2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Formation Control of Multi-agent Systems with Double Integrator Dynamics using Delayed Static Output Feedback

Paresh Deshpande, Prathyush P Menon[1], Christopher Edwards[2] and Ian Postlethwaite[3]

Abstract— **In this paper a network of vehicles moving in a two dimensional plane, described by double integrator dynamics, is stabilized by a novel distributed control methodology, to maintain a formation. The distributed control architecture employs static output feedback using an artificial delay. Delays in communication of the relative information are exploited to stabilize the network system using state output feedback of position information only. The synthesis of the controller gains and the level of artificial delay, is posed as an optimization problem subject to the feasibility of a set of Linear Matrix Inequalities based on a discretized Lyapunov-Krasovskii functional.**

I. Introduction

A large number of studies based on consensus algorithms consider either single or double integrator dynamics as potential representative models for a wide range of applications. See [1]-[10] and the references therein for details. In this paper, the focus is on the double integrator systems, which can be a simple representative model for rendezvous [8], mobile robots [12], single axis spacecraft rotation [13], etc. The double integrator model is more representative in comparison with a single integrator model which cannot for example model motion in a plane in which acceleration is the control input. Consensus algorithms for double integrator models have been previously studied in [8]-[11]. In [8], the rendezvous of vehicles described by double integrator dynamics is studied using local speed feedback along with information communicated from neighbours to analyze fixed and switching topologies. In [9], necessary and sufficient conditions on the network topologies to achieve consensus are derived. In [10], consensus algorithms for double integrator dynamics are analyzed with respect to bounds on the control inputs, the absence of relative measurements and the availability of reference states. In [11], a consensus strategy based on exchange of delayed position and velocity information for a network of double integrators has been studied. Consensus algorithms for double integrator dynamics have been extended to the problem of stabilization of formations in [14]-[17]. In [14], behaviour based methods for maneuvering mobile robots in a formation are presented. The paper proposes a strategy based on relative information to maintain formation. In [15], the correlation between the stability of a network of dynamical systems and the Laplacian eigenvalues is studied. In [16], it is shown that a necessary and sufficient condition for decentralized linear stabilization of formations, is the presence of a directed spanning tree in the network topology. In [17], necessary and sufficient conditions for formation and alignment in the presence of saturation constraints are developed. Another related area of study is flocking, since flocking problems are often modelled as double integrator dynamics. See references [18]-[20] for further details.

Delay effects on consensus strategies have been studied in [15], [21]-[25]. In [21], stability criteria for consensus in a network of agents are derived using a frequency based approach and a Lyapunov-Krasovskii technique. The authors conclude that the position of an equilibrium point depends strongly on the value of a delay and the initial conditions. In [22], a stability criteria for consensus in a network of double integrator systems based on Lyapunov-Krasovskii techniques is derived. In [23], a decentralized control algorithm is developed for a network of vehicles, with second order dynamics, with different position coupling gains at each node. It is proved that for a connected topology, rendezvous is achieved for sufficiently small coupling gains. In [24], the effects of communication delays on consensus in large scale multi-agent systems with nonlinearities is studied. In [25], robustness of consensus in multi-agent systems is investigated. It is shown that delays in communication which only affect an agent's neighbours are less restrictive than delays which affect the agent along with its neighbours.

In many practical cases, the measurement of all the states of a system is not viable. In such scenarios control by output feedback is necessary. Although (static) output feedback control is a well studied problem, no complete solution has been found [26], [27]. Typically the problem must be posed as an optimization and solved using numerical methods. Furthermore it is well known that double integrator systems cannot be asymptotically stabilized via static output feedback. One solution is to create a dynamical feedback control law - possibly based on observers. In [28], it is shown that some systems stabilize due to delays. Consequently it maybe possible to stabilize a system, which cannot be stabilized by static output feedback, by introducing terms based on a delayed version of the output. This design methodology has been considered by a few researchers in [29]-[33]. In [29], necessary and sufficient conditions for achieving stability of systems which cannot be stabilized without delays

Paresh Deshpande is a PhD student in the Department of Engineering, University of Leicester, LE1 7RH, UK. pd101@le.ac.uk

[1]Lecturer at Centre for Systems, Dynamics and Control, University of Exeter, UK, EX4 4QF. P.M.Prathyush@exeter.ac.uk

[2]Professor in the Department of Engineering, University of Leicester, UK, LE1 7RH. ce14@le.ac.uk

[3]Deputy Vice-Chancellor at the University of Northumbria, UK. ian.postlethwaite@northumbria.ac.uk

This work is supported by a UKIERI Grant.

are developed. In [30], issues related to robustness with respect to parametric and delay uncertainty are considered. In [31], necessary conditions for the existence of stabilizing static output feedback controllers with multiple delays are developed. In particular, stabilization of a double integrator by using delays in output feedback terms is briefly described. In [32], a new method for control synthesis, based on descriptor discretized Lyapunov-Krasovskii functionals, which can stabilize systems not stabilizable by static output feedback has been developed. In [32], the method is illustrated on double integrator dynamics. In [33], an artificial delay is used to develop a static output feedback sliding mode control law. It is stated by the authors that an advantage of such a method is that it does not increase the order of the system, and is computationally less complex compared to methods involving the use of compensators.

