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Quantized cooperative control using relative state measurements

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Abstract— **We consider cooperative control of multi-agent systems under limited communication between neighboring agents. In particular, quantized values of the relatives states are used as the control parameters of each agent. The results are derived for both uniform and logarithmic quantizers. Both static and time-varying communication topologies are considered. The stability conditions derived are less conservative than the corresponding ones in our previous work. Moreover, the case of second order dynamics is taken into account. The derived results are supported through computer simulations.**

I. INTRODUCTION

In the vast recent literature concerning multi-agent control, several results have been obtained by utilizing the spectral properties of the graph Laplacian matrix and under the assumption of perfect communication, as in [12], [10], [14]. But imperfect information exchange and communication constraints may have a considerable impact on the performance of a multi-agent system and also the implementation of the control algorithms. Relevant topics have received attention for different system dynamics and with different constrained models, as in [1], [3], [5] and [4] for discrete-time dynamics and [8], [2] for continuous-time models.

When a quantization function is included in the closed-loop dynamics, one consequence is that the right hand side of the differential equations is discontinuous, which may not have solutions in the classic sense [6], [3]. We first show here that starting from certain manifolds a classic or Carathédory solution of the system concerned may not exist, which makes it unavoidable to take into account the general solutions in the Filippov sense [7]. As a result, nonsmooth analysis [6] and discontinuous differential equations are used. We consider the same first-order model under quantized information as in [2] and less conservative results are obtained thanks to the new Lyapunov function candidate and the nonsmooth version of LaSalle's Invariance Principle [6]. Both static and time-varying communication topologies are tackled. Moreover, similar results are established for second-order systems that take quantized relative states as control parameters. The explicit convergence set and conditions for quantization gains to guarantee stability can be found for both uniform and logarithmic quantizers.

The rest of the paper is organized as follows: Section II presents some background on algebraic graph theory and Filippov solutions. In Section III, we treat the first order system under static tree topology, then extend the results to switching trees and general undirected graphs. Section IV is devoted to quantized second order systems under general topologies. The paper concludes with computer simulations in Section V and a summary of the results in Section VI.

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II. SYSTEM MODEL AND BACKGROUND

We first consider N single integrator agents: $\dot{z}_i = u_i,\ i \in \{1, \dots, N\}$, where $z_i = [x_i, y_i]^T \in \mathbb{R}^2$ denotes the position and $u_i \in \mathbb{R}^2$ the control input of agent i. Since a broad class of vehicles requires a second-order dynamic model, a double integrator system is also considered $\dot{z}_i = v_i,\ \dot{v}_i = u_i,\ i \in \{1, \dots, N\}$, where $z_i \in \mathbb{R}^2, v_i \in \mathbb{R}^2$ denotes the position and velocity, and $u_i \in \mathbb{R}^2$ the acceleration input. In the case of first order system, the control objective is to construct feedback controllers that lead the multi-agent system to the consensus, i.e., all agents converge to an agreement point. While for the second order system, the desired configuration is the same position and velocity, i.e., all agents moving with the common speed as one point.

For an undirected graph $G = (V, E)$ with N vertices, $V = 1, \dots, N$ and edges $E = \{(i, j) \in V \times V | i \in \mathcal{N}_j\}$, where $\mathcal{N}_j$ denotes agent j's *communication set* that includes the agents with which it can communicate. Each agent only has access to the state of agents that belong to its communication set. The communication graph is assumed to be *undirected*, $i \in \mathcal{N}_j \Leftrightarrow j \in \mathcal{N}_i, \forall (i, j) \in E$. When the communication topology is static, the sets $\mathcal{N}_i$ are static and G is time-invariant. When the communication topology is time-varying, the sets $\mathcal{N}_i$ change over time and $G = G(t)$.

The *adjacency matrix* [11] $A = A(G) = a_{ij}$ is the $N \times N$ matrix given by $a_{ij} = 1$, if $(i, j) \in E$, and $a_{ij} = 0$, otherwise. $i,\ j$ are called *adjacent* if $(i, j) \in E$. A *path* of length r from i to j is a sequence of $r+1$ distinct vertices starting with i and ending with j such that consecutive vertices are adjacent. If there is a path between any two vertices, then G is called *connected*. A connected graph is called a *tree* if it contains no cycles. Let Δ be the $N \times N$ diagonal matrix of d_i's, where the *degree* d_i of each vertex i is given by $d_i = \sum_j a_{ij}$. The *Laplacian* matrix of G is the symmetric positive semidefinite matrix given by $L = \Delta - A$. For a connected graph, L has a single zero eigenvalue with the corresponding eigenvector **1** [14].

