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Periodic FIR Controller Synthesis for Discrete-Time Uncertain Linear Systems

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Abstract— **This paper is concerned with robust state-feedback controller synthesis for discrete-time linear periodic/time-invariant systems subject to polytopic-type parametric uncertainties. In recent studies, some of the authors conceived an LMI-based approach to periodically time-varying memory controller (PTVMC) synthesis and proved that this approach is indeed effective to get less conservative robust controller design procedures. However, since the peculiar controller structure requires to reset memory to zero in a periodic way, it is pointed out that the control performance depends on the timing of implementation. In this paper we tackle this issue and propose a reset-less state-feedback Periodic FIR Controller (PFIRC), which turns out to be suitable to improve robustness on periodic and time-invariant systems. Moreover, as a special case, a design condition is provided for FIR-type LTI controllers that robustly stabilize uncertain LTI systems. Numerical examples illustrate the efficiency of the proposed approaches.**

Keywords: discrete-time systems, periodic FIR controller, parametric uncertainties, LMI.

I. INTRODUCTION

Uncertain parameters and even some types of nonlinearities in physical systems can be treated effectively by means of polytopic uncertainties [2]. The usefulness of linear matrix inequalities (LMIs) was suggested more than two decades ago [3] for this study area. In this framework, the problem of robust stabilization of uncertain linear discrete-time periodic systems was first tackled in [9]. This work is part of the renewed interest for periodic systems since the end of the eighties [6], mainly due to the variety and the originality of the possible applications. One can first recall the now classic examples of control of vibrations in helicopters [4] as well as autonomous orbit control [13] or the attitude control systems of satellites equipped with magnetorquers [15].

Following this line, a synthesis condition leading to an efficient static periodic controller for discrete-time periodic systems is provided in [1]. Nevertheless, as mentioned in [11], if we persist in this kind of static control law, it should be hard to obtain a systematic single-shot LMI-based design method that outperforms the existing results. Therefore, in [10], [11], it was proposed to rather consider a new kind of periodic controller (the so-called PTVMC for Periodic Time-Varying Memory Controller) in which past states of the plant are kept in memory and actively used to construct current inputs. Using the basic idea of extended LMI proposed in [8] and [7], a particular effort has been made to render the synthesis problem convex and effective for handling polytopic uncertainties. However, for the convexity requirement, stored information are required to be reset to zero in a periodic fashion and, as pointed out in [10], the level of robust performance depends on this timing. This might cause several problems in practical implementation since the designer should first determine the timing of the resets over time instances during one period, which is not an obvious issue in general.

In this paper, we tackle this issue and propose a reset-less state-feedback Periodic Finite Impulse Response Controller (PFIRC). In addition, because of the FIR structure, the number of degrees-of-freedom increases which improves robustness. Relying on a time-invariant representation [12], an extended sufficient LMI-based condition of robust stabilization is provided. Through a simple academic example, the proposed controller is indeed shown to be effective for robust stabilization of periodic systems.

Moreover, by regarding time-invariant models as N-periodic models with constant matrices, it is shown that PFIRCs allow to enlarge the closed-loop stability margins for uncertain LTI systems as well. In this case, N is a tuning parameter. Finally, as a special case, a constructive design condition for FIR-type LTI controllers that robustly stabilize uncertain LTI systems is provided. This is indeed beyond reach in the existing PTVMC approach where the controllers become inherently periodic because of the resets.

We use the following notations in this paper. The symbols **1** and **0** stand for the identity and zero matrices of appropriate dimensions, respectively. The set of symmetric matrices and positive-definite symmetric matrices of the size l are denoted by $\mathbb{S}^l$ and $\mathbb{S}^l_+$, respectively. For a real square matrix A, we define $\text{He}\{A\} = A + A^T$. The operator diag builds block diagonal matrix from input arguments. The convex hull of the collection of N elements $A^{[1]}, \cdots, A^{[N]}$ is denoted by $\text{co}\left\{A^{[1]}, \cdots, A^{[N]}\right\}$. Variables n and m refer to the size of the state vector x_k and the input vector u_k, respectively. The time instant q is expressed as a multiple of the period N plus a reminder such that $q = Nk + r$.

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II. PFIRC SYNTHESIS FOR 2-PERIODIC MODELS

A. A new structure for periodic systems

First of all, we describe our underlying ideas for PFIRC synthesis. For simplicity, we confine our discussion to the 2-periodic case for the time being. The standard discrete-time state-space description of such model is

$$\begin{cases} x_{2k+1} = A_0 x_{2k} + B_0 u_{2k} \\ x_{2k+2} = A_1 x_{2k+1} + B_1 u_{2k+1} \end{cases} \tag{1}$$

where $k = 0, 1, \cdots$ and the corresponding classical Periodic State-Feedback Controller (PSFC), used in [1], is

$$\text{PSFC}: \begin{cases} u_{2k} &= K_0 x_{2k} \\ u_{2k+1} &= K_1 x_{2k+1} \end{cases} \tag{2}$$

First proposed in [10], the PTVMC enriches this control law by allowing u_q to depend, not only on the current state x_q, but also on the state history since the beginning of the period kept in memory:

$$\text{PTVMC}: \begin{cases} u_{2k} &= K_{0,0} x_{2k} \\ u_{2k+1} &= K_{1,0} x_{2k+1} + K_{1,1} x_{2k} \end{cases} \tag{3}$$

The idea is to increase the number of degrees of freedom to improve performance and robustness of the closed-loop system. It is now a dynamic controller whose order increases along the period. As shown in [11], its main drawback comes from the dependency of closed-loop robustness achievement on the timing of the resets of memory that is determined beforehand. Indeed, another PFIRC is

$$\begin{cases} u_{2k} &= K_{0,0} x_{2k} + K_{0,1} x_{2k-1} \\ u_{2k+1} &= K_{1,0} x_{2k+1} \end{cases} \tag{4}$$

which happens to be distinct of (3).