The contribution of this paper is the development of a distributed output feedback control law, making use of an artificial fixed delay, to stabilize a formation of vehicles, moving in a two dimensional plane, and described by double integrator dynamics. Most literature on consensus algorithms and formation control of double integrator systems, for example [8], [9], [14], [16] (and many others), is based on state feedback, whereas this paper is based on static output feedback where the only measurements are position information in a 2-dimensional plane. The network topology is assumed to be fixed. The relative information communicated by the neighbours of each vehicle is provided with precalculated offsets such that the vehicles stabilize in a formation. To find the controller gains and the artificial delay simultaneously, an available numerical optimization method (DIRECT) [38] is employed subject to the constraints emanating from the feasibility of linear matrix inequalities based on the discretized Lyapunov-Krasovskii functional from [37].

II. Preliminaries

Standard notation has been used in this paper. The set of real numbers is denoted by $\mathbb{R}$. The expression $\mathbb{R}^m$ denotes real valued vectors of length m and $\mathbb{R}^{m\times n}$ denotes the set of arbitrary real-valued $m \times n$ matrices. The expression $\mathcal{C}ol(.)$ denotes a column vector and $\mathcal{D}iag(.)$ denotes a diagonal matrix. A symmetric positive definite (s.p.d) matrix will be written as $P = P^{\mathrm{T}} > 0$ and I_n denotes an identity matrix of dimension $n \times n$. The symbol $\otimes$ denotes Kronecker product.

Standard concepts from graph theory are quoted in this section. Please refer [36] for further reading on graph theory. A graph $\mathcal{G}$ consists of a set of vertices denoted by $\mathcal{V}$, and a set of edges $\mathcal{E} \subset \mathcal{V}^2$ where $e = (\alpha, \beta) \in \mathcal{V}^2$, i.e, an unordered pair, denotes an edge. A network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, represents a simple, finite graph consisting of N vertices and k edges. The graphs are assumed to be undirected. It is also assumed that the graph contains no multiple identical edges between two nodes and no loops. For the graph $\mathcal{G}$, the adjacency matrix $\mathcal{A}(\mathcal{G}) = [a_{ij}]$, is defined by setting $a_{ij} = 1$ if i and j are adjacent nodes of the graph, and $a_{ij} = 0$ otherwise. This creates a symmetric matrix. The symbol $\Delta(\mathcal{G}) = [\delta_{ij}]$ represents the degree matrix, and is an $N \times N$ diagonal matrix, where δ_{ii} is the degree of the vertex i. The Laplacian of $\mathcal{G}$, $\mathcal{L}$, is defined as the difference $\Delta(\mathcal{G}) - \mathcal{A}(\mathcal{G})$. The Laplacian for an undirected graph is symmetric. The smallest eigenvalue of $\mathcal{L}$ is exactly zero and the corresponding eigenvector is given by $\mathbf{1} = \mathcal{C}ol(1, \dots 1)$. The Laplacian $\mathcal{L}$ is always rank deficient and positive semi-definite. Moreover, the rank of $\mathcal{L}$ is $n-1$ if and only if $\mathcal{G}$ is connected.

III. Problem Formulation

Consider N identical vehicles moving in a 2-dimensional plane with the dynamics of each dimension described by a double integrator. The linear state space representation for each vehicle is then given by

$$\dot{\xi}_i(t) = A\xi_i(t) + Bu_i(t) \tag{1}$$

$$\vartheta_i = C\xi_i \tag{2}$$

where $\xi_i = Col[x_i, \dot{x}_i, y_i, \dot{y}_i]$ and

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \; B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{3}$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{4}$$

This is equivalent to two decoupled double integrator systems. Here ϑ_i is the measured output i.e the position (x_i, y_i) of the i^{th} vehicle in the $x-y$ plane. In [8], although only relative position information is communicated in the network, local velocity information is considered for feedback. In this paper, each vehicle is assumed to have access only to its output information and delayed relative output information from agents which it can interact with: i.e its neighbours. Here bidirectional communication is assumed between an agent and its neighbours. This interconnected system can be represented by a graph with N vertices (nodes) each representing a vehicle. The existence of relative sensing and communication between two vehicles is represented by an edge in the graph. The signals representing the exchange of relative position information are assumed to have the form

$$z_i(t) = \sum_{j \in \mathcal{J}_i} (\vartheta_i(t) - \vartheta_j(t)) \tag{5}$$