An orientation on G is the assignment of a direction to each edge. An oriented graph has the the *incidence matrix* $B = B(G) = (b_{ij})$, which is the $\{0, \pm 1\}$ matrix with rows and columns indexed by the vertices and edges of G, respectively, such that $b_{ij} = 1$ if the vertex i is the head

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of the edge j, and $b_{ij} = -1$ if vertex i is the tail of the edge j, and $b_{ij} = 0$ otherwise. Obviously, the matrix B varies with different assignment of the edges' orientation. The Laplacian matrix is also given by $L = BB^T$ in [11]. Moreover, we denote by $\bar{x}$ the m-dimensional stack vector of relative differences (*head–tail*) of pairs of agents that form an edge in G, where m is the number of edges. The following relations are easily verified: $Lx = B\bar{x}, \bar{x} = B^T x$. Since $\bar{x} = 0 \Rightarrow B\bar{x} = 0 \Rightarrow Lx = 0$, then if G is connected, the requirement $Lx = 0$ guarantees that x has all its elements equal [11], [13]. In this paper, we treat only the system behavior in the x-coordinates but the analysis that follows holds *mutatis mutandis* in higher dimensions.

The consensus protocol in [12],[13] for the first order system, is given by $u_i = -\sum_{j\in\mathcal{N}_i}(x_i - x_j)$ and the closed-loop nominal system (without quantization) is $\dot{x}_i = -\sum_{j\in\mathcal{N}_i}(x_i - x_j,), \, i \in \{1,\ldots,N\}$, or equivalently $\dot{x} = -Lx$. It holds that $\dot{\bar{x}} = B^T\dot{x} = -B^T Lx = -B^T B\bar{x}$. On the other hand, the agreement protocol defined in [14] for the second-order system was $u_i = -\sum_{j\in\mathcal{N}_i}[(x_i - x_j) + \gamma(v_i - v_j)]$, where $\gamma > 0$ is the control gain. Similarly we have $\dot{\bar{x}} = \bar{v}, \dot{\bar{v}} = -B^T B\,\bar{x} - \gamma B^T B\,\bar{v}$, where $\bar{x} = B^T x$ and $\bar{v} = B^T v$. As stated in [2], each agent i is assumed to have only quantized measurements of the relative position $q(x_i - x_j)$ for $j \in \mathcal{N}_i$. $q(.) : \mathbb{R} \to \mathbb{R}$ is the *quantization function*. In this paper, we mainly consider two types of quantization models, uniform and logarithmic, which are defined as: The *uniform* quantizer, $q_u, \mathbb{R} \to \mathbb{R}$: $q_u(x) = \delta_u \; [\frac{x}{\delta_u}]$, where $[\cdot]$ denotes the nearest integer operation and $[\frac{1}{2}] = 1$. So the following relations hold: (i) $x \cdot q_u(x) \geq 0$, (ii) $|q_u(x) - x| \leq \frac{\delta_u}{2}$, (iii) $q_u(-x) = -q_u(x)$. 4. $q_u(0) = 0$. The *logarithmic* quantizer [18] $q_l, \mathbb{R} \to \mathbb{R}$: $q_l(x) = \mathrm{sign}(x) \cdot \exp(q_u(\ln(|x|))) \quad (x \neq 0)$, where q_u is the uniform quantizer with gain δ_u and $q_l(0)$ is defined to be 0. Similarly we have (i) $x \cdot q_l(x) \geq 0$, (ii) $q_l(-x) = -q_l(x)$, (iii) $|q_l(x) - x| \leq \delta_l |x|$, where $\delta_l = e^{\frac{\delta_u}{2}} - 1$.

As mentioned in the introductory part, we need to consider the Filippov solution of a nonsmooth system [6]. Given the system $\dot{x} = f(x,t)$ where $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is measurable and essentially locally bounded, its Filippov solution $x(\cdot)$ is absolutely continuous on $[t_0, t_1]$ and for almost all $t \in [t_0, t_1]$, $\dot{x} \in \mathcal{K}[f](x, t)$ where $\mathcal{K}[f](x_0, t) = \overline{\mathrm{co}}\{\lim_{x\to x_0} f(x,t) | x \notin N_0\}$; $\overline{\mathrm{co}}$ denotes the convex closure of a set and N_0 is a set of Lebesgue measure zero.