To tackle this problem, we propose to allow u_q to *always* depend on the complete state history over one period such that (3) becomes the so-called PFIRC:

$$\text{PFIRC}: \begin{cases} u_{2k} &= K_{0,0} x_{2k} + K_{0,1} x_{2k-1} \\ u_{2k+1} &= K_{1,0} x_{2k+1} + K_{1,1} x_{2k} \end{cases} \tag{5}$$

This equation highlights that the PFIRC is indeed a FIR controller with periodically time-varying coefficients.

Under this control law, (1) becomes

$$\begin{cases} x_{2k+1} = A_{0,0} x_{2k} + A_{0,1} x_{2k-1} \\ x_{2k+2} = A_{1,0} x_{2k+1} + A_{1,1} x_{2k} \end{cases} \tag{6}$$

with

$$\begin{array}{ll} A_{0,0} = A_0 + B_0 K_{0,0} & A_{0,1} = B_0 K_{0,1} \\ A_{1,0} = A_1 + B_1 K_{1,0} & A_{1,1} = B_1 K_{1,1} \end{array} \tag{7}$$

For this reason, this paper is concerned with closed-loop periodic model described by (6) instead of $x_{q+1} = (A_q + K_q B_q) x_q$ corresponding to the standard PSFCs.

B. Stability analysis

1) The lifted model: In order to directly apply theories coming from the time-invariant world (or at least be inspired by it), it is often useful to find a time-invariant representation of periodic systems. Following this line, the peculiar periodic model (6) is shown to be equivalent to

$$\mathfrak{E} \xi_{k+1} = \mathfrak{A} \xi_k \tag{8}$$

where

$$\mathfrak{E} = \begin{bmatrix} \mathbf{1} & -A_{1,0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}, \mathfrak{A} = \begin{bmatrix} A_{1,1} & \mathbf{0} \\ A_{0,0} & A_{0,1} \end{bmatrix}, \xi_{k+1} = \begin{bmatrix} x_{2k+2} \\ x_{2k+1} \end{bmatrix} \tag{9}$$

This readily follows by rewriting (6) as

$$\begin{bmatrix} -\mathbf{1} & A_{1,0} & A_{1,1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} & A_{0,0} & A_{0,1} \end{bmatrix} \begin{bmatrix} x_{2k+2} \\ x_{2k+1} \\ x_{2k} \\ x_{2k-1} \end{bmatrix} = 0 \tag{10}$$

Moreover, since $\mathfrak{E}$ is non-singular, (6) can be reformulated as another (and more classical) time-invariant model which state matrix, denoted by Ψ, is

$$\Psi = \mathfrak{E}^{-1} \mathfrak{A} = -\begin{bmatrix} A_{1,0} A_{0,0} + A_{1,1} & A_{1,0} A_{0,1} \\ A_{0,0} & A_{0,1} \end{bmatrix} \tag{11}$$

2) Building a suitable stability condition: The matrix Ψ is the monodromy matrix of (6). According to [5], Schur stability of Ψ is equivalent to the stability of the periodic model (6). Therefore, stability condition for time-invariant models can be directly applied to Ψ. However, using this matrix is unsuitable neither for state-feedback stabilization nor for robust stability analysis in case of polytopic uncertainties due to multiplications between $A_{i,j}$ matrices. Consequently, an LMI that preserves the structure of (10) is desirable. This purpose can be achieved by first considering a dual model of (8).

Theorem 2.1: A dual version of the model (8) is

$$\mathfrak{A}^T \xi^d_{k+1} = \mathfrak{E}^T \xi^d_k \tag{12}$$

Proof: By first multiplying (8) by $\mathfrak{E}^{-1}$ on the left and then using the usual definition of system duality, one gets

$$\eta^d_k = \left(\mathfrak{E}^{-1} \mathfrak{A}\right)^T \eta^d_{k+1} \tag{13}$$

which can be rewritten as

$$\mathfrak{E}^T \left(\mathfrak{E}^{-1}\right)^T \eta^d_k = \mathfrak{A}^T \left(\mathfrak{E}^{-1}\right)^T \eta^d_{k+1} \tag{14}$$

To avoid inversion of $\mathfrak{E}$, the change of variables $\xi^d_k = \left(\mathfrak{E}^{-1}\right)^T \eta^d_k$ is introduced and one gets (12). ■

Since duality preserves stability, the following theorem gives a more suitable stability condition for our analysis and synthesis purpose.

