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On Decentralized Connectivity Maintenance for Mobile Robotic Systems

Lorenzo Sabattini, Nikhil Chopra and Cristian Secchi

***Abstract*— To accomplish cooperative tasks, robotic systems are often required to communicate with each other. Thus, maintaining connectivity of the interagent communication graph is a fundamental issue in the field of multi–robot systems. In this paper we present a completely decentralized control strategy for global connectivity maintenance of the interagent communication graph. We describe a gradient–based control strategy that exploits decentralized estimation of the algebraic connectivity. The proposed control algorithm guarantees the global connectivity of the communication graph without requiring maintenance of the local connectivity between the robotic systems. The control strategy is validated by means of an analytical proof and simulative results.**

I. INTRODUCTION

In this paper we study decentralized control strategies for guaranteeing connectivity maintenance in multi–robot systems.

In the literature, several approaches to connectivity maintenance have been proposed. These approaches can be divided into two categories: approaches to maintain the local connectivity, and approaches to maintain the global connectivity.

Maintaining the local connectivity entails designing a controller that ensures that, if a communication link is active at time $t = 0$, it will be active $\forall t \geq 0$. Examples of decentralized algorithms for local connectivity maintenance can be found in [1], [2], [3], and [4]. The main advantage of these control algorithms is that the maintenance of the connectivity is formally proven. Nevertheless, imposing the maintenance of each single communication link is often too restrictive. In fact, to ensure that information exchange among all the robots is possible, it is necessary to guarantee only the *global* connectivity of the communication graph. Loosely speaking, it is acceptable that a few links are broken, as long as the overall graph is still connected: if necessary, redundant links can be removed, and new ones can be introduced. As shown in [5], a measure of the connectivity of a graph is the value of the second–smallest eigenvalue of the Laplacian matrix of the graph.

In [6] a gradient based control strategy was proposed to guarantee that the second–smallest eigenvalue of the Laplacian matrix is greater than zero. The main drawback of this control strategy is the fact that the eigenvalue was computed in a centralized way. To overcome this problem, decentralized estimation algorithms have recently been introduced [7], [8]. In particular, strategy described in [8] introduced a decentralized estimation procedure, that allowed each agent to obtain an estimate of the second–smallest eigenvalue of the Laplacian matrix and of its gradient. This estimates were then used in a gradient based control strategy that aimed at increasing the value of the second–smallest eigenvalue. However, as we discuss in Section V, it can be demonstrated via simulations that, in presence of certain (bounded) external control laws, the control strategy described in [8] may not guarantee the connectivity of the communication graph.

Motivated by the above discussion, in this paper we propose a decentralized control strategy to provably guarantee maintenance of the global connectivity, that relies on a decentralized estimation procedure of second–smallest eigenvalue of the Laplacian matrix. Specifically, the contribution of the paper is the following:

1) We develop a new estimation algorithm, inspired by [8], and we demonstrate the boundedness of the estimation error.
2) We analytically prove that, since the estimation error is bounded, our control strategy guarantees the maintenance of the connectivity of the communication graph.

The outline of the paper is as follows. A preliminary control strategy is introduced in Section III for a simplified case, i.e. we provide a solution to the connectivity maintenance problem in absence of estimation errors. The presence of estimation errors is explicitly taken into account in Section IV. The effectiveness of the control strategy is validated by simulations in Section V and the results are summarized in Section VI.

II. BACKGROUND ON GRAPH THEORY

In this section we summarize some of the main notions on graph theory used in the paper. Further details can be found for instance in [9]. Given N mobile robots, we describe the communication architecture among them as an undirected graph. Each robot corresponds to a node of the graph, and each link between two robots corresponds to an edge of the graph. Let $\mathcal{N}_i$ be the neighborhood of the i–th robot, i.e. the set of robots that can exchange information with the i–th one. The communication graph can be described by means of the adjacency matrix $A \in \mathbb{R}^{N\times N}$. Each element a_{ij} is defined as the weight of the edge between the i–th and the j–th robot, and is a positive number if $j \in \mathcal{N}_i$, zero otherwise. Since we are considering undirected graphs, we assume $a_{ij} = a_{ji}$. The degree matrix of the graph is defined as $D = \text{diag}\left(\{d_i\}\right)$, where d_i is the degree of the i–th node of the graph, i.e.

L. Sabattini is with the Department of Electronics, Computer Sciences and Systems (DEIS), University of Bologna, Italy. lorenzo.sabattini2@unibo.it

N. Chopra is with the Department of Mechanical Engineering and Institute for Systems Research, University of Maryland, College Park, MD, USA nchopra@umd.edu

C. Secchi is with the Department of Sciences and Methods of Engineering (DISMI), University of Modena and Reggio Emilia, Italy cristian.secchi@unimore.it

$d_i = \sum_{j=1}^{N} a_{ij}$. The (weighted) Laplacian matrix of the graph is defined as $L = D - A$. The unweighted Laplacian matrix, L_*, is defined as a special case of Laplacian matrix, where all non–zero entries of the adjacency matrix are equal to one. The Laplacian matrix exhibits some remarkable properties:

1) Let $\mathbf{1}$ be the column vector of all ones. Then, $L\mathbf{1} = \mathbf{0}$.
2) Let $\lambda_i, i = 1, \ldots, N$ be the eigenvalues of the Laplacian matrix.
   - The eigenvalues can be ordered such that

$$0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N \tag{1}$$

   - $\lambda_2 > 0$ if and only if the graph is connected. Then, λ_2 is defined as the algebraic connectivity of the graph.