III. FIRST-ORDER QUANTIZED AGREEMENT UNDER TIME-VARYING TOPOLOGY

Before we state any conclusion about the stability and convergence of the closed-loop system

$$\dot{x}_i = -\sum_{j\in\mathcal{N}_i} q\,(x_i - x_j), \tag{1}$$

we need to show the fact that the solution of (1) may not exist in the classic or Carathédory sense. Similar arguments as in Proposition 1 in [16] will be used. In our model (1) with uniform quantizers, let $x_i - x_{j^*} = (k + \frac{1}{2})\Delta$ for certain $j^* \in \mathcal{N}_i$ and any $k \in \mathbb{Z}$. For $j \in \mathcal{N}_i$ and $j \neq j^*$, $x_i - x_j \neq (h + \frac{1}{2})\Delta, \forall h \in \mathbb{Z}$ and $\sum_{j\in\mathcal{N}_i, j\neq j^*} q(x_i - x_j) \in (-(k+1)a_{ij^*}\Delta, \, -ka_{ij^*}\Delta)$. Then the Carathédory solution starting from $x_i - x_{j^*} = (k+\frac{1}{2})\Delta$ can not leave the surface in the direction of decreasing x_i as $\dot{x}_i^- = -[\sum_{j\in\mathcal{N}_i, j\neq j^*} q(x_i - x_j) + a_{ij^*}k\Delta] > 0$ and neither in the direction of increasing x_i as $\dot{x}_i^+ = -[\sum_{j\in\mathcal{N}_i, j\neq j^*} q(x_i - x_j) + a_{ij^*}(k+1)\Delta] < 0$. Thus the possible solution would remain on the surface, which doesn't satisfy the definition of Carathédory solution. As a result, there are no Carathédory solutions starting from these hyperplanes. The same analysis holds for the case of logarithmic quantizers. So we need to consider more general solutions in the Filippov sense. The local existence of Filippov solution is guaranteed as the right hand of (1) is measurable and locally bounded [6].

A. Static Tree Topology

Assume all agents have the same quantizer and parameters, i.e., $q(\cdot)$ is the same for all i. Since it holds for both quantizers that $q\,(-a) = -q\,(a)\,, \forall a \in \mathbb{R}$, it can be verified that (1) is equivalent to

$$\dot{\bar{x}} = -B^T Bq\,(\bar{x}) \tag{2}$$

where $q(\bar{x})$ is the stack vector of all pairs $q\,(x_i - x_j)\,, \forall (i,\, j) \in E$. By Lemma 1 in [10], $B^T B$ is always positive definite with a tree graph. For brevity, we denote $M = B^T B$. Let $V = \bar{x}^T M^{-1} \bar{x}$ be a candidate Lyapunov function, where M^{-1} is the inverse of $B^T B$. Since M is positive definite, M^{-1} exists and is also positive definite. Consider the Filippov solution of system (2) that $\dot{\bar{x}} \in \mathcal{K}[-Mq(\bar{x})]$. Note that due to Statement 5 in [6] $\mathcal{K}[-Mq(\bar{x})] = -M\mathcal{K}[q](\bar{x})$. Since V is smooth and regular, the generalized time derivative of V satisfies

$$\begin{aligned} \dot{\tilde{V}} &\subset (2\,M^{-1}\bar{x})^T(-M\mathcal{K}[q](\bar{x})) \\ &= -2\,\bar{x}^T\,\mathcal{K}[q](\bar{x}) = -2\sum_{i=1}^{m} \bar{x}_i\,\mathcal{K}[q](\bar{x}_i) \end{aligned}$$

where $\mathcal{K}[q](a) = \overline{\mathrm{co}}\{\lim_{x\to a} q(x) | x \notin N_0\}$ and N_0 is the set of the discontinuous points of $q(\cdot)$. We denote a set $\mathcal{S} \subset \mathbb{R}$ by $\mathcal{S} \geq 0$ if all elements $v \in \mathcal{S}$ satisfy $v \geq 0$.