Theorem 2.2: Any model described by (8) and (9) is stable if and only if there exists $\mathcal{F} \in \mathbb{R}^{2n \times 4n}$ and $X \in \mathbb{S}^{2n}_+$ such that

$$\mathcal{X}(X) + \mathrm{He}\left\{ \begin{bmatrix} \mathfrak{A} \\ -\mathfrak{E} \end{bmatrix} \mathcal{F} \right\} \prec 0, \ \mathcal{X}(X) := \begin{bmatrix} -X & \mathbf{0} \\ \mathbf{0} & X \end{bmatrix} \tag{15}$$

Proof: For a quadratic Lyapunov function $V_k = \left(\xi^d_k\right)^T X \xi^d_k$, the system (12) is stable if and only if there exists $X \in \mathbb{S}^{2n}_+$ such that

$$\begin{bmatrix} \xi^d_{k+1} \\ \xi^d_k \end{bmatrix}^T \mathcal{X}(X) \begin{bmatrix} \xi^d_{k+1} \\ \xi^d_k \end{bmatrix} \prec 0, \ \forall \begin{bmatrix} \mathfrak{A}^T & -\mathfrak{E}^T \end{bmatrix} \begin{bmatrix} \xi^d_{k+1} \\ \xi^d_k \end{bmatrix} = 0. \tag{16}$$

Thus, the end of the proof follows immediately from Elimination Lemma (see [14]) which leads to (15). ■

Remark: By multiplying (15) by $\mathcal{R} = \begin{bmatrix} \mathbf{1} & \mathfrak{A}\mathfrak{E}^{-1} \end{bmatrix}$ from left and its transpose from right, it can be stated that (15) is nothing but a Schur stability condition of Ψ.

3) A robust stability condition: Let us consider now the case where the system (1) is subject to polytopic uncertainties as follows:

$$\begin{bmatrix} A_0 & B_0 \\ A_1 & B_1 \end{bmatrix} \in \text{co}\left\{ \begin{bmatrix} A_0^{[1]} & B_0^{[1]} \\ A_1^{[1]} & B_1^{[1]} \end{bmatrix}, \cdots, \begin{bmatrix} A_0^{[L]} & B_0^{[L]} \\ A_1^{[L]} & B_1^{[L]} \end{bmatrix} \right\}. \quad (17)$$

Controlled by a PFIRC, the closed-loop system can be reformulated as (8) with

$$\left[\, \mathfrak{E} \;\; \mathfrak{A} \,\right] \in \text{co}\left\{ \left[\, \mathfrak{E}^{[1]} \;\; \mathfrak{A}^{[1]} \,\right], \cdots, \left[\, \mathfrak{E}^{[L]} \;\; \mathfrak{A}^{[L]} \,\right] \right\} \quad (18)$$

Here the definitions of $\mathfrak{E}^{[i]}$ and $\mathfrak{A}^{[i]}$ $(i = 1, \cdots, L)$ are obvious. As these matrices appear linearly in (15), the Th 2.2 can be directly extended to the robust case by repeating (15) for each of the L vertices of the polytope. Moreover, to reduce the conservatism of the condition, the Lyapunov matrix P can be allowed to depend on uncertainties. This leads to the following sufficient robust stability condition.

Theorem 2.3: Any model described by (8), (9) and (18) is robustly stable *if* there exists $\mathcal{F} \in \mathbb{R}^{2n\times 4n}$ and $X^{[i]} \in \mathbb{S}_+^{2n}$ such that, for $i = 1, \cdots, L$

$$\mathcal{X}(X^{[i]}) + \text{He}\left\{ \begin{bmatrix} \mathfrak{A}^{[i]} \\ -\mathfrak{E}^{[i]} \end{bmatrix} \mathcal{F} \right\} \prec 0 \quad (19)$$

Obviously if (15) holds then (19) holds for constant matrices $P^{[i]} = P$. Hence, using Th 2.3 instead of Th 2.2 can only reduce the conservatism of robust stability analysis.

C. Robust stabilization of periodic systems via PFIRC

1) The nominal case: To move on to robust PFIRC synthesis, we first consider the nominal (i.e uncertainty free) case. Using (7), the necessary and sufficient condition for closed-loop stability of Th 2.2 is rewritten as:

$$\mathcal{X}(X) + \text{He}\left\{ (\mathcal{A} + \mathcal{B}\mathcal{K})\, \mathcal{F} \right\} \prec 0 \quad (20)$$

where

$$\mathcal{A} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ A_0 & \mathbf{0} \\ -\mathbf{1} & A_1 \\ \mathbf{0} & -\mathbf{1} \end{bmatrix},\; \mathcal{B} = \begin{bmatrix} B_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & B_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & B_1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix},\; \mathcal{K} = \begin{bmatrix} K_{1,1} & \mathbf{0} \\ K_{0,0} & K_{0,1} \\ \mathbf{0} & K_{1,0} \end{bmatrix}. \quad (21)$$

Unfortunately, the bilinear term $\mathcal{K}\mathcal{F}$ appearing when expanding (20) makes this inequality unsuitable for controller synthesis. To get around this difficulty, we apply the classical change of variables $\mathcal{W} = \mathcal{K}\mathcal{F}$. More precisely, in order to allow the recovery of $\mathcal{K}$ from the knowledge of $\mathcal{W}$ and $\mathcal{F}$, we propose to constraint $\mathcal{F}$ such that $\mathcal{F} = \left[\, \mathbf{0} \;\; \mathcal{G} \,\right]$ with $\mathcal{G}$ block diagonal. Consequently, $\mathcal{K}\mathcal{F}$ is written as $\left[\, \mathbf{0} \;\; \mathcal{Y} \,\right]$ with $\mathcal{K}\mathcal{G} = \mathcal{Y}$, where the structure of $\mathcal{Y}$ is inherited from $\mathcal{K}$.