III. Connectivity maintenance

In this section we will introduce our connectivity maintenance algorithm in a simplified framework: we suppose that each agent can compute the actual value of the algebraic connectivity of the communication graph. This assumption will be removed in the next section, where a decentralized estimation procedure will be introduced.

In this paper we consider a group of N single–integrator agents, i.e.:

$$\dot{p}_i = u_i^c \tag{2}$$

where $p_i \in \mathbb{R}^m$ is the position of the i–th agent, and u_i^c is the control input. Let $p = \left[p_1^T \ldots p_N^T\right]^T \in \mathbb{R}^{Nm}$ be the state vector of the multi–agent system.

The following connectivity maintenance control strategy was first introduced in [8]:

$$u_i^c = \frac{\partial \lambda_2}{\partial p_i} \tag{3}$$

We will demonstrate using simulations (Section V) that in the presence of certain (bounded) external control laws, the above control strategy may not be able to guarantee the connectivity maintenance. Thus, in order to guarantee the connectivity maintenance under arbitrary initial conditions and bounded external control laws, we define a control action whose magnitude increases, as the algebraic connectivity of the graph deteriorates. Hence, we modify the control law in Eq. (3) by adding a multiplicative coefficient $K(p)$:

$$u_i^c = K(p) \frac{\partial \lambda_2}{\partial p_i} \tag{4}$$

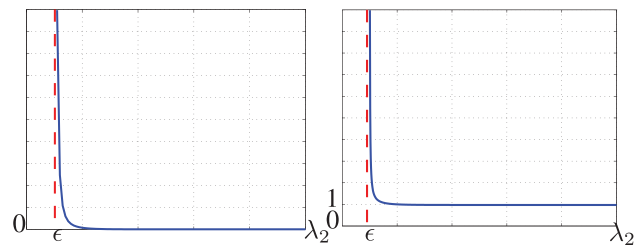
The function $K(p)$ is defined as follows (Fig. 1(a)):

$$K(p) = \operatorname{csch}^2(\lambda_2 - \epsilon) \tag{5}$$

where ϵ is the desired lower–bound for the value of λ_2. The magnitude of this multiplicative coefficient (see Fig. 1(a)) increases suddenly as λ_2 decreases: we will show in the simulations described in Section V that this property is fundamental for guaranteeing connectivity maintenance in presence of external control laws. In the sequel, we also demonstrate that a correct choice of the lower–bound ϵ guarantees connectivity maintenance when dealing with estimation errors and external control laws as well.

From Eqs. (4), (5), the control law can be rewritten as follows:

$$u_i^c = \operatorname{csch}^2(\lambda_2 - \epsilon) \frac{\partial \lambda_2}{\partial p_i} \tag{6}$$



(a) $K(p) = \operatorname{csch}^2(\lambda_2 - \epsilon)$ (b) $V(p) = \coth(\lambda_2 - \epsilon)$

Fig. 1. Functions $K(p)$ and $V(p)$, with respect to λ_2

Let R be the maximum communication range for each agent, i.e. the j–th agent is inside $\mathcal{N}_i$ if $\|p_i - p_j\| \leq R$. We define the edge–weights for the inter–agent communication graph as in [8]:

$$a_{ij} = \begin{cases} e^{-\left(\|p_i - p_j\|^2\right)/\left(2\sigma^2\right)} & \text{if } \|p_i - p_j\| \leq R \\ 0 & otherwise \end{cases} \tag{7}$$

The scalar parameter σ is chosen to satisfy the threshold condition $e^{-\left(R^2\right)/\left(2\sigma^2\right)} = \Delta$, where Δ is a small predefined threshold.

Let v_2 be the eigenvector corresponding to the eigenvalue λ_2. Given the definition of the edge–weights in Eq. (7), the value of $\frac{\partial \lambda_2}{\partial p_i}$ can be computed as [8]:

$$\frac{\partial \lambda_2}{\partial p_i} = \sum_{j \in \mathcal{N}_i} -a_{ij} \left(v_2^i - v_2^j\right)^2 \frac{p_i - p_j}{\sigma^2} \tag{8}$$

where v_2^i and v_2^j are the i–th and the j–th components of v_2, respectively.

Thus, the control law in Eq. (6) can be rewritten as follows:

$$u_i^c = -\operatorname{csch}^2(\lambda_2 - \epsilon) \sum_{j \in \mathcal{N}_i} a_{ij} \left(v_2^i - v_2^j\right)^2 \frac{p_i - p_j}{\sigma^2} \tag{9}$$

Inspired by [1], we define the following non–negative energy function:

$$V(p) = \coth(\lambda_2 - \epsilon) \tag{10}$$

The energy function (Fig. 1(b)) is non–increasing (with respect to λ_2) and non–negative, for any $\lambda_2 > \epsilon$.