for $i = 1 \dots N$. The nonempty set $\mathcal{J}_i \subset \{1, 2, \dots N\}/\{i\}$ denotes the vehicles, for which the i^{th} vehicle has information. The signals $z_i(t)$ represent the sum of the external output measurements relative to the other vehicles which the i^{th} vehicle can sense. At a network level, the system given in (1) is represented by

$$\dot{X}(t) = (I_N \otimes A)X(t) + (I_N \otimes B)U(t) \tag{6}$$

where

$$X(t) = \mathcal{C}ol(\xi_1(t), ..., \xi_N(t)) \tag{7}$$

$$U(t) = \mathcal{C}ol(u_1(t), ..., u_N(t)) \tag{8}$$

At network level, (5) can be represented as

$$Z(t) = (\mathcal{L} \otimes C)X(t) \quad (9)$$

where $Z(t) = \mathcal{C}ol(z_1(t), \ldots, z_N(t))$. As in references [3], [6], [15], [16] (and many others) an assumption is made that each vehicle has information about at least one other vehicle which ensures $rank(\mathcal{L}) = N - 1$.

It is well known that the two decoupled double integrators associated with (A, B, C) cannot be stabilized by static output feedback (position information alone). To circumvent this issue, this paper considers distributed static output feedback control laws involving delay terms of the form

$$u_i(t) = -K_1\vartheta_i(t) + K_2\vartheta_i(t-\tau) + K_2(\beta z_i(t-\tau) - d_i) \quad (10)$$

where $K_1 = k_1 I_2$ and $K_2 = k_2 I_2$, and k_1, k_2 are non-zero scalars. The scalar $\beta > 0$ represents a scalar weighting for z_i and τ is a fixed delay. In equation (10) $z_i(t-\tau)$ is the delayed relative position information given by

$$z_i(t-\tau) = \sum_{j \in \mathcal{J}_i} (\vartheta_i(t-\tau) - \vartheta_j(t-\tau)) \quad (11)$$

The scalar d_i in (10) is the offset in the relative information at each node so that each agent maintains a desired relative distance from its neighbours. For a given β, the gains K_1 and K_2 together with τ must be chosen such that the closed loop network system is stable.

Remark: Since it is assumed that each agent is described by two decoupled double integrator systems, for motion in each planar direction the gains can be chosen as $K_1 = k_1 I_2$ and $K_2 = k_2 I_2$ where k_1 and k_2 are scalars.

Remark: In real engineering systems relative sensing and communication of relative information will incur delays. Here it is assumed that a minimum delay of $\tau_{min} > 0$ will be present in relative sensing and communication. Since it is assumed that each node has access to its own output information it is assumed, in addition, that it is possible to store this information and use it in delayed feedback.

The control law (10) at a network level is given by

$$\begin{aligned} U(t) = &-(I_N \otimes K_1 C)X(t) + (I_N \otimes K_2 C)X(t-\tau) \\ &+(\beta\mathcal{L} \otimes K_2 C)X(t-\tau) + (I_N \otimes K_2)D \end{aligned} \quad (12)$$

where $D \in \mathbb{R}^{2N}$ is the offset for the network such that $D = \mathcal{C}ol(d_1, \ldots, d_N)$. This can be represented as

$$\begin{aligned} U(t) = &-(I_N \otimes K_1 C)X(t) + (I_N \otimes K_2)D \\ &+((I_N + \beta\mathcal{L}) \otimes K_2 C)X(t-\tau) \end{aligned} \quad (13)$$

Substituting (13) in (6), the closed loop system is given by

$$\dot{X}(t) = \bar{A}_0 X(t) + \bar{A}_1 X(t-\tau) + (I_N \otimes BK_2)D \quad (14)$$

where

$$\bar{A}_0 = I_N \otimes (A - BK_1 C) \quad (15)$$
$$\bar{A}_1 = (I_N + \beta\mathcal{L}) \otimes BK_2 C \quad (16)$$

Since the system (A, B, C) is not stabilizable by static output feedback, the system in (14) is not stable for $\tau = 0$. The problem which will be addressed in the sequel is to design the control law in (10) with τ chosen to satisfy $\tau > \tau_{min}$, such that the closed loop system in (14) is stable. In other words, for a given $\beta > 0$ find the triplet (k_1, k_2, τ) with $\tau > \tau_{min}$ such that the system (14) is stable.