1. Uniform Quantizers. In the case of $q(x)$ being uniform quantizers, the Filippov set-valued map for $q_u(x)$ is given as

$$\mathcal{K}[q_u](x) = \begin{cases} q_u(x) & x \neq (k - \frac{1}{2})\delta_u, \; k \in \mathbb{Z} \\ [(k-1)\delta_u, \; k\delta_u] & \text{otherwise} \end{cases}$$

Note that $a\,\mathcal{K}[q_u](a) \geq 0, \forall a \in \mathbb{R}$. Furthermore, for $|a| < \frac{\delta_u}{2}$, $\mathcal{K}[q_u](a) = q_u(a) = 0$ and $\mathcal{K}[q_u](\frac{\delta_u}{2}) \in [0, \, \delta_u]$. Thus $\bar{x}^T\,\mathcal{K}[q_u](\bar{x}) \geq 0 \Rightarrow \dot{\tilde{V}} \leq 0$ and the equality holds only when $|x_i - x_j| \leq \frac{\delta_u}{2}, \forall (i,\, j) \in E$. Since the level sets of V are obviously compact, we can apply the nonsmooth version of the LaSalle's invariance principle [6]. System (2) converges to the consensus set $\mathcal{I} = \{x | \, |x_i - x_j| \leq \frac{\delta_u}{2}, (i,j) \in E\}$, which implies convergence to the set $\{x | \, |\bar{x}| \leq \frac{\delta_u}{2}\sqrt{m}\}$, a ball centered in the desired equilibrium point, with radius $\frac{\delta_u}{2}\sqrt{m}$. This point is the average of the initial states by virtue of Lemma 3 which will be stated later. When $x \in \mathcal{I}$, we have $u = 0$, keeping all agents in the set $\mathcal{I}$.

2. Logarithmic Quantizers

In the case of $q(x)$ being logarithmic quantizers, the Filippov set-valued map for $q_l(x)$ is given as

$$\mathcal{K}[q_l](x) = \begin{cases} q_l(x) & x \geq 0 \text{ and } x \neq e^{(k-\frac{1}{2})\delta_u},\ k \in \mathbb{Z} \\ [e^{(k-1)\delta_u}, \quad e^{k\delta_u}] & x = e^{(k-\frac{1}{2})\delta_u},\ k \in \mathbb{Z} \end{cases}$$

and $\mathcal{K}[q_l](-x) = -\mathcal{K}[q_l](x)$. Applying the same arguments as before, but now $\bar{x}^T \mathcal{K}[q_l](\bar{x}) = 0$ only when $\bar{x} = 0$ because $q_l(a) = 0$ only if $a = 0$. So $\dot{\tilde{V}} \leq 0$ and equality holds when $x_i = x_j, \forall (i, j) \in E$. For a connected graph like tree, this corresponds to an agreement point. Consequently, for any logarithmic gain satisfying $\delta_l = e^{\frac{\delta_u}{2}} - 1 > 0$, $\dot{\tilde{V}}$ is strictly negative before an agreement is reached. The nonsmooth version of LaSalle's invariance principle guarantees that (1) converges to consensus asymptotically for any $\delta_l > 0$.

Compared with the results in [2], a smaller convergence set is obtained for the uniform quantizer and the bounds on the logarithmic gain that guarantee consensus are less conservative. The previous analysis is summarized as:

Theorem 1: Assume that G is a static tree. Then (1) has the following properties:

1) In the case of uniform quantizers, the system converges to the consensus set $\mathcal{I}$.
2) With logarithmic quantizers, the system asymptotically converges to the average consensus for all $\delta_l > 0$.

Remark: It will be shown in Theorem 3 that the convergence in case (1) occurs in finite time.

B. Time-varying Communication Topology

In this section we treat the case when the communication topology is time-varying or in particular switching among different tree topologies. Now the stack vector $\bar{x}$ changes discontinuously whenever edges are added or deleted. But we will show that the same $V = \bar{x}^T (B^T B)^{-1} \bar{x}$ can serve as a common Lyapunov function to prove convergence under time varying topologies.

Using $\bar{x} = B^T x$ we get $V = x^T B(B^T B)^{-1} B^T x$. Since the state vector x is continuous at all switching instants, V would also be continuous at all switches if the matrix $H = B(B^T B)^{-1} B^T$ remains invariant for different B.

