpha(\tau)\delta_\tau \tag{6}$$

$$\alpha\delta_\tau^{(r)} = 0 \quad \forall \alpha \text{ s.t. } \alpha^{(k)}(\tau) = 0, \quad k = 0, \ldots, r. \tag{7}$$

This property will be used for the annihilation of singular distributions. We complete this introductory section with some well-known definitions and results from the convolution products, and as usual, denote $\mathcal{D}'_+$ the space of distributions with support contained in $[0, \infty)$. It is an algebra with respect to convolution with identity $\delta$. For $u$, $v \in \mathcal{D}'_+$, this product is defined as $\langle u * v, \varphi\rangle = \langle u(x)v(y), \varphi(x+y)\rangle$, and can be identified with the familiar convolution product $(u * v)(t) = \int_0^\infty u(\theta)v(t-\theta)d\theta$ in case of locally bounded functions $u$ and $v$. Derivation, integration and translation can also be defined from the convolutions $\dot{u} = \dot{\delta} * u$, $\int u = H * u$, $u(t-\tau) = \delta_\tau * u$, where $H$ is the familiar Heaviside step function. As for the supports, one has for $u$, $v \in \mathcal{D}'_+$:

$$\text{supp } u * v \subset \text{supp } u + \text{supp } v, \tag{8}$$

where the sum in the righthand side is defined by $A + B = \{x + y = x \in A, y \in B\}$.

Finally, we introduce the following notation making use of the adjoint action of Lie algebras:

$$\begin{array}{l} ad_A^0 B = B \\ ad_A^{i+1} B = [A, ad_A^i B] \end{array} \tag{9}$$

where $A$ and $B$ are two square matrices of the same dimension and $[\cdot, \cdot]$ is the commutator, defined as $[A, B] = AB - BA$.

## III. SWITCHING TIMES IDENTIFICATION

The goal of this work is to obtain an algebraic relation of the measured variables which involves only known quantities in order to explicitly obtain the unknown switching times. To do so, first, the original system is rewritten in the form of an auxiliary impulsive model whose general equation is given in the next subsection. Then, an annihilating algebraic manipulation based on (7) is provided to get the desired differential algebraic relation in which the unknown switching times explicitly appear. Finally, using appropriate filters, the switching times are recovered without using any derivative measurement.

### A. Impulsive model

Consider the $\mu$th order differential equations subject to impulsive righthand sides, described in a distributional framework, with $z \in \mathbb{R}^n$ and each component of $z$ in $\mathcal{D}'_+$ :

$$\sum_{i=0}^{\mu} g_i^{(i)}(z) = \sum_{j=1}^{\infty} \Theta_{t_j} + \psi_0. \tag{10}$$

Here, $g_i^{(i)}(z)$ are known and possibly nonlinear functions of $z$. The function $g_i(z)$ is assumed to be continuous, and the distribution $\psi_0$, of order $\mu - 1$ and support $\{0\}$, gathers all the singular terms derived from the initial conditions as described in the previous section. Finally, $\Theta_{t_j}$ is a Dirac distribution with support $\{t_j\}_{j=1}^{\infty}$.

In Section III-C the procedure to obtain from (10) the estimation of the times sequence $\{t_j\}_{j=1}^{\infty}$ is provided.
The next subsection will show how to rewrite, in a distributional framework, the original switched system (1) in the form (10), where the left-hand side contains available and continuous functions and the right-hand side involves Dirac distributions centered at the switching times. Once this equivalent representation is obtained, the switching times can be estimated as explained in section III-C.

### B. An equivalent representation of the switched systems: the case of 2 modes

Consider the system (1) with two modes and assume that it switches from one subsystem to the other at the unknown switching times $\{t_j\}_{j=1}^{\infty}$. Its dynamical behavior can be written as follows, including the initial conditions which in the distribution sense occur as an impulsive term:

$$\dot{y} = \Gamma y + y_0 \delta_0 \quad \Gamma \in \{A_1, A_2\}. \tag{11}$$

Under the change of variable $z = e^{Gt} y$, where $G$ will be defined later, system (11) is transformed into

$$\dot{z} = \bar{\Gamma} z + z_0 \delta_0 \quad \bar{\Gamma} \in \{M_1, M_2\} \tag{12}$$

with $M_i = G + e^{Gt} A_i e^{-Gt}$, for $i = 1, 2$. Then, the matrix $G$ is chosen such that

$$M_1 + M_2 = 0. \tag{13}$$

Since $G$ and $e^{Gt}$ commute, (13) is fulfilled when $G = -\frac{A_1 + A_2}{2}$. In fact, by this choice of the matrix $G$, one has:

$$\begin{aligned} M_1 &= -\frac{A_1 - A_2}{2} + e^{Gt} A_1 e^{-Gt} \\ &= e^{Gt} \left( \frac{A_1 - A_2}{2} \right) e^{-Gt} \\ &= -M_2 \end{aligned} \tag{14}$$