Theorem 2.4: Any 2-periodic model described by (1) can be stabilized by a PFIRC *if* there exists $\mathcal{G}$, $\mathcal{Y}$ and $X \in \mathbb{S}_+^{2n}$ such that

$$\mathcal{X}(X) + \text{He}\left\{ (\mathcal{A}\mathcal{G} + \mathcal{B}\mathcal{Y}) \left[\, \mathbf{0}_{2n} \;\; \mathbf{1}_{2n} \,\right] \right\} \prec 0 \quad (22)$$

with $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{K}$ given by (21) and $\mathcal{G}$ and $\mathcal{Y}$ such that

$$\mathcal{G} = \begin{bmatrix} G_0 & \mathbf{0} \\ \mathbf{0} & G_1 \end{bmatrix},\quad \mathcal{Y} = \begin{bmatrix} Y_{1,1} & \mathbf{0} \\ Y_{0,0} & Y_{0,1} \\ \mathbf{0} & Y_{1,0} \end{bmatrix} \quad (23)$$

The controller can then be recovered by solving $\mathcal{K} = \mathcal{Y}\mathcal{G}^{-1}$.

Proof: The proof that (22) implies closed-loop stability follows directly by rewriting $\mathcal{A}\mathcal{G} + \mathcal{B}\mathcal{Y}$ as $(\mathcal{A} + \mathcal{B}\mathcal{K})\,\mathcal{F}$ which leads to (20) with $\mathcal{F} = \left[\, \mathbf{0} \;\; \mathcal{G} \,\right]$. ■

This theorem provides a sufficient condition for the existence of stabilizing state-feedback PFIRCs. Of course the lack of the necessity stems from the restriction $\mathcal{F} = \left[\, \mathbf{0} \;\; \mathcal{G} \,\right]$ with $\mathcal{G}$ being block-diagonal. Because of this restriction, we cannot ensure that all the stabilizing PFIRCs can be parametrized by (22).

However, we can ensure at least that (22) is always feasible if the nominal system (1) is stabilizable. Indeed, in this case, a stabilizing PSFC (2) exists, which is equivalent to the feasability of the following condition [10]:

$$\begin{aligned} &\text{diag}\left\{ -X_s, \mathbf{0}, X_s \right\} + \\ &\text{He}\left\{ \begin{bmatrix} A_0 + B_0K_0 & \mathbf{0} \\ -\mathbf{1} & A_1 + B_1K_1 \\ \mathbf{0} & -\mathbf{1} \end{bmatrix} \left[\, \mathbf{0}\; \mathcal{G} \,\right] \right\} \prec 0 \end{aligned} \quad (24)$$

where $X_s \in \mathbb{S}_+^n$ and $\mathcal{G}$ is given by (23). This clearly validates the assertion since, if this LMI holds, then there exists $\epsilon > 0$ such that

$$\begin{aligned} &\text{diag}\left\{ -\epsilon\mathbf{1}, -X_s, \epsilon\mathbf{1}, X_s \right\} + \\ &\text{He}\left\{ \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ A_0 + B_0K_0 & \mathbf{0} \\ -\mathbf{1} & A_1 + B_1K_1 \\ \mathbf{0} & -\mathbf{1} \end{bmatrix} \left[\, \mathbf{0}\; \mathbf{0}\; \mathcal{G} \,\right] \right\} \prec 0 \end{aligned} \quad (25)$$

which complies with the structure of (22) with $X = \text{diag}\,(\epsilon\mathbf{1}, X_s) \in \mathbb{S}_+^{2n}$ and $\mathcal{G}$ and $\mathcal{Y}$ given by (23) with $Y_{1,1} = Y_{0,1} = \mathbf{0}$.

As clarified later on, the extra freedom introduced by $K_{1,1}$ and $K_{0,1}$ is effective for the robust stabilization of uncertain systems. Hence, by relaxing $\mathcal{Y}$, Th 2.4 cannot increase the conservatism of condition (24).

2) Robust controller synthesis: We are now ready to state the main result of this paper, namely the robust version of Th 2.4 which readily follows with simple convexity arguments.

Theorem 2.5: Any uncertain 2-periodic model described by (1) and (17), can be stabilized by a PFIRC *if* there exists $\mathcal{G}$, $\mathcal{Y}$ and $X^{[i]} \in \mathbb{S}_+^{2n}$ such that, for $i = 1, \cdots, L$

$$\mathcal{X}(X^{[i]}) + \text{He}\left\{ \left(\mathcal{A}^{[i]}\mathcal{G} + \mathcal{B}^{[i]}\mathcal{Y}\right) \left[\, \mathbf{0}_{2n} \;\; \mathbf{1}_{2n} \,\right] \right\} \prec 0. \quad (26)$$

Here, $\mathcal{G}$ and $\mathcal{Y}$ are given by (23) and $\mathcal{A}^{[i]}$ and $\mathcal{B}^{[i]}$ are given by (21) and correspond to the vertex matrices of the polytope. The controller can then be recovered by solving $\mathcal{K} = \mathcal{Y}\mathcal{G}^{-1}$.