Note that for undirected graphs with N vertices, it is always the case that tree topologies have $N - 1$ edges, i.e., $m = N - 1$. Thus the incident matrix B of any tree topology has dimension $N \times (N-1)$. We assume its singular value decomposition to be $B = U \Sigma W^T$, where $U_{(N\times N)}$ is the left singular matrix, composed by the normalized eigenvectors of $BB^T_{(N\times N)}$, $W_{((N-1)\times(N-1))}$ is the right singular matrix composed by the normalized eigenvectors of $B^T B_{((N-1)\times(N-1))}$. The singular value matrix $\Sigma_{(N\times(N-1))}$ has the structure

$$\Sigma = \left[\begin{array}{cccc} \lambda_{N-1} & 0 & \cdots & 0 \\ 0 & \lambda_{N-2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_1 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]_{N\times(N-1)}$$

where λ_i for $i = (N-1), \ldots, 1$ are the singular values of B in descending order.

Lemma 1: BB^T has N nonnegative eigenvalues: one of them is zero, corresponding to the eigenvector $\mathbf{1}$ and the others are the same as the eigenvalues of $B^T B$.

Proof: Using $B = U\Sigma W^T$ we get $B^T B = (U\Sigma W^T)^T U \Sigma W^T = W \Sigma^T U^T U \Sigma W^T = W \Sigma^T \Sigma W^T = WTW^T$. Let

$$T = \Sigma^T \Sigma = \begin{bmatrix} \lambda_{N-1}^2 & 0 & \cdots & 0 \\ 0 & \lambda_{N-2}^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_1^2 \end{bmatrix}_{(N-1)\times(N-1)}$$

which implies that WTW^T is the singular value decomposition of $B^T B$. Moreover, using similar calculations for the matrix BB^T, and denoting $S = \Sigma\ \Sigma^T$, we get $BB^T = U\Sigma W^T (U \Sigma W^T)^T = U \Sigma W^T W \Sigma^T U^T = U \Sigma \Sigma^T U^T = USU^T$. Let

$$S = \Sigma\ \Sigma^T = \left[\begin{array}{cccc|c} \lambda_{N-1}^2 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_{N-2}^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_1^2 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right]_{N\times N}$$

Thus USU^T is the singular value decomposition of BB^T, which has N nonnegative eigenvalues: one of them is zero and the others are identical to the eigenvalues of $B^T B$.

Consider the zero eigenvalue of BB^T. Since $BB^T = L$ and $L\mathbf{1} = \mathbf{0} \Rightarrow BB^T\mathbf{1} = 0 \cdot \mathbf{1}$. This shows that $\mathbf{1}$ is the eigenvector associated with the eigenvalue zero. The corresponding normalized eigenvector is $\frac{1}{\sqrt{N}}\mathbf{1}$. ■

Lemma 2: $H = B(B^T B)^{-1} B^T$ is the same for all incident matrices B corresponding to undirected trees.

Proof: Inserting $B = U\Sigma W^T$ into H we have $H = B(B^T B)^{-1} B^T = U\Sigma W^T (W \Sigma^T \Sigma W^T)^{-1} W \Sigma^T U^T = U \Sigma W^T (WTW^T)^{-1} W \Sigma^T U^T = U \Sigma\, T^{-1} \Sigma^T U^T$. Let

$$G = \Sigma\, T^{-1} \Sigma^T = \left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right]_{N\times N}$$

Consider the matrix $U_{N\times N} = [u_{N-1}\ u_{N-2} \ldots u_0]$, composed of the normalized column eigenvectors u_i of BB^T. Denote by $u_i(k)$ the k_{th} element of u_i and since U is the left singular orthogonal matrix and $UU^T = I_N$, we get $\sum_{k=0}^{N-1} u_k(i)\, u_k(i) = 1$ and $\sum_{k=0}^{N-1} u_k(i)\, u_k(j) = 0\ (i \neq j)$ for all $i,\ j \in \{1, \ldots, N\}$. In Lemma 1 we have shown that the last eigenvector corresponding to the eigenvalue zero is $u_0 = \frac{1}{\sqrt{N}}\mathbf{1}$. Computing each entry of $H = UGU^T$ element-wise, we have $H(i,j) = \sum_{k=1}^{N-1} u_k(i)\, u_k(j) = \sum_{k=0}^{N-1} u_k(i)\, u_k(j) - u_0(i)\, u_0(j)$

$$= \begin{cases} 1 - \frac{1}{\sqrt{N}}\frac{1}{\sqrt{N}} = \frac{N-1}{N} & (i = j) \\ 0 - \frac{1}{\sqrt{N}}\frac{1}{\sqrt{N}} = -\frac{1}{N} & (i \neq j) \end{cases}$$

by which we can conclude that $H = B(B^TB)^{-1}B^T$ is identical for any B corresponding to a tree graph. This completes the proof. ■