From Eqs. (2) (10), (8), (6) it follows that:

$$\frac{\partial V}{\partial p_i} = \frac{\partial V}{\partial \lambda_2} \frac{\partial \lambda_2}{\partial p_i} = -\operatorname{csch}^2(\lambda_2 - \epsilon) \frac{\partial \lambda_2}{\partial p_i} = -u_i^c = -\dot{p}_i \tag{11}$$

Let $\mathcal{D}_\epsilon$ be a set where the communication graph is connected, above a desired connectivity threshold ϵ, i.e.:

$$\mathcal{D}_\epsilon = \left\{p \in \mathbb{R}^{Nm} \text{ s. t. } \lambda_2 > \epsilon\right\} \tag{12}$$

Proposition 1 *Consider the dynamical system described by Eq.* (2)*. Given an initial configuration* $p_0 \in \mathcal{D}_\epsilon$*, for some* $\epsilon > 0$*, then, if the system is driven by the control law in Eq.* (9)*, the energy function defined in Eq.* (10) *does not increase.*

Proof: To prove the statement, we compute the time derivative of the energy function. From Eq. (11):

$$\dot{V}(p) = \nabla_p V(p)^T \cdot \dot{p} = \sum_{i=1}^{N} \frac{\partial V}{\partial p_i}^T \cdot \dot{p}_i = -\sum_{i=1}^{N} \dot{p}_i^T \dot{p}_i \leq 0 \tag{13}$$

Thus, the energy function does not increase over time. ■

Hence, Proposition 1 guarantees that $V(p)$ does not increase over time. Consequently, if the initial condition for the dynamical system ensures that $\lambda_2 > \epsilon$, the value of λ_2 will never decreases, and the connectivity of the graph is always maintained.

IV. Connectivity maintenance in presence of estimation errors

In this section the main results of the paper are presented. A decentralized estimation procedure is introduced, that allows each agent to compute its own estimate of the algebraic connectivity of the communication graph. We will demonstrate the boundedness of the estimation errors, and provide analytical proofs of the connectivity maintenance.

A. Estimation of the algebraic connectivity of the graph

In this section, for the sake of clarity we describe the estimation procedure introduced in [8], that allows each agent to compute its own estimate of the algebraic connectivity of the communication graph. Specifically, the estimate of λ_2 is computed by exploiting the estimate of the corresponding eigenvector v_2. The power iteration procedure described in [10] is utilized to design the following update law:

$$\begin{aligned}&\dot{\tilde{v}}_2 = \\ &-k_1 \text{Ave}\left(\left\{\tilde{v}_2^i\right\}\right)\mathbf{1} - k_2 L \tilde{v}_2 - k_3\left(\text{Ave}\left(\left\{\left(\tilde{v}_2^i\right)^2\right\}\right) - 1\right)\tilde{v}_2\end{aligned} \tag{14}$$

where $k_1, k_2, k_3 > 0$ are the control gains, and $\text{Ave}(\cdot)$ is the averaging operation. Furthermore, $\tilde{v}_2^i$ is defined as the i–th agent's estimate of v_2^i, the i–th component of the eigenvector v_2, and $\tilde{v}_2 = \left[\tilde{v}_2^1, \ldots, \tilde{v}_2^N\right]^T$. Further details can be found in [8].

To implement the update law in Eq. (14) in a decentralized way, the averaging operation is implemented by means of the PI average consensus estimator described in [11]:

$$\begin{aligned}\dot{z}^i &= \gamma\left(\alpha^i - z^i\right) - K_p \sum_{j\in\mathcal{N}_i}\left(z^i - z^j\right) + K_i \sum_{j\in\mathcal{N}_i}\left(w^i - w^j\right) \\ \dot{w}^i &= -K_i \sum_{j\in\mathcal{N}_i}\left(z^i - z^j\right)\end{aligned} \tag{15}$$

Further details can be found in [11].

Since there are two averaging operations in the update law in Eq. (14), two PI consensus estimators must be run:

- the first one, with input $\alpha^{i,1} = \tilde{v}_2^i$, provides z_1^i as the i–th agent's estimate of $\text{Ave}\left(\left\{\tilde{v}_2^i\right\}\right)$;
- the second one, with input $\alpha^{i,2} = \left(\tilde{v}_2^i\right)^2$, provides z_2^i as the i–th agent's estimate of $\text{Ave}\left(\left\{\left(\tilde{v}_2^i\right)^2\right\}\right)$.