IV. Control Design Procedure

First, introduce a linear coordinate transformation of the form

$$\bar{X}(t) = X(t) - X_f \quad (17)$$

where $X_f \in \mathbb{R}^{4N}$ is the desired final state of the network. This will have the form

$$X_f = \mathcal{C}ol\begin{pmatrix} x_1^f, & 0, & y_1^f, & 0, & \ldots & x_N^f, & 0, & y_N^f, & 0 \end{pmatrix} \quad (18)$$

where x_i^f and y_i^f are the desired final steady state positions for all $i = 1, \ldots, N$. The system in (14) with the transformation in (17) is given by

$$\begin{aligned} \dot{\bar{X}}(t) = \bar{A}_0\bar{X}(t) + \bar{A}_1\bar{X}(t-\tau) + (\bar{A}_0 + \bar{A}_1)X_f \\ +(I_N \otimes BK_2)D \end{aligned} \quad (19)$$

Here the offsets in D are chosen to satisfy

$$(I_N \otimes K_2)D = (I_N \otimes K_1 C)X_f - ((I_N + \beta\mathcal{L}) \otimes K_2 C)X_f \quad (20)$$

Then exploiting the fact that from (18) and (3)

$$(I_N \otimes A)X_f = 0, \quad (21)$$

multiplying both sides of (20) on the left by $(I_N \otimes B)$ and adding $(I_N \otimes A)X_f$ to the left hand side yields

$$(\bar{A}_0 + \bar{A}_1)X_f + (I_N \otimes BK_2)D = 0 \quad (22)$$

when exploiting the definitions of $\bar{A}_0$ and $\bar{A}_1$ from (15) and (16). The system in (19) is then given by

$$\dot{\bar{X}}(t) = \bar{A}_0\bar{X}(t) + \bar{A}_1\bar{X}(t-\tau) \quad (23)$$

Since $K_2 = k_2 I_2$, from (20):

$$D = \frac{1}{k_2}((I_N \otimes K_1 C) - ((I_N + \beta\mathcal{L}) \otimes K_2 C))X_f \quad (24)$$

Remark: Note that the offset D in (14) is chosen as in (24) once the gains k_1 and k_2 are designed. The offset D also depends on the desired formation encapsulated in X_f. Note that the provision of d_i in (10) has a typical feedforward signal architecture. Because $\mathcal{L}$ is symmetric positive semi-definite, $(I_N + \beta\mathcal{L})$ is symmetric positive definite as $\beta > 0$. Since $(I_N + \beta\mathcal{L})$ is symmetric positive definite, by spectral decomposition $(I_N + \beta\mathcal{L}) = V\Lambda V^T$ where $V \in \mathbb{R}^{N \times N}$ is an orthogonal matrix formed from the eigenvectors of $(I_N + \beta\mathcal{L})$ and $\Lambda = Diag(\lambda_1, \ldots \lambda_N)$ is the matrix of the eigenvalues of $(I_N + \beta\mathcal{L})$. Note that all $\lambda_i \geq 1$ for all $i = 1, \ldots, N$ and that the smallest eigenvalue of $(I_N + \beta\mathcal{L})$ is $\lambda_1 = 1$. As in [36], consider an orthogonal transformation

$$\bar{X} \mapsto (V^T \otimes I_4)\bar{X} = \tilde{X} \quad (25)$$

In the new coordinates equation (23) is given by

$$\dot{\tilde{X}}(t) = \tilde{A}_0\tilde{X}(t) + \tilde{A}_1\tilde{X}(t-\tau) \tag{26}$$

where

$$\tilde{A}_0 = (I_N \otimes (A - BK_1C)) \tag{27}$$
$$\tilde{A}_1 = (\Lambda \otimes BK_2C) \tag{28}$$

since $V^T(I_N + \beta\mathcal{L})V = \Lambda$ because V is orthogonal. Equation (26) can be represented at node level in transformed coordinates as

$$\dot{\tilde{\xi}}_i(t) = A_0\tilde{\xi}_i(t) + A_i\tilde{\xi}_i(t-\tau) \tag{29}$$

where

$$A_0 = (A - BK_1C) \tag{30}$$
$$A_i = \lambda_i BK_2C \tag{31}$$

for all $i = 1, \dots, N$. In order to ensure a level of performance in the closed loop system, as suggested in [34] consider the transformation

$$\tilde{\xi}_{i\alpha}(t) = e^{\alpha t}\tilde{\xi}_i(t) \tag{32}$$

for $i = 1, \dots, N$ where $\alpha > 0$. Asymptotic convergence of the $\tilde{\xi}_{i\alpha}$ implies exponential convergence of $\tilde{\xi}_i$ at a decay rate α. Further details appear in [33]- [35] and the references therein. With this transformation, the system represented in (29) becomes

$$\dot{\tilde{\xi}}_{i\alpha}(t) = (A_0 + \alpha I_4)\tilde{\xi}_{i\alpha}(t) + e^{\alpha\tau}A_i\tilde{\xi}_{i\alpha}(t-\tau) \tag{33}$$

where A_0 and A_i are as given in (30) and (31) for all $i = 1, \dots, N$. The stability of system (33) will be ascertained using *Proposition 5.22* in [37]. In [37], *Proposition 5.22* divides the delay interval $[-\tau, 0]$ into M partitions and then employs a discretized Lyapunov functional to test for stability. Formally: (for completeness)