In the next section, we compare the effectiveness of robust PFIRC synthesis by this theorem with those proposed in the literature.

III. Comparisons with other robust state-feedback controller synthesis

A. The PSFCs case [1]

The robust PSFC synthesis condition proposed in [1] is shown to be equivalent to the LMI of the following theorem [10].

Theorem 3.1: Any uncertain 2-periodic model described by (1) and (17), can be stabilized by a PSFC *if* there exists $\mathcal{G}$, $\mathcal{Y}_s$ and $X_s^{[i]} \in \mathbb{S}_+^n$ such that, for $i = 1, \cdots, L$

$$\begin{aligned} &\text{diag}\left\{ -X_s^{[i]}, \mathbf{0}, X_s^{[i]} \right\} + \\ &\text{He}\left\{ (\bar{\mathcal{A}}^{[i]}\mathcal{G} + \bar{\mathcal{B}}^{[i]}\mathcal{Y}_s) \left[\, \mathbf{0}_{2n\times n} \;\; \mathbf{1}_{2n} \,\right] \right\} \prec 0 \end{aligned} \quad (27)$$

$$\bar{\mathcal{A}}^{[i]} := \begin{bmatrix} A_0^{[i]} & \mathbf{0} \\ -\mathbf{1} & A_1^{[i]} \\ \mathbf{0} & -\mathbf{1} \end{bmatrix},\ \bar{\mathcal{B}}^{[i]} := \begin{bmatrix} B_0^{[i]} & \mathbf{0} \\ \mathbf{0} & B_1^{[i]} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},\ \mathcal{Y}_s := \begin{bmatrix} Y_0 & \mathbf{0} \\ \mathbf{0} & Y_1 \end{bmatrix}. \quad (28)$$

where $\mathcal{G}$ is given by (23). The controller can then be recovered by solving $K_q = \mathcal{Y}_q G_q^{-1}$ $(q = 0, 1)$.

Following the same lines as for establishing (25), we can prove that, if (27) holds, then there exists $\varepsilon > 0$ such that (26) holds with the same $G_0,\ G_1$ and with $Y_{0,0} = Y_0,\ Y_{1,0} = Y_1,\ Y_{0,1} = 0,\ Y_{1,1} = 0$ and $X^{[i]} = \mathrm{diag}\{\epsilon \mathbf{1}, X_s^{[i]}\}$. It follows that the condition (26) for robust PFIRC synthesis is not more conservative than condition (27).

B. The PTVMCs case [11]

We first remind the LMI condition for robust synthesis of the PTVMC (4) established in [11].

Theorem 3.2: Any uncertain 2-periodic model described by (1) and (17), can be stabilized by the PTVMC (4) *if* there exists $\mathcal{G}_m$, $\mathcal{Y}_m$ and $X_m^{[i]} \in \mathbb{S}_+^n$ such that, for $i = 1, \cdots, L$

$$\begin{aligned} &\mathrm{diag}\left\{-X_m^{[i]}, \mathbf{0}, X_m^{[i]}\right\} + \\ &\mathrm{He}\left\{(\bar{\mathcal{A}}^{[i]}\mathcal{G}_m + \bar{\mathcal{B}}^{[i]}\mathcal{Y}_m)\begin{bmatrix} \mathbf{0}_{2n\times n} & \mathbf{1}_{2n} \end{bmatrix}\right\} \prec 0, \\ &\mathcal{G}_m := \begin{bmatrix} G_{0,0} & G_{0,1} \\ \mathbf{0} & G_{1,0} \end{bmatrix},\ \mathcal{Y}_m := \begin{bmatrix} Y_{0,0} & Y_{0,1} \\ \mathbf{0} & Y_{1,0} \end{bmatrix}. \end{aligned} \quad (29)$$

The controller can then be recovered by solving

$$\begin{bmatrix} K_{0,0} & K_{0,1} \\ \mathbf{0} & K_{1,0} \end{bmatrix} = \mathcal{Y}_m \mathcal{G}_m^{-1}. \quad (30)$$

In contrast with the case of (27), it seems hard to derive explicit inclusion relationship among (26) and (29) since in (29), the variable $\mathcal{G}_m$ is allowed to be block upper triangular. However, similarly to (25), we can prove that if (29) holds with $\mathcal{G}_m$ restricted to be block-diagonal (i.e., $G_{0,1} = 0$), then there exists $\varepsilon > 0$ such that (26) holds with $G_0 = G_{0,0},\ G_1 = G_{1,0},\ X^{[i]} = \mathrm{diag}\{\epsilon\mathbf{1}, X_m^{[i]}\}$ $(i = 1, \cdots, L)$ and with the same $Y_{i,j}$ except for $Y_{1,1} = 0$. Therefore, the condition (26) for robust PFIRC synthesis is at least no more conservative than (29) *with diagonal* $\mathcal{G}_m$.