Thus, each agent can run the decentralized version of the update law in Eq. (14):

$$\dot{\tilde{v}}_2^i = -k_1 z_1^i - k_2 \sum_{j\in\mathcal{N}_i} a_{ij}\left(\tilde{v}_2^i - \tilde{v}_2^j\right) - k_3\left(z_2^i - 1\right)v_2^i \tag{16}$$

As demonstrated in [8], the i–th agent can compute its estimate of λ_2, namely λ_2^i, as follows:

$$\lambda_2^i = \frac{k_3}{k_2}\left(1 - z_2^i\right) \tag{17}$$

B. Estimates of λ_2

Exploiting the estimation procedure introduced in Section IV-A, each agent computes an estimate of a component of the eigenvector v_2, namely $\tilde{v}_2^i$. Let $\tilde{v}_2 = \left[\tilde{v}_2^1 \ldots \tilde{v}_2^N\right]^T$, and let $\tilde{\lambda}_2$ be the value that the second smallest eigenvalue of the Laplacian matrix would take if $\tilde{v}_2$ were the corresponding eigenvector. As proved in [8], $\tilde{\lambda}_2$ can be computed as follows:

$$\tilde{\lambda}_2 = \frac{k_3}{k_2}\left[1 - \text{Ave}\left(\left\{\left(\tilde{v}_2^i\right)\right\}\right)\right] \tag{18}$$

As shown in [8], $\frac{\partial \tilde{\lambda}_2}{\partial p_i}$ can be computed as follows

$$\frac{\partial \tilde{\lambda}_2}{\partial p_i} = \tilde{v}_2^T \frac{\partial L}{\partial p_i}\tilde{v}_2 = \sum_{j\in\mathcal{N}_i} \frac{\partial a_{ij}}{\partial p_i}\left(\tilde{v}_2^i - \tilde{v}_2^j\right)^2 \tag{19}$$

Then, from the definition of the edge–weights a_{ij} given in Eq. (7):

$$\frac{\partial \tilde{\lambda}_2}{\partial p_i} = \sum_{j\in\mathcal{N}_i} -a_{ij}\left(\tilde{v}_2^i - \tilde{v}_2^j\right)^2 \frac{p_i - p_j}{\sigma^2} \tag{20}$$

Further details can be found in in [8].

The actual value of $\tilde{\lambda}_2$ can not be computed by each agent. In fact, the real value of $\text{Ave}\left(\left\{\left(\tilde{v}_2^i\right)\right\}\right)$ is not available. Nevertheless, an estimate of this average, namely z_2^i, is available to each agent. According to Eq. (17), each agent can compute λ_2^i, that is indeed different from both λ_2 and $\tilde{\lambda}_2$.

We will show in the Section IV-D that λ_2^i is a good estimate of both λ_2 and $\tilde{\lambda}_2$. More specifically, we will show that $\exists \Xi, \Xi' > 0$ such that

$$\begin{aligned}\left|\lambda_2 - \lambda_2^i\right| &\leq \Xi \quad \forall i = 1, \ldots, N \\ \left|\tilde{\lambda}_2 - \lambda_2^i\right| &\leq \Xi' \quad \forall i = 1, \ldots, N\end{aligned} \tag{21}$$

From Eq. (21), we can conclude that

$$\left|\lambda_2 - \tilde{\lambda}_2\right| \leq \Xi + \Xi' \tag{22}$$

C. Connectivity maintenance

Consider the control law introduced in Eq. (4). Since the real values of λ_2 and $\frac{\partial \lambda_2}{\partial p_i}$ are not available, the agents will actually implement the following control law:

$$u_i^c = \operatorname{csch}^2\left(\lambda_2^i - \tilde{\epsilon}\right)\frac{\partial \tilde{\lambda}_2}{\partial p_i} \tag{23}$$

where $\tilde{\epsilon} = \epsilon + 2\Xi + \Xi'$.

We now introduce the following energy function

$$\tilde{V}(p) = \coth\left(\tilde{\lambda}_2 - \tilde{\epsilon}\right) \tag{24}$$

The following proposition provides the main result of the paper.

Proposition 2 *Consider the dynamical system described by Eqs.* (2), (23). *Let Ξ, Ξ' be defined according to Eq.* (21). *$\exists \epsilon, \tilde{\epsilon} \in \mathbb{R}$, such that, if the initial value of $\lambda_2 > \tilde{\epsilon} + \Xi + \Xi'$, then the control law defined in Eq.* (23) *ensures that the value of λ_2 never goes below ϵ.*

Proof: To prove the statement, we compute the time derivative of the energy function introduced in Eq. (24).

From Eq. (24) it follows that:

$$\frac{\partial \tilde{V}}{\partial p_i} = \frac{\partial \tilde{V}}{\partial \tilde{\lambda}_2}\frac{\partial \tilde{\lambda}_2}{\partial p_i} = -\operatorname{csch}^2\left(\tilde{\lambda}_2 - \tilde{\epsilon}\right)\frac{\partial \tilde{\lambda}_2}{\partial p_i} \tag{25}$$

From Eqs. (2), (23), (25), the time derivative of $\tilde{V}(p)$ can be computed as follows:

$$\begin{aligned}\dot{\tilde{V}}(p) &= \nabla_p \tilde{V}(p)^T \dot{p} = \sum_{i=1}^{N} \frac{\partial \tilde{V}}{\partial p_i}^T \dot{p}_i = \\ &= -\sum_{i=1}^{N} \operatorname{csch}^2\left(\tilde{\lambda}_2 - \tilde{\epsilon}\right)\operatorname{csch}^2\left(\lambda_2^i - \tilde{\epsilon}\right)\left\|\frac{\partial \tilde{\lambda}_2}{\partial p_i}\right\|^2 \leq 0\end{aligned} \tag{26}$$