Denoting this time-varying matrix $M$ and assuming that the first active mode is $A_1$, without loss of generality, the system (12) can be written as follows:

$$\dot{z} = \sigma M z + z_0 \delta_0 \tag{15}$$

with

$$\sigma(t) = \sum_{j=0}^{\infty} (-1)^j \chi_{[t_j, t_{j+1})}(t) \tag{16}$$

where $\chi$ is the characteristic function:

$$\chi_{[t_j, t_{j+1})}(t) = \begin{cases} 1 & \text{if } t \in [t_j, t_{j+1}) \\ 0 & \text{otherwise} \end{cases} \tag{17}$$

The system (11) whose modes are described by the two constant matrices $A_1$ and $A_2$ has been transformed into the systems (15) whose modes are described by a time-variant matrix $M$, that inverts its sign at each switching.

Since $\sigma(t)$ is constant outside the set $\{t_j\}_{j=1}^{\infty}$, the distributional derivative of the product $\sigma u$ is well defined when $u$ is a smooth function. Considering that the jumps of $\sigma$ at the switching times are $\sigma(t_j^+) - \sigma(t_j^-) = \mp 2$ and using the properties (3) and (5), one has:

$$\dot{(\sigma u)} = \dot{\sigma} u + \sigma \dot{u} = 2 \sum_{j=1}^{\infty} (-1)^j u(t_j) \delta_{t_j} + \sigma \dot{u} \tag{18}$$

Since $\sigma^2 = 1$ and using (18), one differentiation of (15) yields:

$$\ddot{z} - M^2 z = \sigma \dot{M} z + 2 \sum_{j=1}^{\infty} (-1)^j M(t_j) z(t_j) \delta_{t_j} + \psi(\delta_0, \dot{\delta}_0)$$

where the distribution $\psi$ gathers the singularities stemming from the origin $t_0 = 0$. Differentiating one more time:

$$\begin{aligned} z^{(3)} - \dot{M} M z - (\dot{M^2 z}) &= \sigma M^{(2)} z \\ &+ 2 \sum_{j=1}^{\infty} (-1)^j \left( \dot{M}(t_j) z(t_j) \delta_{t_j} + M(t_j) z(t_j) \dot{\delta}_{t_j} \right) \\ &+ \psi(\delta_0, \dot{\delta}_0, \ddot{\delta}_0) \end{aligned}$$

After $k$ differentiations, the following equation is obtained:

$$z^{(k+1)} - \sum_{i=0}^{k-1} g_i^{(i)}(z) = \sigma M^{(k)} z + 2 \sum_{j=1}^{\infty} (-1)^j \Theta_{t_j} + \psi_0 \tag{19}$$

where

$$\begin{aligned} g_i(z) &= M^{(k-1-i)} M z \\ \Theta_{t_j} &= \sum_{i=0}^{k-1} M(t_j)^{(k-1-i)} z(t_j) \delta_{t_j}^{(i)} \end{aligned}$$

Here, the Dirac distribution $\Theta_{t_j}$ is of order $k-1$ and support $\{t_j\}_{j=1}^{\infty}$, and the distribution $\psi_0$ of order $k$ and support $\{0\}$ gathers the singularities stemming from the origin $t_0 = 0$.

In what follows, we are interested in the conditions guaranteeing the possibility to write (19) in the form of the impulsive model given in (10). Note that what makes (19) different from (10) is only the term $\sigma M^{(k)} z$, which is unknown and discontinuous.

*Lemma 1:* The $k$th-order derivative of the matrix $M$ can be written as:

$$M^{(k)} = e^{Gt} (ad_G^k A) e^{-Gt} \tag{20}$$

with $A = \frac{A_1 - A_2}{2}$.

*Proof:* For $k=1$, from equation (14), one has:

$$\begin{aligned} M^{(1)} &= Ge^{Gt}Ae^{-Gt} - e^{Gt}AGe^{-Gt} \\ &= e^{Gt}[G,A]e^{-Gt} \\ &= e^{Gt}(ad^1_G A)e^{-Gt} \end{aligned}$$

Assume (20) holds for $k$, then for $k+1$:

$$\begin{aligned} M^{(k+1)} &= Ge^{Gt}(ad^k_G A)e^{-Gt} - e^{Gt}(ad^k_G A)Ge^{-Gt} \\ &= e^{Gt}\left(G(ad^k_G A) - (ad^k_G A)G\right)e^{-Gt} \\ &= e^{Gt}[G, ad^k_G A]e^{-Gt} \\ &= e^{Gt}(ad^{k+1}_G A)e^{-Gt} \end{aligned}$$

■

Assume that there exists a $\bar{k} \geq 0$ such that the matrix $ad^{\bar{k}}_G A$ is null or full rank, then it is possible to rewrite the original switched system in the form of impulsive model given in (10) and consequently the switching times can be detected, as it will be shown in the next subsection.