Since $G(t)$ remains a tree for continuous evolution, the notation $\mathcal{T} = \{t_1, t_2, \cdots t_j\}$ is used of the set of switching instants. We consider $V_g = x^THx$. Since $V_g = x^TB(B^TB)^{-1}B^Tx = \bar{x}^T(B^TB)^{-1}\bar{x} \geq 0$, V_g is continuously varying and positive semidefinite. If V_g is decreasing between any two consecutive switching instants, it holds that V_g is decreasing over all time. Assume that the tree topology has incidence matrix B_j for $t \in [t_j, t_{j+1}]$. Let $V_j = \bar{x}^T(B_j^TB_j)^{-1}\bar{x}$. We have $\dot{\tilde{V}}_g = \dot{\tilde{V}}_j \in -2\bar{x}^T\mathcal{K}[q](\bar{x}) \leq 0$. So $\dot{\tilde{V}}_g \leq 0$ for $t \geq 0$ and $\dot{V}_g = 0$ when $\mathcal{K}[q](\bar{x}) = 0$. Following the same reasoning for convergence as in the static case, we have the following conclusions:

Theorem 2: Assume that $G(t)$ remains a tree within the finite switching sequence. Then (2) has the following convergence properties:

1) In the case of uniform quantizers, the system converges to the consensus set $\mathcal{I}_n = \{x|\, |x_i - x_j| < \frac{\delta_u}{2}, (i,j) \in E_n\}$ in finite time.
2) In the case of logarithmic quantizers, the system asymptotically converges to the consensus point for all logarithmic gains $\delta_l > 0$.

All results above are independent of the way that the topology changes as long as the tree structure is kept.

C. General Undirected Graphs

The above results are useful whenever the communication graph retains the tree structure. A more practical situation however occurs if we allow for the tree assumption to be lost or even disconnected for certain periods. First of all, we need to establish the result that the average of the states is invariant with the Filippov solutions of (1).

Lemma 3: Let $x(t)$ be a Filippov solution of system (1). The average of all agent states $\frac{1}{N}\sum_{i=1}^N x_i$ remains constant over all time in the case of undirected topologies.

Proof: We can easily verify that (1) is equivalent to $\dot{x} = -B\,q(\bar{x})$. By definition, the Filippov solution $x(t)$ satisfies $\dot{x}(t) \in \mathcal{K}(-B\,q(\bar{x}))$. The derivative of $\frac{1}{N}\sum_{i=1}^N x_i(t)$ is given by $\frac{1}{N}\sum_{i=1}^N \dot{x}_i = \frac{1}{N}\mathbf{1}^T\dot{x} \in \frac{1}{N}\mathbf{1}^T\mathcal{K}(-Bq(\bar{x})) = -\frac{1}{N}\mathbf{1}^TB\,\mathcal{K}[q](\bar{x}) = \{0\}$, where the final equality is due to the fact that $\mathbf{1}^TB = \mathbf{0}$. Hence the centroid is preserved during the evolution, which is denoted by constant C. ■

We propose a new Lyapunov function candidate for (1) $V = \sum_{i=1}^N(x_i - \frac{1}{N}\sum_{i=1}^N x_i)^2 = \sum_{i=1}^N(x_i - C)^2$, which is a quadratic disagreement function to the invariant centroid. V is continuously differentiable and $V = 0$ when all states equal to the initial average. The level sets of V define compact sets with respect to the agents' state. In particular, $V \leq c \Rightarrow |x_i - C| \leq \sqrt{c} \Rightarrow C + \sqrt{c} \leq x_i \leq C - \sqrt{c}$ for all i. Since V is smooth, the generalized time derivative of V is $\dot{\tilde{V}} = (\nabla V)^T\dot{x}$, where $\nabla V = 2(x - C\mathbf{1})$ and $\dot{x} \in -B\mathcal{K}[q](\bar{x})$. Thus $\dot{\tilde{V}} \in -2(x^T - C\mathbf{1}^T)B\,\mathcal{K}[q](\bar{x}) = -2\,\bar{x}^T\,\mathcal{K}[q](\bar{x})$, where $\bar{x}^T\,\mathcal{K}[q](\bar{x})$ has been proved to be positive semidefinite in Theorem 1. So $v \leq 0$ for all $v \in \dot{\tilde{V}}$ with both quantizers. Consequently, we can derive the same convergence properties as before: System (2) converges to the invariant set $\mathcal{I}_n = \{x|\,\mathcal{K}[q](x_i - x_j) = 0,\, \forall (i,j) \in E_n\}$.