On the other hand, as previously pointed out, the achievement of robust stabilization by PTVMCs depends on the timing of the reset and the best choice cannot be determine *a priori*. This might be problematic if the period becomes large. In the following, we proved that it is not the case anymore with the synthesis condition (26) leading to a PFIRC.

Let us introduce the first Proposition 3.3 where, in relation to (9), we define

$$\widetilde{\mathfrak{E}} = \begin{bmatrix} \mathbf{1} & -A_{0,0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix},\quad \widetilde{\mathfrak{A}} = \begin{bmatrix} A_{0,1} & \mathbf{0} \\ A_{1,0} & A_{1,1} \end{bmatrix}. \quad (31)$$

Proposition 3.3: For the two periodic system (6), suppose there exists $X \in \mathbb{S}_+^{2n}$, G_0, G_1 such that

$$\mathcal{X}(X) + \mathrm{He}\left\{\begin{bmatrix} \mathfrak{A} \\ -\mathfrak{E} \end{bmatrix}\begin{bmatrix} 0 & \mathcal{G} \end{bmatrix}\right\} \prec 0,\ \mathcal{G} := \begin{bmatrix} G_0 & 0 \\ 0 & G_1 \end{bmatrix}. \quad (32)$$

Then, there exists $\widetilde{X} \in \mathbb{S}_+^{2n}$ such that

$$\mathcal{X}(\widetilde{X}) + \mathrm{He}\left\{\begin{bmatrix} \widetilde{\mathfrak{A}} \\ -\widetilde{\mathfrak{E}} \end{bmatrix}\begin{bmatrix} 0 & \widetilde{\mathcal{G}} \end{bmatrix}\right\} \prec 0,\ \widetilde{\mathcal{G}} = \begin{bmatrix} G_1 & 0 \\ 0 & G_0 \end{bmatrix}. \quad (33)$$

Proof: The condition (32) can be written as

$$\begin{bmatrix} -X & 0 \\ 0 & X \end{bmatrix} + \begin{bmatrix} 0 & 0 & A_{11}G_0 & 0 \\ * & 0 & A_{00}G_0 & A_{01}G_1 \\ * & * & -G_0 - G_0^T & A_{10}G_1 \\ * & * & * & -G_1 - G_1^T \end{bmatrix} \prec 0.$$

Then, from Lemma 1 in [10], the above condition holds if and only if there exists $\widetilde{X} \in \mathbb{S}^{2n}$ such that

$$\begin{bmatrix} -X & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \widetilde{X} \end{bmatrix} + \begin{bmatrix} 0 & 0 & A_{11}G_0 \\ * & 0 & A_{00}G_0 \\ * & * & -G_0 - G_0^T \end{bmatrix} \prec 0,$$

$$\begin{bmatrix} 0 & 0 \\ 0 & X \end{bmatrix} + \begin{bmatrix} -\widetilde{X} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & A_{01}G_1 \\ * & 0 & A_{10}G_1 \\ * & * & -G_1 - G_1^T \end{bmatrix} \prec 0.$$

Noticing that the second inequality proves that $\widetilde{X} \succ 0$ and eliminating X from these two inequalities by Lemma 1 in [10], we obtain (33). This completes the proof. ■

This proposition implies that, if the stability of the monodromy matrix Ψ is ensured by G_0 and G_1, then, the stability of $\widetilde{\Psi}$, defined by changing the starting time instance, can be proved by the same matrices. This property holds even for robustly stabilizing PFIRC synthesis. Therefore, the possibility of robust stabilization is independent of the choice of the starting time instance for defining the monodromy matrix.

IV. Extension to the N-periodic case

To extend the previous results to general N-periodic case, let us consider the following N-periodic system

$$x_{q+1} = A_q x_q + B_q u_q,\ A_{q+N} = A_q,\ B_{q+N} = B_q. \quad (34)$$

For this plant, we design a PFIRC of period N of the form

$$\begin{aligned} &u_q = K_{q,0}x_q + \cdots + K_{q,N-1}x_{q-N+1}, \\ &K_{q+N,j} = K_{q,j}\ (j = 0, \cdots, N-1). \end{aligned} \quad (35)$$

The closed-loop system can be written as

$$x_{q+1} = A_{q,0}x_q + \cdots + A_{q,N-1}x_{q-N+1} \quad (36)$$

with $A_{q,0} = A_q + B_q K_{q,0}$ and $A_{q,j} = B_q K_{q,j}$ $\quad (j \neq 0)$. In this case as well, (36) can also be regarded as the time-invariant lifted system (8) with

$$\xi_k = \begin{bmatrix} x_{kN+N}^T & \cdots & x_{kN+1}^T \end{bmatrix}^T \quad (37)$$

$$\mathfrak{E} = -\begin{bmatrix} -\mathbf{1} & A_{N-1,0} & \cdots & \cdots & A_{N-1,N-2} \\ \mathbf{0} & -\mathbf{1} & A_{N-2,0} & \cdots & A_{N-2,N-3} \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & -\mathbf{1} & A_{1,0} \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & -\mathbf{1} \end{bmatrix} \quad (38)$$

$$\mathfrak{A} = \begin{bmatrix} A_{N-1,N-1} & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ A_{N-2,N-2} & A_{N-2,N-1} & \mathbf{0} & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \mathbf{0} \\ A_{0,0} & \cdots & \cdots & \cdots & A_{0,N-1} \end{bmatrix} \quad (39)$$