Thus, the energy function does not increase over time. According to Eq. (22), the fact that the initial value of λ_2 is greater than $\tilde{\epsilon} + \Xi + \Xi'$ ensures that the initial value of $\tilde{\lambda}_2$ is greater than $\tilde{\epsilon}$. Hence, we can conclude that the value of $\tilde{\lambda}_2$ does not decrease over time. Then, $\tilde{\lambda}_2 \geq \tilde{\epsilon}$.

Hence, according to Eq. (22), we conclude that $\lambda_2 \geq \epsilon = \tilde{\epsilon} - 2\Xi - \Xi'$.

■

D. Boundedness of the estimation errors

In order to prove the boundedness of the estimation error of λ_2, we will first show the boundedness of the estimation system's state.

For this purposes, we slightly modify the decentralized update law in Eq. (16) as follows:

$$\begin{aligned}\dot{\tilde{v}}_2^i = \; & -k_1 z_1^i - k_2 \sum_{j \in \mathcal{N}_i} a_{ij}\left(\tilde{v}_2^i - \tilde{v}_2^j\right) \\ & -k_3\left(z_2^i - 1\right)\tilde{v}_2^i - k_4\left|\tilde{v}_2^i\right|\tilde{v}_2^i\end{aligned} \tag{27}$$

for some value $k_4 > 0$. The introduction of this additional term worsens the estimation (with respect to the original update law introduced in [8]), but is necessary to guarantee the connectivity maintenance, which is the goal of the control strategy presented in this paper.

Let $\chi = \left[\tilde{v}_2^T z_1{}^T w_1{}^T z_2{}^T w_2{}^T\right]^T$ be the state vector of the estimation system.Thus, the estimation dynamics can be represented as the feedback interconnection of a linear dynamic system Σ with a memoryless nonlinearity $\psi(\cdot)$. More specifically, the linear dynamic system Σ is defined as follows:

$$\Sigma : \begin{cases} \dot{\chi}(t) & = \Lambda\chi(t) + B\nu(t) \\ y(t) & = C\chi(t) \end{cases} \tag{28}$$

where

$$\begin{aligned}&\Lambda = \\ &\begin{bmatrix} -k_2 L & -k_1 I_N & 0_N & 0_N & 0_N \\ \gamma I_N & -\gamma I_N - K_p L_* & K_i L_* & 0_N & 0_N \\ 0_N & -KiL_* & 0_N & 0_N & 0_N \\ 0_N & 0_N & 0_N & -\gamma I_N - K_p L_* & K_i L_* \\ 0_N & 0_N & 0_N & -KiL_* & 0_N \end{bmatrix} \\ &B = \begin{bmatrix} I_N & 0_N \\ 0_N & 0_N \\ 0_N & 0_N \\ 0_N & I_N \\ 0_N & 0_N \end{bmatrix} \quad C = \begin{bmatrix} I_N & 0_N & 0_N & 0_N & 0_N \\ 0_N & 0_N & 0_N & I_N & 0_N \end{bmatrix}\end{aligned} \tag{29}$$

where I_N is the identity matrix of size N, and 0_N is the zero matrix of size N. The input ν is defined as $\nu(t) = -\psi(y(t))$, where $\psi(\cdot)$ will be defined later on.

From the definition of the matrix C in Eq. (29), it follows that

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \tilde{v}_2 \\ z_2 \end{bmatrix} \tag{30}$$

Given a vector $\xi \in \mathbb{R}^N$, let $\operatorname{diag}(\xi)$ be the diagonal matrix whose diagonal elements are the entries of the vector ξ. Let $\xi_s \in \mathbb{R}^N$ be a vector whose entries are the square of the corresponding entries of ξ, namely $\xi_s = \left\{(\xi_i)^2\right\}$. It is easy to prove that $\xi_s = \operatorname{diag}(\xi)\,\xi = \xi^T \operatorname{diag}(\xi)$.

The memoryless nonlinearity $\psi(\cdot)$ is then defined as follows:

$$\psi(y) = \begin{bmatrix} k_3\left(\operatorname{diag}(y_2) - I_N\right)y_1 + k_4 \operatorname{diag}\left(\left\{\left|y_1^i\right|\right\}\right)y_1 \\ -\gamma \operatorname{diag}(y_1)\, y_1 \end{bmatrix} \tag{31}$$

The following proposition proves the boundedness of the estimation system's state.

Proposition 3 *Consider the dynamics of the estimation system, described by Eqs.* (28), (31). *Given any initial condition $\chi(0)$, the norm of the state vector of the estimation system, $\|\chi(t)\|$, is bounded.*

Proof: We will first prove that $\exists S > 0$ such that, if $\|\tilde{v}_2\| \geq S$, then $\|\chi\|$ does not increase over time.