Proposition 5.22 [37] *The system with single time delay τ described by (33) is asymptotically stable if there exist matrices P_i, S_{pi}, $R_{pqi} \in \mathbb{R}^{4\times 4}$, $P_i = P_i^T$; $S_{pi} = S_{pi}^T$, $R_{pqi} = R_{qpi}^T$, $p = 0, \dots, M$, $q = 0, \dots, M$; and such that*

$$\begin{pmatrix} P_i & \tilde{Q}_i \\ * & \tilde{R}_i + \tilde{S}_i \end{pmatrix} < 0 \tag{34}$$

$$\begin{pmatrix} \Delta_i & -D_i^s & -D_i^a \\ * & R_{di} + S_{di} & 0 \\ * & * & 3S_{di} \end{pmatrix} < 0 \tag{35}$$

where

$$\tilde{Q}_i = \begin{pmatrix} Q_{0i} & Q_{1i} & \dots & Q_{Mi} \end{pmatrix} \tag{36}$$

$$\tilde{R}_i = \begin{pmatrix} R_{00i} & R_{01i} & \dots & R_{0Mi} \\ R_{10i} & R_{11i} & \dots & R_{1Mi} \\ . & . & \dots & . \\ R_{M0i} & R_{M1i} & \dots & R_{MMi} \end{pmatrix} \tag{37}$$

$$\tilde{S}_i = \begin{pmatrix} \frac{1}{h}S_{0i} & \frac{1}{h}S_{1i} & \dots & \frac{1}{h}S_{Mi} \end{pmatrix} \tag{38}$$

$$\Delta_i = \begin{pmatrix} \Delta_{00i} & \Delta_{01i} \\ * & \Delta_{11i} \end{pmatrix} \tag{39}$$

$$\Delta_{00i} = -P_i(A_0 + \alpha I_4) - (A_0 + \alpha I_4)^T P_i - Q_{0i} - Q_{0i}^T - S_{0i} \tag{40}$$

$$\Delta_{01i} = Q_{Mi} - P(e^{\alpha\tau}A_i) \tag{41}$$
$$\Delta_{11i} = S_{Mi} \tag{42}$$
$$S_{di} = Diag\begin{pmatrix} S_{d1i} & S_{d2i} & \dots & S_{dMi} \end{pmatrix} \tag{43}$$
$$S_{dpi} = S_{(p-1)i} - S_{pi} \tag{44}$$

$$R_{di} = \begin{pmatrix} R_{d11i} & R_{d12i} & \dots & R_{d1Mi} \\ R_{d21i} & R_{d22i} & \dots & R_{d2Mi} \\ . & . & \dots & . \\ R_{dM1i} & R_{dM2i} & \dots & R_{dMMi} \end{pmatrix} \tag{45}$$

$$R_{dpqi} = h(R_{(p-1,q-1)i} - R_{pqi}) \tag{46}$$
$$D_i^s = \begin{pmatrix} D_{1i}^s & D_{2i}^s & \dots & D_{Mi}^s \end{pmatrix} \tag{47}$$
$$D_{pi}^s = \begin{pmatrix} D_{0pi}^s \\ D_{1pi}^s \end{pmatrix} \tag{48}$$

$$D_{0pi}^s = \frac{h}{2}(A_0 + \alpha I_4)^T(Q_{(p-1)i} + Q_{pi}) + \frac{h}{2}(R_{(0,p-1)i} + R_{0pi}) - (Q_{(p-1)i} - Q_{pi}) \tag{49}$$

$$D_{1pi}^s = \frac{h}{2}(e^{\alpha\tau}A_i)^T(Q_{(p-1)i} + Q_{pi}) - \frac{h}{2}(R_{(M,p-1)i} + R_{Mpi}) \tag{50}$$

$$D_i^a = \begin{pmatrix} D_{1i}^a & D_{2i}^a & \dots & D_{Mi}^a \end{pmatrix} \tag{51}$$
$$D_{pi}^a = \begin{pmatrix} D_{0pi}^a \\ D_{1pi}^a \end{pmatrix} \tag{52}$$

$$D_{0pi}^a = -\frac{h}{2}(A_0 + \alpha I_4)^T(Q_{(p-1)i} - Q_{pi}) - \frac{h}{2}(R_{(0,p-1)i} - R_{0pi}) \tag{53}$$

$$D_{1pi}^a = -\frac{h}{2}(e^{\alpha\tau}A_i)^T(Q_{(p-1)i} + Q_{pi}) + \frac{h}{2}(R_{(M,p-1)i} - R_{Mpi}) \tag{54}$$