As far as stability analysis is concerned, Th 2.2 and its robust extension Th 2.3 remain valid for this more general definition of $\mathfrak{E}$ and $\mathfrak{A}$. Correspondingly, Th 2.4 and Th 2.5 still hold for stabilizing PFIRCs synthesis with the following definition of the matrices

$$\mathcal{A} = \left[\begin{array}{cccc} \multicolumn{4}{c}{\mathbf{0}_{n(N-1)\times nN}} \\ \hline A_0 & \mathbf{0} & \cdots & \mathbf{0} \\ -\mathbf{1} & A_{N-1} & \ddots & \vdots \\ \mathbf{0} & \ddots & \ddots & \mathbf{0} \\ \vdots & \ddots & \ddots & A_1 \\ \mathbf{0} & \cdots & \mathbf{0} & -\mathbf{1} \end{array} \right] \tag{40}$$

$$\mathcal{K} = \begin{bmatrix} K_{N-1,N-1} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \mathbf{0} \\ K_{0,0} & \cdots & \cdots & K_{0,N-1} \\ \mathbf{0} & K_{N-1,0} & \cdots & K_{N-1,N-2} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & K_{1,0} \end{bmatrix} \tag{41}$$

$$\mathcal{Y} = \begin{bmatrix} Y_{N-1,N-1} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \mathbf{0} \\ Y_{0,0} & \cdots & \cdots & Y_{0,N-1} \\ \mathbf{0} & Y_{N-1,0} & \cdots & Y_{N-1,N-2} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & Y_{1,0} \end{bmatrix} \tag{42}$$

$$\mathcal{G} = \text{diag}\left\{G_0, \cdots, G_{N-1}\right\} \tag{43}$$

$$\mathcal{B} = \text{diag}\left\{B_{N-1}, \cdots, B_0, B_{N-1}, \cdots, \begin{bmatrix} B_1 \\ \mathbf{0}_{n\times m} \end{bmatrix}\right\} \tag{44}$$

In the general N-periodic case, inclusion relationships among synthesis theorems established in the preceding section still hold: if Th 3.1 or Th 3.2 with $\mathcal{G}_m$ restricted to be block-diagonal holds then Th 2.5 holds. Proofs are omitted here for conciseness.

Remark: It is worth noticing that, in the proposed PFIRC design method, the size of the Lyapunov matrix P is equal to Nn and hence tends to increase with N. This is in stark contrast with the PSFC and PTVMC methods, where the size of the Lyapunov matrix remains to be n. We believe that the freedom introduced by the enlarged size of Lyapunov matrix allows us to enhance robustness, at the expense of increased computational burden.

Numerical example: To illustrate the previous results, we use the 3-periodic problem of Example 1 of [9] as a benchmark. The goal is to maximize a properly defined stability margin, i.e. the allowable maximal absolute value of an uncertain parameter.

The 3-periodic controllers are designed by using the new Th 2.5, leading to PFIRCs, and Th 3.2, leading to PTVMCs, with $\mathcal{G}$ diagonal and then upper triangular. Robustness of every resulting closed-loop system is evaluated by means of the new Th 2.3 since it encompasses all previous results. We also designed a PSFC and a PTVMC by means of Th 2.5, which is obviously possible by appropriately restricting the structure of $\mathcal{Y}$. Results are gathered in Table I.

controller	Th. 3.2 with diagonal $\mathcal{G}$	Th. 3.2 with upper triangular $\mathcal{G}$	Th. 2.5
PSFC	0.5571	-	0.5571
PTVMC	0.7715	0.9275	0.7715
PFIRC	-	-	0.8111

TABLE I

CLOSED-LOOP STABILITY MARGIN FOR PERIODIC MODEL

PTVMC	$K_{0,0}$	3.1624	-2.4025	$K_{1,0}$	1.4338	-2.0730
	$K_{1,1}$	0.2105	-0.1657	$K_{2,0}$	-2.3287	-2.5145
	$K_{2,1}$	0.1892	0.0180	$K_{2,2}$	0.0790	0.1183
PFIRC	$K_{0,0}$	3.1597	-2.5285	$K_{0,1}$	-0.3366	0.0449
	$K_{0,2}$	0.1687	0.0028	$K_{1,0}$	1.1496	-2.1476
	$K_{1,1}$	-0.3840	0.1333	$K_{1,2}$	-0.2062	-0.0776
	$K_{2,0}$	-2.8963	-2.5987	$K_{2,1}$	0.1560	-0.0417
	$K_{2,2}$	-0.1091	0.0992			

TABLE II

GAINS OF THE DESIGNED PTVMC AND PFIRC FOR THE 3-PERIODIC MODEL

Note that the first two elements of the first and last columns are equal. Indeed, we have proved that the new Th 2.5 encompasses Th 3.2 *for a diagonal* $\mathcal{G}$. These results are equivalent when applied to the same control structure. Furthermore, designing PFIRC, which can be only done by means of Th 2.5, improves the results as it can be seen on the last line of the last column. Finally, this table brings to light how crucial the choice of $\mathcal{G}$ is. Indeed, PTVMC designed by means of Th 3.2 with $\mathcal{G}$ upper triangular leads to a better result than PFIRCs designed by means of Th 2.5 (where $\mathcal{G}$ is diagonal). Designed feedback gains of the controllers are given by Table II.