Let

$$W(\chi) = \frac{1}{2}\chi^T \chi \geq 0 \tag{32}$$

where, for the sake of simplicity, we have dropped the dependence on time. We can compute the time derivative of this function as follows:

$$\dot{W}(\chi) = \chi^T \dot{\chi} = \chi^T\left[\Lambda\chi + B\nu\right] \tag{33}$$

The matrix Λ can be decomposed as the sum of the matrices Λ_{diag} and Λ_{skew}, defined as follows:

$$\Lambda_{diag} = \begin{bmatrix} -k_2 L & 0_N & 0_N & 0_N & 0_N \\ 0_N & -\gamma I_N - K_p L_* & 0_N & 0_N & 0_N \\ 0_N & 0_N & 0_N & 0_N & 0_N \\ 0_N & 0_N & 0_N & -\gamma I_N - K_p L_* & 0_N \\ 0_N & 0_N & 0_N & 0_N & 0_N \end{bmatrix}$$

$$\Lambda_{skew} = \begin{bmatrix} 0_N & k_1 I_N & 0_N & 0_N & 0_N \\ -\gamma I_N & 0_N & K_i L_* & 0_N & 0_N \\ 0_N & -K_i L_* & 0_N & 0_N & 0_N \\ 0_N & 0_N & 0_N & 0_N & K_i L_* \\ 0_N & 0_N & 0_N & -K_i L_* & 0_N \end{bmatrix} \tag{34}$$

Since L and L_*, as defined in Section II, are symmetric and positive semidefinite, Λ_{diag} is negative semidefinite. Imposing $k_1 = \gamma$, Λ_{skew} is skew–symmetric. Thus, we can rewrite Eq. (33) as follows:

$$\dot{W}(\chi) = \chi^T \Lambda \chi + \chi^T B \nu = \chi^T \Lambda_{diag} \chi + \chi^T B \nu \tag{35}$$

Substituting Eqs. (29), (31) into Eq. (35) we obtain

$$\begin{aligned} \dot{W}(\chi) = & \chi^T \Lambda_{diag}\chi - k_3 \tilde{v}_2^T [\text{diag}(z_2) - I_N]\tilde{v}_2 \\ & + {z_2}^T [\gamma \text{diag}(\tilde{v}_2)]\tilde{v}_2 \\ = & \chi^T \Lambda_{diag}\chi + \left(-k_3 \tilde{v}_2^T \text{diag}(z_2) + \gamma z_2^T \text{diag}(\tilde{v}_2)\right)\tilde{v}_2 \\ & + k_3 \tilde{v}_2^T I_N \tilde{v}_2 - k_4 \tilde{v}_2^T \text{diag}\left(\left\{\left|\tilde{v}_2^i\right|\right\}\right)\tilde{v}_2 \end{aligned} \tag{36}$$

Given two vectors $\xi, \phi \in \mathbb{R}^N$, the vector $\zeta = \xi^T \text{diag}(\phi)$ is the vector whose components are the products of the corresponding components of ξ and ϕ, namely $\zeta = \{\xi_i \phi_i\}$. It is easy to prove that $\zeta = \phi^T \text{diag}(\xi)$ as well.

Then, Eq. (36) can be rewritten as follows:

$$\begin{aligned} \dot{W}(\chi) = & \chi^T \Lambda_{diag}\chi + \left(-k_3 \tilde{v}_2^T \text{diag}(z_2) + \gamma \tilde{v}_2^T \text{diag}(z_2)\right)\tilde{v}_2 \\ & + k_3 \tilde{v}_2^T I_N \tilde{v}_2 - k_4 \tilde{v}_2^T \text{diag}\left(\left\{\left|\tilde{v}_2^i\right|\right\}\right)\tilde{v}_2 \end{aligned} \tag{37}$$

Imposing $k_3 = \gamma$, Eq. (37) can be rewritten as follows:

$$\dot{W}(\chi) = \chi^T \Lambda_{diag}\chi + \gamma \tilde{v}_2^T I_N \tilde{v}_2 - k_4 \tilde{v}_2^T \text{diag}\left(\left\{\left|\tilde{v}_2^i\right|\right\}\right)\tilde{v}_2 \tag{38}$$

From the definition of Λ_{diag} in Eq. (34), Eq. (38) can be rewritten as follows:

$$\begin{aligned} \dot{W}(\chi) = & -\tilde{v}_2^T k_2 L \tilde{v}_2 - {z_1}^T \gamma I_N z_1 - {z_1}^T K_p L_* z_1 \\ & - {z_2}^T \gamma I_N z_2 - {z_2}^T K_p L_* z_2 + \gamma \tilde{v}_2^T I_N \tilde{v}_2 \\ & - k_4 \tilde{v}_2^T \text{diag}\left(\left\{\left|\tilde{v}_2^i\right|\right\}\right)\tilde{v}_2 \end{aligned} \tag{39}$$

From Eq. (39), we can state that:

$$\dot{W}(\chi) \leq -\tilde{v}_2^T \left(k_4 \text{diag}\left(\left\{\left|\tilde{v}_2^i\right|\right\}\right) - \gamma I_N\right)\tilde{v}_2 \tag{40}$$