For uniform quantizers, if at least one pair $(i,\, j) \in E_n$ satisfies $|x_i - x_j| \geq \frac{\delta_u}{2}$, it holds that $\dot{\tilde{V}} \leq -\delta_u^2$. Since $V(t)$ is bounded from below, there exist a settling time $T \in [0,\, \infty)$ that $x(t) \in \{x|\, |x_i - x_j| < \frac{\delta_u}{2},\, \forall (i,j) \in E_n\}$ for $t \geq T$. But it is not the case for logarithmic quantizers due to the fact that $q_l(x) \to 0$ when $x \to 0$.

Theorem 3: Assume that $G(t)$ is undirected with a finite switching sequence and E_n denotes the edge set of the last topology. Then system (1) is guaranteed to converge to the invariant set $\mathcal{I}_n$, namely

1) $\{x|\, |x_i - x_j| \leq \frac{\delta_u}{2},\, \forall (i,j) \in E_n\}$ with uniform quantizers, in finite time.
2) $\{x|\, x_i = x_j,\, \forall (i,j) \in E_n\}$ asymptotically with logarithmic quantizers satisfying $\delta_l > 0$.

Since we didn't put any constraints on the communication topology G but being undirected, the above conclusion is valid for static or time-varying, connected or disconnected information exchange graphs.

IV. SECOND-ORDER QUANTIZED AGREEMENT UNDER STATIC TOPOLOGY

The control law for second-order system with quantized relative states is given by $u_i = -\sum_{j\in\mathcal{N}_i}[\,q(x_i - x_j) + \gamma\,q(v_i - v_j)]$. The closed-loop system is $\dot{\bar{x}} = \bar{v}$ and $\dot{\bar{v}} = -B^TBq(\bar{x}) - \gamma B^TBq(\bar{v})$, where $\bar{x} = B^Tx$ and $\bar{v} = B^Tv$. Denote the stack vector $y = [\bar{x}^T\,\bar{v}^T]^T$ and rewrite the system into matrix form:

$$\begin{aligned} \dot{y} &= \begin{bmatrix} 0_{m\times m} & I_m \\ 0_{m\times m} & 0_{m\times m} \end{bmatrix} y + \begin{bmatrix} 0_{m\times m} \\ I_m \end{bmatrix} \bar{u} \\ \bar{u} &= \begin{bmatrix} -M & -\gamma M \end{bmatrix} q(y) \end{aligned} \tag{3}$$

Due to similar reasons as in the first order system, the Carathédory solution of (3) may not exist. We need to consider its solution in the Filippov sense. The control objective for (3) is to force the group of agents converge to one point and move with the same velocity. Before we state the main results about the stability and convergence of system (3), we need the following lemmas:

Lemma 4: If the undirected graph G is connected, then B^TB and the Laplacian matrix $L = BB^T$ both have nonnegative eigenvalues and moreover the same positive ones.

Proof: The case when G is a tree and $m = N - 1$ has been solved in Lemma 1. When $m \geq N$, there are exactly $m + 1 - N$ cycles in G, which correspond to $m + 1 - N$ dimensional null space of B [11]. Again the singular value decomposition of the incidence matrix $B_{N\times m}$ is $B = U\Sigma W^T$, where $\Sigma_{N\times m}$ has the structure [15]

$$\Sigma = \left[\begin{array}{cccc|cc} \lambda_{N-1} & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \lambda_{N-2} & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \cdots & 0 \\ 0 & 0 & \cdots & \lambda_1 & \cdots & 0 \\ \hline 0 & 0 & \cdots & 0 & \cdots & 0 \end{array}\right]_{N\times m}$$

where $\lambda_{N-1}, \cdots, \lambda_1$ are the non-zero singular values of B in descending order. There are $m+1-N$ zero singular values, corresponding to the cycle space of G. The rest of the proof can be done by following the same steps as in Lemma 1. ■

Lemma 5: $\bar{x}^T B^T B \bar{x} \geq \lambda_2(L)|\bar{x}|^2$ with connected G.