Consider the two cases separately:

1) if $ad^{\bar{k}}_G A$ is null, then $M^{(\bar{k})}$ is null from (20), and the equation (19) becomes:

$$z^{(\bar{k}+1)} - \sum_{i=0}^{\bar{k}-1} g_i^{(i)}(z) = 2\sum_{j=1}^{\infty}(-1)^j \Theta_{t_j} + \psi_0 \quad (21)$$

Note that the case $\bar{k}=0$ is trivial, because one has $A_1 = A_2$.

2) if $ad^{\bar{k}}_G A$ is full rank, then $M^{(\bar{k})}$ is full rank from (20). So multiplying both side of equation (19) by the matrix $W = (M^{(\bar{k})})^{-1}$ one has:

$$\begin{aligned} W\left(z^{(\bar{k}+1)} - \sum_{i=0}^{\bar{k}-1} g_i^{(i)}(z)\right) &= \sigma z \\ &+ 2\sum_{j=1}^{\infty}(-1)^j W(t_j)\Theta_{t_j} + W(0)\psi_0 \quad (22) \end{aligned}$$

In this case a further differentiation is necessary in order to obtain the form given in (10):

$$\begin{aligned} &\dot{W}\left(z^{(\bar{k}+1)} - \sum_{i=0}^{\bar{k}-1} g_i^{(i)}(z)\right) \\ &+ W\left(z^{(\bar{k}+2)} - \sum_{i=0}^{\bar{k}-1} g_i^{(i+1)}(z)\right) - Mz \\ &= 2\sum_{j=1}^{\infty}(-1)^j \bar{\Theta}_{t_j} + \bar{\psi}_0 \quad (23) \end{aligned}$$

where

$$\bar{\Theta}_{t_j} = \textstyle\sum_{i=0}^{\bar{k}} W(t_j)M(t_j)^{(\bar{k}-i)}z(t_j)\delta_{t_j}^{(i)}$$

Here, the distribution $\bar{\Theta}_{t_j}$ is of order $\bar{k}$ and support $\{t_j\}_{j=1}^{\infty}$, and the distribution $\bar{\psi}_0$ is of order $\bar{k}+1$ and support $\{0\}$.

Both (21) and (23) are particular cases of the impulsive differential equation (10) of order:

$$\mu = \begin{cases} \bar{k}+1 & \text{if } ad^{\bar{k}}_G A \text{ is null} \\ \bar{k}+2 & \text{if } ad^{\bar{k}}_G A \text{ is full rank} \end{cases} \quad (24)$$

### *C. Explicit computation of the switching times*

In this section the procedure is presented with the goal to estimate the times sequence $\{t_j\}_{j=1}^{\infty}$ of (10). The method has been formulated in [23] for the identification of a class of impulsive systems, and it is mainly based on two steps consisting in: (i) a multiplication with smooth functions $\alpha_k$, $k=1,2$ and (ii) a convolution with an appropriate filter $h$. From the multiplication properties, given in (7), we first get that multiplying equation (10) by an arbitrary smooth function $\alpha_k$, such that $\alpha_k^{(i)}(0)=0$, for $i=0,\ldots,\mu-1$, the initial conditions are canceled. Thus:

$$\alpha_k \times \sum_{i=0}^{\mu} g_i^{(i)}(z) = \sum_{j=1}^{\infty}\alpha_k(t_j)\Theta_{t_j}, \quad k=1,2 \quad (25)$$

Hence, the next step consists in a convolution with a filter $h$, yielding an equality of functions that is more appropriate for measurements manipulations:

$$h*\left[\alpha_k \times \sum_{i=0}^{\mu} g_i^{(i)}(z)\right] = \sum_{j=1}^{\infty}\alpha_k(t_j)h*\Theta_{t_j}, \quad k=1,2 \quad (26)$$