Let

$$\Omega_i(\chi) = -k_4 \left|\tilde{v}_2^i\right|^3 + \gamma \left|\tilde{v}_2^i\right|^2 \quad \forall i = 1, \ldots, N \tag{41}$$

and let $\Omega(\chi) = \sum_{i=1}^{N} \Omega_i(\chi)$, namely:

$$\begin{aligned} \Omega(\chi) & = -k_4 \sum_{i=1}^{N} \left|\tilde{v}_2^i\right|^3 + \gamma \sum_{i=1}^{N} \left|\tilde{v}_2^i\right|^2 \\ & = -\tilde{v}_2^T \left(k_4 \text{diag}\left(\left\{\left|\tilde{v}_2^i\right|\right\}\right) - \gamma I_N\right)\tilde{v}_2 \end{aligned} \tag{42}$$

Thus, from Eqs. (39), (42) it follows that $\dot{W}(\chi) \leq \Omega(\chi)$. The function $\Omega(\chi)$ has a strict maximum Ω^M when $\left|\tilde{v}_2^i\right| = \frac{2\gamma}{3k_4} < \frac{\gamma}{k_4}$ $\forall i = 1, \ldots, N$.

Namely, $\Omega^M = N \cdot \bar{\Omega}$, where:

$$\bar{\Omega} = \left[-k_4 \left(\frac{2\gamma}{3k_4}\right)^3 + \gamma \left(\frac{2\gamma}{3k_4}\right)^2\right] \tag{43}$$

In order to compute an upper–bound on $\left|\tilde{v}_2^i\right|$ $\forall i = 1, \ldots, N$, we consider the worst case. More specifically, we will show that each entry of the vector $\tilde{v}_2$ is bounded. To do this, we suppose that all the entries of the vector $\tilde{v}_2$ are bounded, such that $\left|\tilde{v}_2^i\right| < \frac{\gamma}{k_4}$, except the j–th one.

In this case, the following inequality holds:

$$\Omega(\chi) \leq (N-1)\bar{\Omega} + \Omega_j(\chi) = (N-1)\bar{\Omega} - k_4 \left|\tilde{v}_2^j\right|^3 + \gamma \left|\tilde{v}_2^j\right|^2 \tag{44}$$

By *worst case* we mean that letting more than one components of $\tilde{v}_2$ be greater than $\frac{\gamma}{k_4}$ would decrease the value on the right–hand side of Eq. (44). We show now that a value α exists such that, if $\left|\tilde{v}_2^j\right| > \alpha$, then $\Omega_j(\chi) > (N-1)\bar{\Omega}$, and then $\Omega(\chi) < 0$. More specifically, $\Omega(\chi) < 0$ if $\left|\tilde{v}_2^j\right| > \alpha > 0$ such that:

$$\alpha^3 > \frac{\gamma}{k_4}\alpha^2 + \frac{(N-1)\bar{\Omega}}{k_4} \tag{45}$$

Hence, $\dot{W}(\chi) \leq \Omega(\chi) < 0$ if $\left|\tilde{v}_2^i\right| > \alpha$ for at least one value of $i = 1, \ldots, N$. Thus, $\exists S > 0$ such that, if $\|\tilde{v}_2\| \geq S$, then $W(\chi)$ does not increase over time, which implies that $\|\chi\|$ does not increase over time as well.

We will now show that, if $\|\tilde{v}_2\| < S$, then $\|\chi\|$ is bounded as well. Let $\zeta_1 = \left[{z_1}^T {w_1}^T\right]^T$ and $\zeta_2 = \left[{z_2}^T {w_2}^T\right]^T$ be the state vectors of the PI average consensus estimators. Thus, $\chi = \left[\tilde{v}_2^T \zeta_1^T \zeta_2^T\right]^T$. As proved in [11], the PI average consensus estimators are input–to–state stable (ISS) systems. The boundedness of $\|\tilde{v}_2\|$ implies the boundedness of the inputs of the PI average consensus estimators. In fact, as stated in Section III, these inputs are v_2^i and $\left(v_2^i\right)^2$, respectively. Thus, both $\|\zeta_1\|$ and $\|\zeta_2\|$ are bounded, given $\|\tilde{v}_2\| < S$. ■

From Proposition 3 we can state that $\exists M > 0$ such that $\|\chi(t)\| \leq M$, $\forall t \geq 0$.

Since $\|\tilde{v}_2(t)\| \leq \|\chi(t)\|$ and $\|z_2(t)\| \leq \|\chi(t)\|$, it follows that $\|\tilde{v}_2(t)\| \leq M$ and $\|z_2(t)\| \leq M$, $\forall t \geq 0$.

The following proposition proves the boundedness of the estimation error of λ_2.

Proposition 4 *Consider the equation for the computation of the estimate of λ_2, namely Eq.* (17)*, and consider the results given in Proposition 3. Then, the error on the estimation of λ_2 is bounded.*

Proof: Let $\hat{\lambda}_2 = \left[\lambda_2^1, \ldots, \lambda_2^N\right]^T \in \mathbb{R}^N$ be the vector containing the estimates of λ_2 performed by each agent.