and

$$h = \tau/M$$

for all $i = 1, \dots, N$ Q.E.D

In order to have a convex representation in (34) and (35), the matrices A_0 and A_i in (30) and (31) must be fixed. Since the gains K_1 and K_2 are not known a-priori, the matrices A_0 and A_i in (30) and (31) are dependent on the feedback gains. However, if the gains K_1 and K_2 are fixed, Proposition 5.22 stated above, provides a feasibility check for stability for a fixed known delay value τ. The design problem associated here is to identify minimum gain values for K_1 and K_2 and an associated minimum possible delay τ such that Proposition 5.22 is satisfied. The solution to such a problem is not straightforward, and often depends on fine gridding of the search space (or similar technique). However, there is no guarantee of finding the optimal solution, or even a sub-optimal one depending on the type of non-convex surface. In this paper a solution is sought by making use of a deterministic global optimization algorithm, Dividing Rectangles (DIRECT). The method does not require any derivative information to be supplied, and uses a center point sampling strategy. The method was originally developed in [38] as a modification of the classical one dimensional Lipschitzian optimization algorithm known as the Shubert algorithm. The search space is an n-dimensional hypercube or box, defined as $H = \{x \in R^n : 0 \leq x_i \leq 1\}$. The algorithm works in the normalized parametric space, transforming to the actual search space as and when the cost function has to be evaluated. The principle idea can be summarized as: while the algorithm proceeds, the search space is partitioned into smaller hypercubes or boxes and each hypercube is sampled at the center point of the interval. Over many iterations, the algorithm tries to find all the 'potentially optimal' hypercubes or boxes in the search space and then partitions them, (see [38] for details on the definition of the potentially optimal hypercubes and the division strategies) thereby eventually obtaining the global solution. The algorithm has asymptotic convergence property, and details of the proof are available in [39].

Since there are multiple minimization objectives, a collective optimization objective function is defined with appropriate scaling as follows:

$$J(k_1, k_2, \tau) := W_1|k_1| + W_2|k_2| + W_3|\tau| \qquad (55)$$

subject to feasibility of (34) and (35) and the side constraints on the optimization variables $k_{1_{min}} \leq k_1 \leq k_{1_{max}}$, $k_{2_{min}} \leq k_2 \leq k_{2_{max}}$ and $\tau_{min} \leq \tau \leq \tau_{min}$. In (55) the scalars W_i for $i = 1, 2, 3$ are the weights of the optimization variables k_1, k_2 and τ. Initially the bounds for the variables are chosen arbitrarily large and the bounds will be normalized to be in the range $[0, 1]$ since it is a requirement for the performance of the DIRECT optimization algorithm. The underlying rationale behind this objective function is to obtain the gain set that provides minimum control effort at a minimum possible level of delay. When the feasibility of the LMI constraints in (34) and (35) for a specific set of gains and an artificial delay is not satisfied, the cost associated with such a set is penalized by assigning it a large value. The idea is during the iterations, the DIRECT optimization procedure then eliminates that region from the search space.

Since the transformed systems as in (33) are considered, the positive scalar α, which is design freedom associated with the required exponential convergence decay rate, is fixed a-priori. In a similar way, the other design scaling parameter β in (12) is also fixed a-priori. The MATLAB code used for the DIRECT optimization is available from the authors of [39].

V. Numerical Example

Consider a network of $N = 4$ agents, described by (1) and (2) with matrices A, B and C as given in (3)-(4), connected over a nearest neighbour interconnection topology. In the example the decay rate $\alpha = 1$ and $\beta = 0.1$. The number of partitions of the delay interval for *Proposition 5.22* was considered to be $M = 1$. After employing the transformations (17), (25) and (32) a system of the form (33) is obtained. The DIRECT algorithm has been employed with the following bounds

$$\begin{aligned} k_1 &\in [10\ 20];\ k_2 \in [10\ 20] \\ \tau &\in [0.1\ 0.5] \end{aligned} \qquad (56)$$

The weights for the cost function $J(k_1, k_2, \tau)$ in (55) for this range have been chosen as $W_1 = W_2 = W_3 = 1$. The optimal gains k_1 and k_2 and delay τ obtained from within these bounds are

$$k_1 = 18.33;\ k_2 = 11.29;\ \tau_{opt} = 0.1667 \qquad (57)$$

The offset D is then calculated using (24). The desired formation is a square with $(x_i^f, y_i^f) = (\pm 5, \pm 5)$. The system in (6) has been simulated with the control law in (13) using the values in (57). Fig. 1 shows the agents form a square from random initial conditions. The initial condition for the delayed output in (14) is set as the initial condition in the interval $t \in [-\tau_{opt}, 0]$. Fig. 2 shows the agents settling into a formation as a function of time.