In order to ensure causality (avoiding any measurement derivative), the chosen function $h$ has to be at least of class $C^{\mu}$, which is equivalent for a linear filter $h(s)$ † to have a relative degree $\geq \mu$. Furthermore, in order to get a sequential estimation of the times $\{t_j\}_{j=1}^{\infty}$, the main *a priori* assumption we shall need in the sequel is a lower bound estimate of the dwell time since algebraic relations are obtained only on intervals containing a single switch. If the support of $h$ lies within the smallest dwell time, i.e. supp $h \subset (0, \min_j(t_{j+1}-t_j))$, then the main idea consists in getting the switching instant from the comparison of the left hand sides of (26) for $k=1,2$. For instance, if $\alpha_2 = t\alpha_2$, then the ratio of the left hand sides of (26), when defined, yields precisely, and in real time, the times $\{t_j\}_{j=1}^{\infty}$. More generally, let us introduce the switching function $\zeta(t)$ as:

$$\zeta(t) = \sum_{j=1}^{\infty}\zeta_j \chi_{[t_j,t_{j+1})}(t), \quad \zeta_j = \alpha_1(t_j)/\alpha_2(t_j) \quad (27)$$

By virtue of the support of $h$, the local equality we have just obtained from such a comparison can be extended to the whole real line as:

$$h*\left[\alpha_1 \times \sum_{i=0}^{\mu} g_i^{(i)}(z)\right] = \zeta(t)\left(h*\left[\alpha_2 \times \sum_{i=0}^{\mu} g_i^{(i)}(z)\right]\right)$$

†We shall denote $h(s)$, $s \in \mathbb{C}$, the Laplace transform of $h$ (in a distribution setting).

### D. The case of $N$ modes

In section III-B it has been shown how to rewrite a switched linear system with 2 modes in the form of the impulsive model given in (10). The conditions under which this transformation is possible have been given.
In this section, we will give the conditions in the case of $N$ modes, extending the applicability of the proposed method to the general case.

For each pair of matrices $A_p$ and $A_q$ in system (1) the following matrices can be considered:

$$G_{p,q} = -\frac{A_p + A_q}{2}, \qquad A_{p,q} = \frac{A_p - A_q}{2} \tag{28}$$

Assume that for each pair of matrices $A_p$ and $A_q$, there exists a $\bar{k}_{p,q}$ such that the matrix $ad^{\bar{k}}_{G_{p,q}} A_{p,q}$ is null or full rank, then for each pair of modes the equivalent impulsive model can be considered, and consequently all the switches among the $N$ modes can be detected. In fact an estimator $E_{p,q}$ can be used to determine the times $\{t_j\}_{j=1}^{\infty}(p,q)$ of the switchings that occur between the two subsystems $p$ and $q$ (either from mode $p$ to mode $q$ or from mode $q$ to mode $p$). Hence, in order to identify all the switches, one can use $C_N^2 = \frac{N(N-1)}{2}$ estimators in parallel.

Note that each estimator $E_{p,q}$ can detect the switching time and the pair of modes involved in the switching, but not the direction of the switching, i.e., if the switching is from the mode $p$ to the mode $q$ or vice versa. It is obvious that the knowledge of the first active mode allows to reconstruct the whole sequence of the active subsystems.

## IV. AN ILLUSTRATIVE EXAMPLE

In this section, we present an example that shows the interest and applicability of our method. Consider a switched linear system (1) with two modes described by the matrices:

$$A_1 = \begin{bmatrix} -0.3 & 0.2 \\ 0.5 & -0.3 \end{bmatrix} \quad A_2 = \begin{bmatrix} -0.1 & 0.1 \\ -0.1 & -0.2 \end{bmatrix}.$$

Following the procedure of section III-B, the model of this system is described by equation (12). Since in this case $ad^0_G A$ is full rank, after a multiplication by the matrix $W = M^{-1}$ and a differentiation, (12) reads as:

$$(Wz)^{(2)} - (\dot{W}z)^{(1)} - Mz = 2\sum_{j=1}^{\infty}(-1)^j z(t_j)\delta_{t_j} + \psi_0 \tag{29}$$

which corresponds to the model (10) with $g_2(z) = Wz$, $g_1(z) = \dot{W}z$, $g_0(z) = -Mz$, and with the distribution $\psi_0$ of order 1 and support $\{0\}$. Following the method of section III-C, after a multiplication by functions $\alpha_k$, $k = 1,2$ such that $\alpha_k(0) = \dot{\alpha}_k(0) = 0$ cancels the initial conditions, and a convolution with a filter $h$, the left hand side results in:

$$h * \left[\alpha_k \times \left((Wz)^{(2)} - (\dot{W}z)^{(1)} - Mz\right)\right], \quad k = 1,2. \tag{30}$$