Since each agent computes its estimate of λ_2, namely λ_2^i, according to Eq. (17), the vector $\hat{\lambda}_2$ is defined as follows:

$$\hat{\lambda}_2 = \frac{k_3}{k_2}(\mathbf{1} - z_2) \tag{46}$$

Since, from Proposition 3, we know that $\|z_2\|$ is bounded, then $\left\|\hat{\lambda}_2\right\|$ is bounded as well. Once defined the number of agents in the graph, the real value of λ_2 is bounded, namely $\lambda_2 \in \left[0, \lambda_2^M\right]$. More specifically:

- $\lambda_2 = 0$ if the graph is disconnected;
- $\lambda_2 = \lambda_2^M$ if the graph is complete (i.e. an edge exist between each couple of agents), and the distance between each couple of agents is such that the edge–weights a_{ij} defined in Eq. (7) assume their maximum value. Namely, the distance between each couple of agents is zero, and $a_{ij} = 1\ \forall i = 1, \ldots, N$. Then, for any value of the number of agents N, λ_2^M is well defined.

Let $\delta \in \mathbb{R}^N$ be the estimation error vector, i.e. $\delta = \hat{\lambda}_2 - \lambda_2 \mathbf{1}$.

Since both $\left\|\hat{\lambda}_2\right\|$ and $\|\lambda_2 \mathbf{1}\| = \lambda_2$ are bounded, we can conclude that $\exists \Xi > 0$ such that $\|\delta\| \leq \Xi$. Hence, $\left|\lambda_2^i - \lambda\right| \leq \Xi$, $\forall i = 1, \ldots, N$. ■

The following proposition proves the boundedness of the estimation error $\left|\lambda_2^i - \tilde{\lambda}_2\right|$, $\forall i = 1, \ldots, N$.

Proposition 5 *Consider the equation for the computation of λ_2^i, namely Eq.* (17)*, the definition of $\tilde{\lambda}_2$, given in Eq.* (18)*, and consider the results given in Proposition 3. Then, the estimation error $\left|\lambda_2^i - \tilde{\lambda}_2\right|$ is bounded, $\forall i = 1, \ldots, N$.*

Proof: The proof is analogous to that of Proposition 4. Hence, $\exists \Xi' > 0$ such that $\left|\lambda_2^i - \tilde{\lambda}\right| \leq \Xi'$, $\forall i = 1, \ldots, N$. ■

V. SIMULATIONS

To show the effectiveness of the control strategy presented in this paper, we implemented Matlab simulations where a bounded external control action is added, with the objective of disconnecting the group. More specifically, we implemented the following control law:

$$\dot{p}_i = u_i^c + u_i^e \tag{47}$$

where $p_i \in \mathbb{R}^2$, and the external controller u_i^e is defined as follows:

$$u_i^e = \begin{bmatrix} k \cos\left(\dfrac{2\pi}{N+1} i\right) \\ k \sin\left(\dfrac{2\pi}{N+1} i\right) \end{bmatrix} \tag{48}$$

for different values of $k > 0$. Hereafter, we will provide some results for $k = 5$.

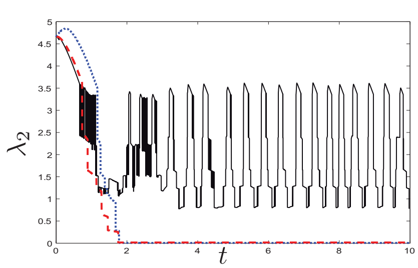


Fig. 2. Value of λ_2 with a *disconnecting* external controller, *with* the connectivity maintenance controller described in this paper (black solid line), *with* the connectivity maintenance controller described in [8] (blue dotted line), and *without* any connectivity maintenance controller (red dashed line)

Without the connectivity maintenance controller (i.e. $u_i^c = 0$), the external control law makes the agents move away from each other. As shown in Fig. 2 (red dashed line), the value of λ_2 decreases, until the connectivity of the communication graph is lost. Simulations give a similar result implementing the connectivity maintenance controller described in [8], as shown in Fig. 2 (blue dotted line).

As expected, using the connectivity maintenance controller described in this paper (i.e. u_i^c as described in Eq. (23)), the connectivity of the communication graph is never lost (Fig. 2, black solid line).

VI. CONCLUSIONS

In this paper we presented a control algorithm that for arbitrary initial conditions, and by means of a decentralized estimator for the algebraic connectivity of the communication graph, ensures maintenance of the connectivity among a group of single integrator agents, .

Utilizing analytical proofs and simulative validation, we demonstrated that by means of the proposed control strategy, the value of the algebraic connectivity of the graph, that is λ_2, is bounded away from zero, and consequently the graph is connected. Connectivity maintenance in presence of estimation errors has also been formally proved.

Current work aims at providing a constructive procedure for defining the smallest possible bound $\tilde{\epsilon}$ for ensuring connectivity maintenance.

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