As a conclusion, designing PFIRCs is promising considering the additional number of degrees-of-freedom and the lack of dependency on the starting point. However, robustness obtained by means of this controller does not overcome results of PTVMC designed with an upper triangular $\mathcal{G}$. Therefore, reduction of conservatism in PTVMCs synthesis, obtained by relaxing $\mathcal{G}$, paves the way for further works in PFIRCs synthesis.

V. SYNTHESIS FOR LTI SYSTEM

In this section, we clarify that the suggested PFIRC structure and the associated LMI-based design method are promising when dealing with LTI systems as well. As nominal system stabilization can always be achieved by means of PSFCs, it is of course meaningless to complexify the control law in this case. However, a complete definite solution for robust stabilization problems of polytopic uncertain systems

N	[8]	Th. 3.2 with upper triangular $\mathcal{G}$	Th. 2.5	Th. 5.1
1	0.9228	0.9228	0.9228	0.9228
2	-	1.0272	0.9842	0.9842
3	-	1.1587	1.1276	1.1276

TABLE III

CLOSED-LOOP STABILITY MARGIN FOR LTI MODEL

[8] is not currently available and PFIRCs give a relatively simple way, over the existing methods, to compute dynamic state-feedback controller robust to parametric uncertainties.

Let us consider the polytopic-type uncertain LTI system

$$x_{q+1} = Ax_q + Bu_q \tag{45}$$

where

$$\begin{bmatrix} A & B \end{bmatrix} \in \text{co}\left\{ \begin{bmatrix} A^{[1]} & B^{[1]} \end{bmatrix}, \cdots, \begin{bmatrix} A^{[L]} & B^{[L]} \end{bmatrix} \right\}. \tag{46}$$

By regarding it as an N-periodic (i.e, $A_q = A, B_q = B (q = 0, \cdots, N-1)$ in (34)), a robust stabilizing PFIRC (35) can be designed by means of the Th 2.5. In this case, N is a tuning parameter.

From numerical examples shown below, we can confirm that it is possible to reduce the conservatism of [8] by designing PFIRCs. Its complicated structure is nevertheless the price to pay for this improvement. However, by constraining (35) to be time-invariant, Th 2.5 gives a sufficient condition to design the following *time-invariant* FIR controller:

$$u_q = K_0 x_q + \cdots + K_N x_{q-N} \tag{47}$$

Theorem 5.1: Any uncertain time-invariant model (45) with (46) can be stabilized by the FIR state-feedback controller (47) *if* LMI (26) in Th 2.5 with $A_i^{[l]} = A^{[l]}$, $B_i^{[l]} = B^{[l]}$ $(l = 1, \cdots, L)$, $K_{j,i} = K_i$, $G_i = G$ and $Y_{j,i} = Y_i$ for $(i = 0, \cdots, N-1)$ is feasible. The robustly stabilizing feedback gains of the controller is computed by $K_i = Y_i G^{-1}$.

Remark: As clearly illustrated by numerical examples below, we can prove that the LMI condition of Th 5.1 encompasses the extended-LMI condition proposed in [8]. This is validated easily by following a similar idea to the one used to get (25).

Numerical example: To illustrate the effectiveness of the suggested design methods, we first solve the robust state-feedback stabilization problem for polytopic-type uncertain LTI systems discussed in Section 4 of [8]. By regarding the system as N-periodic, we maximize the closed-loop stability margin γ_N. Results are evaluated by means of Th 2.3 and gathered in Table III.

As expected, when N increases, the controllers tend to improve the stability margin obtained in [8]. Moreover, for this example, hierarchy between synthesis theorems seems to be [8] < Th. 5.1 = Th. 2.5 < Th. 3.2. Once again, it proves that a trade-off has to be made between the closed-loop robustness and the complexity of the controller. Furthermore, it highlights the importance of the structure imposed to the design matrix $\mathcal{G}$. Finally, these results seem to show that Th. 5.1 gives the same results as Th. 2.5 in the case of time-invariant system. Further works will investigate this point.

VI. CONCLUSIONS AND FUTURE WORKS

In this paper, we proposed an LMI-based design method of Periodically time-varying Finite Impulse Response state-feedback Controllers (PFIRC). Compared to existing controllers of this framework, this new control law offers more degrees-of-freedom which contribute to reduce the conservatism of the obtained stability condition. Moreover, for a given system, the best PFIRC is obtained by solving a single-shot LMI condition. As a special case, an extended sufficient condition to design time-invariant FIR state-feedback controllers has been provided as well.

Even though we have successfully established a basic strategy for robust PFIRC synthesis, some issues remain under investigation to improve the proposed design theorem. Among them, a challenging topic is to relax somehow the block-diagonal restriction on $\mathcal{G}$, composed of additional variables, without breaking convexity of the synthesis condition.

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