Considering the sequence of switching times $\{t_j\} = \{2.1, 4.3, 7, 8.5\}$, Figure 1 shows the measured trajectories starting from the mode $A_1$ with initial condition $x_0 = \begin{bmatrix} 3 & 1 \end{bmatrix}$. A lower bound for the smallest dwell time has been

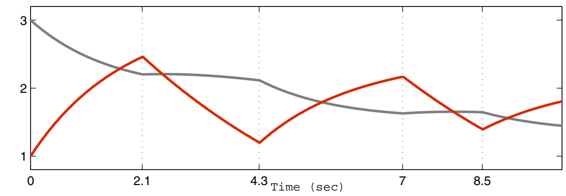


Fig. 1. Measured signals

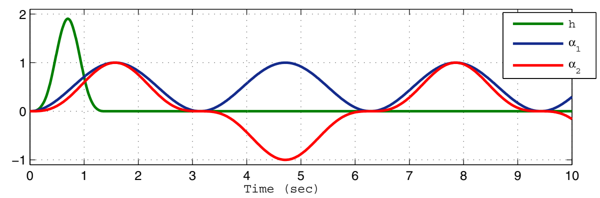


Fig. 2. Adopted filter and multiplicative functions

fixed to $T = 1.4$ sec., and the adopted filter and functions $\alpha_k$ are given by:

$$h = ((1 - e^{-sT/4})/s)^4,$$
$$\alpha_1 = sin^2(t), \quad \alpha_2 = sin^3(t),$$

and are depicted in Figure 2. The choice of the filter is such that its support lies within the smallest dwell time, i.e. supp $h \subset (0, 1.4)$. Concerning the candidate functions $\alpha_k$, polynomial functions are simpler to manipulate but, in noisy situations, it is recommended to use bounded functions. Figure 3 illustrates the realization of (30). Hence, on each (unknown) interval $(t_j, t_{j+1})$, the formed functions on Figure 3 are linked by the relation:

$$\begin{aligned} &\left(h * \left[\alpha_2 \times \left((Wz)^{(2)} - (\dot{W}z)^{(1)} - Mz\right)\right]\right)(t) \\ &= sin(t_j)\left(h * \left[\alpha_1 \times \left((Wz)^{(2)} - (\dot{W}z)^{(1)} - Mz\right)\right]\right)(t), \\ &\qquad t \in (t_j, t_{j+1}) \end{aligned} \tag{31}$$

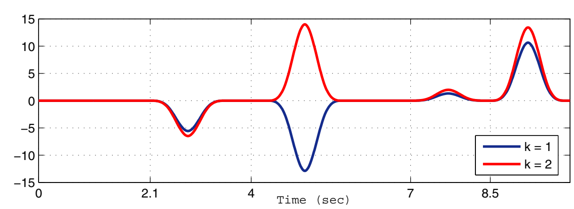


Fig. 3. Realization of equations (30)

Note that from the properties of the support of a convolution product (8), the estimation problem of the switching sequence is not consistent for all $t > 0$. In other words, and as illustrated in Figure 3, both formed functions vanish locally, yielding $0 = \zeta(t)0$ outside the latter intervals and leading to a local loss of identifiability of $\zeta$. This drawback may

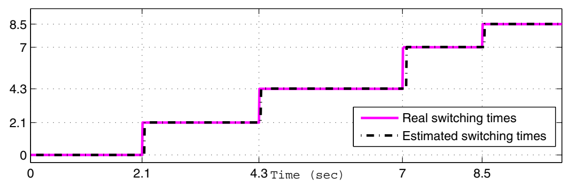


Fig. 4. Switching times estimation

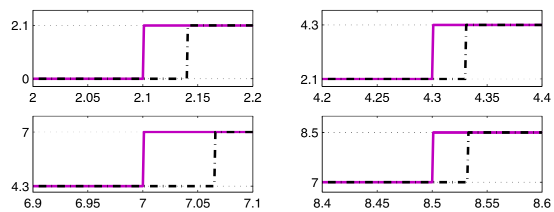

Fig. 5. Switching times estimation (zoom)

require the use of *a priori* information (threshold) testing the consistency of (30). Although the chosen functions $\alpha_k$ are not bijective, the true values of the switching times can be easily obtained by means of equation (31) by taking into account the current running time (Figure 4).

## V. CONCLUSIONS

In this paper, an algebraic approach for switching times estimation of a class of hybrid systems has been presented. An explicit algorithm based on distributions, annihilation of singular terms and filtering has been derived to compute on-line the switching time instants. The combination of optimal multiplicative functions and filters adopted in case of noisy data is under active investigations. Moreover, our approach can be developed in order to extend the method to more general cases of systems with partial state measurements.

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