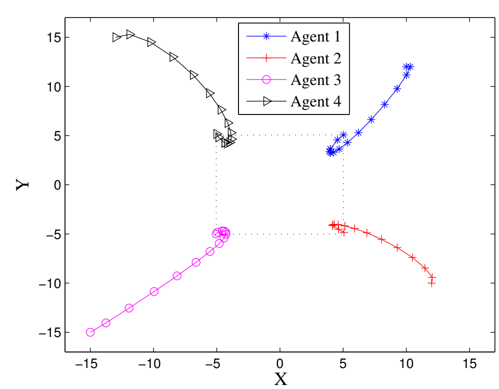

Fig. 1. Formation of agents

VI. Conclusions

In this paper, a novel distributed control law to stabilize a formation of multi-agent systems described by double integrator dynamics is proposed. The distributed control law employs delays in static output feedback to stabilize the network of double integrators. Optimal gains that guarantee performance in terms of a pre-specified decay rate along

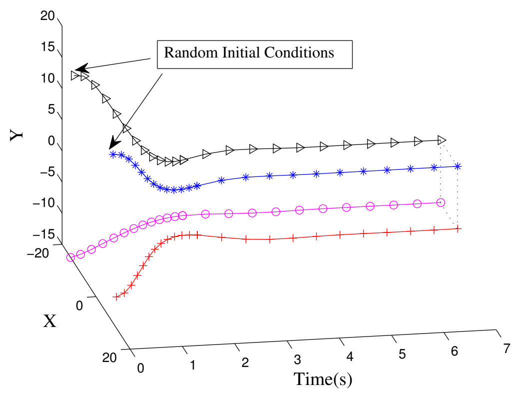

Fig. 2. Formation of agents with time

with minimum control effort at a minimum possible level of delay were obtained by employing the DIRECT optimization algorithm.

REFERENCES

[1] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of Groups of Mobile Autonomous Agents using Nearest Neighbour Rules," *Proc. of IEEE CDC*, 2002.

[2] J. Lin, A.S. Morse, and B.D.O. Anderson, "The multi-agent rendezvous problem," *IEEE CDC*, 2003.

[3] R. Olfati-Saber, and R. M. Murray, "Consensus Problems in Networks of Agents with Switching Topology and Time-Delays", *IEEE Trans. on Aut. Control*, Vol. 49, pp. 1520-1533, 2004.

[4] L. Moreau, "Stability of multiagent systems with time dependent communication links," *IEEE Trans. Aut. Control*, vol. 50, pp. 169-182, 2005.

[5] Z. Lin, B. Francis, and M. Maggiore, "Necessary and sufficient graphical conditions for formation control of unicycles," *IEEE Trans. Automat. Contr.*, Vol. 50, pp. 121127, 2005.

[6] R. Olfati-Saber, J.A. Fax, and R. M. Murray, "Consensus and Cooperation in Networked Multi-Agent Systems," *Proceedings of the IEEE*, Vol. 95, pp. 215-233, 2007.

[7] W. Ren, R. W. Beard, and E. M. Atkins, "Information consensus in multivehicle cooperative control,", *IEEE Control Sys. Mag.*, Vol. 27, pp. 71-82, 2007.

[8] G. Xie and L. Wang, "Consensus control for a class of networks of dynamic agents," *Int. J. Robust Nonlinear Contr.*, Vol. 17, pp. 941959, 2007.

[9] W. Ren, and E. M. Atkins, "Distributed multi-vehicle coordinated control via local information exchange," *Int. J. Robust Nonlinear Contr.*, Vol. 17, pp. 10021033, 2007.

[10] W. Ren, "On Consensus Algorithms for Double-Integrator Dynamics," *IEEE Transactions on Automatic Control*, Vol. 53, pp. 15031509, 2008.

[11] W. Yang, and A. L. Bertozzi, X. F. Wang, "Stability of a second order consensus algorithm with time delay," *Proc. of IEEE CDC*, 2008.

[12] A. Regmi, R. Sandoval, R. Byrne, H. Tanner, and C. T. Abdallah, "Experimental implementation of flocking algorithms in wheeled mobile robots," *Proc. of ACC*, 2005.

[13] P. C. Hughes, "Spacecraft Attitude Dynamics,", New York: Wiley, 1986.

[14] J. R. Lawton, R. W. Beard, and B. Young, "A decentralized approach to formation maneuvers," *IEEE Trans. Robot. Autom.*, Vol. 19, pp.933941, 2003.

[15] J. A. Fax, and R. M. Murray, "Information Flow and Cooperative Control of Vehicle Formations", *IEEE Trans. on Aut. Control*, Vol. 49, pp. 1465-1476, 2004.

[16] G. Lafferriere, A. Williams, J. Caughman, and J. J. P. Veerman, "Decentralized control of vehicle formations," *Syst. Contr. Lett.*, Vol. 54, pp. 899910, 2005.

[17] S. Roy, A. Saberi, and K. Herlugson, "Formation and alignment of distributed sensing agents with double-integrator dynamics and actuator saturation," *Sensor Network Operations. Piscataway*, NJ: IEEE Press, 2006.

[18] R. Olfati-Saber, "Flocking for multi-agent dynamic systems: Algorithms and theory," *IEEE Trans. Autom. Control*, Vol. 51, pp. 401420, 2006.

[19] D. Lee and M. W. Spong, "Stable flocking of multiple inertial agents on balanced graphs," *IEEE Transactions on Automatic Control*, Vol. 52, pp. 14691475, 2007.

[20] H. G. Tanner, A. Jadbabaie, and G. J. Pappas, "Flocking in fixed and switching networks," *IEEE Transactions on Automatic Control*, Vol. 52, pp. 863868, 2007.

[21] A. Seuret, D. V. Dimarogonas and K. H. Johanson, "Consensus under Communication Delays," *Proc. of IEEE CDC*, 2008.

[22] A. Seuret, D. V. Dimarogonas and K. H. Johanson, "Consensus of double integrator multi-agents under communication delay," *Proc. IFAC Time Delay Systems Workshop*, 2009.

[23] U. Münz, A. Papachristodoulou, and F. Allgöwer, "Delay-Dependent Rendezvous and Flocking of Large Scale Multi-Agent Systems with Communication Delays," *Proc. of IEEE CDC*, 2008.

[24] A. Papachristodoulou, A. Jadbabaie, and U. Münz, "Effects of Delay in Multi-Agent Consensus and Oscillator Synchronization," *IEEE Trans. Aut. Control*, Vol. 55, pp. 1471-1477, 2010.

[25] U. Münz, A. Papachristodoulou, and F. Allgöwer, "Delay robustness in consensus," *Automatica*, Vol. 46, pp. 1252-1265, 2010.

[26] A. Trofino-Neto, and V. Kucera, "Stabilization via Static Output Feedback," *IEEE Trans. Aut. Control*, Vol. 38, pp. 764-765, 1993.

[27] V. L. Syrmos, C. T. Abdallah, P. Dorato, and K. Grigoriadis, "Static Output Feedback - A survey," *Automatica*, Vol. 33, pp. 125-137, 1997.

[28] S-I. Niculescu and C. T. Abdallah, "Delay effects on static output feedback stabilization,", *Proc. of IEEE CDC*, 2000.

[29] S-I. Niculescu, K.Gu and C. T. Abdallah, "Some remarks on the delay stabilizing effect in SISO systems,", *Proc. of ACC*, 2003.

[30] W. Michiels, S.-I. Niculescu and L. Moreau, "Using delays and time-varying gains to improve the static output feedback stabilizability of linear systems: A comparison", *IMA J. Math. Control Inform.*, Vol. 21, No. 4, pp. 393-418, 2004.

[31] V. L. Kharitonov, S. I. Niculescu, J. Moreno, and W. Michiels, "Static Output Feedback Stabilization: Necessary Conditions for Multiple Delay Controllers," *IEEE Trans. on Aut. Control*, Vol. 50, 2005.

[32] E. Fridman, "Descriptor descritized Lyapunov functional method: Analysis and design," *IEEE Trans. on Aut. Control*, Vol. 51, pp. 890-897, 2006.

[33] A. Seuret, C. Edwards, S. K. Spurgeon, and E. Fridman "Static output feedback sliding mode control design via an aritifial stabilizing delay," *IEEE Trans. on Aut. Control*, Vol. 54 , pp. 256-265, 2006.

[34] S. Mondié, and V. L. Kharitonov, "Exponential estimates for retarded time-delay systems: An LMI approach," *IEEE Trans. on Aut. Control*, Vol. 50, pp. 666-670, 2005.

[35] A. Seuret, M. Dambrine, and J-P Richard, "Robust exponential stabilization for systems with time-varying delays," *Proc. IFAC Time Delay Systems Workshop*, 2004.

[36] C. Godsil, and G. Royle, "Algebraic Graph Theory", Springer, 2001.

[37] K. Gu, V. L. Kharitonov, and J. Chen, "Stability of Time-Delay Systems", Birkhauser, 2003.

[38] D.R. Jones, C.D. Perttunen, B.E. Stuckman, "Lipschitzian optimization without the Lipschitz constant," *J. Optim. Theor. Appl.*, Vol 79, pp. 157181, 1993.

[39] D.E. Finkel, and C.T. Kelley, "Convergence analysis of the DIRECT algorithm," Technical Report, North Carolina State University, Center for Research in Scientific Computation and Department of Mathematics, 2004.

[40] S.P. Boyd and L. El Ghaoui and E. Feron and V. Balakrishnan, "Linear Matrix Inequalities in Systems and Control Theory", SIAM: Philadelphia, 1994.

[41] P. Gahinet, A. Nemirovski, A. J. Laub and M. Chilali, "LMI Control ToolBox," The MathWorks Inc., 1995.