Proof: First we need to clarify that $B\bar{x} = 0$ only if $\bar{x} = 0$ because $B\bar{x} = 0 \Rightarrow BB^T x = 0 \Rightarrow Lx = 0 \Rightarrow x \in span\{\mathbf{1}\} \Rightarrow \bar{x} = 0$. Denote the $m+1-N$ column eigenvectors of $B^T B$ associated with the $m+1-N$ zero eigenvalues by $[c_1, c_2, \cdots, c_{m+1-N}]$, which can be obtained as signed edge vectors depending on the direction of cycles in G [11]. By definition $Bc_i = 0$. Moreover, since $c_i^T \bar{x} = c_i^T B^T x = (Bc_i)^T x = 0$, we have $c_i \perp \bar{x}$, $\forall i = 1, \cdots, m+1-N$. By the Courant-Fischer Theorem [15]

$$\min_{\substack{c_i \perp \bar{x},\, \bar{x} \neq 0 \\ i=1,\cdots,m+1-N}} \frac{\bar{x}^T B^T B \bar{x}}{\bar{x}^T \bar{x}} = \lambda_{m+2-N}(B^T B)$$

where $\lambda_{m+2-N}(B^T B) = \lambda_1^2 = \lambda_2(L)$ by Lemma 4. Thus $\bar{x}^T B^T B \bar{x} \geq \lambda_2(L)|\bar{x}|^2$, which completes the proof. ■

Denote $V(y) = y^T \begin{bmatrix} \gamma M & \frac{1}{2} I_m \\ \frac{1}{2} I_m & \frac{\gamma}{2} I_m \end{bmatrix} y$, which is continuously differentiable for a static graph G. The generalized time derivative of $V(y)$ along the Filippov solution of 3 is given by $\dot{\tilde{V}}(y) \in -y^T Q y + y^T W \left(\mathcal{K}[q](y) - y\right)$, where

$$Q = \begin{bmatrix} M & 0_{m\times m} \\ 0_{m\times m} & \gamma^2 M - I_m \end{bmatrix}, \quad W = \begin{bmatrix} -M & -\gamma M \\ -\gamma M & -\gamma^2 M \end{bmatrix}.$$

Lemma 6: $y^T Q y \geq \min\{\lambda_2(L), \gamma^2 \lambda_2(L) - 1\}|y|^2$ if $\gamma > \sqrt{\frac{1}{\lambda_2(L)}}$.

Proof: By virtue of Lemma 5, we have $\bar{x}^T M \bar{x} \geq \lambda_2(L)|\bar{x}|^2$. The first part of $\dot{\tilde{V}}(y)$ can be lower bounded by $y^T Q y = \bar{x}^T M \bar{x} + \gamma^2 \bar{v}^T M \bar{v} - \bar{v}^T \bar{v} \geq \lambda_2(L)|\bar{x}|^2 + (\gamma^2 \lambda_2(L) - 1)|\bar{v}|^2$. Thus if $\gamma > \sqrt{\frac{1}{\lambda_2(L)}}$, we have $\gamma^2 \lambda_2(L) - 1 > 0$, which ensures $y^T Q y$ to be positive semidefinite. Moreover, $y^T Q y \geq \min\{\lambda_2(L), \gamma^2 \lambda_2(L) - 1\}(|\bar{x}|^2 + |\bar{v}|^2) = \min\{\lambda_2(L), \gamma^2 \lambda_2(L) - 1\}|y|^2 = \lambda_{\min}(Q)|y|^2$, where $\lambda_{\min}(Q) = \min\{\lambda_2(L), \gamma^2 \lambda_2(L) - 1\}$. ■

Lemma 7: $\|W\|_2 = (1+\gamma^2)\lambda_{\max}(L)$

Proof: $\det(\lambda I_{2m} - W^T W) = \det(\begin{bmatrix} \lambda I_m - (1+\gamma^2) M^2 & -\gamma(1+\gamma^2) M^2 \\ \gamma(1+\gamma^2) M^2 & \lambda I_m - (1+\gamma^2)\gamma^2 M^2 \end{bmatrix})$
$= \det(\lambda^2 I_m - \lambda M^2 (1+\gamma^2)^2) = \prod_{i=1}^{m} [\lambda\,(\lambda - \theta_i^2 (1+\gamma^2)^2)] = 0$

Thus $W^T W$ has an eigenvalue at 0 with multiplicity m and other m non-zero eigenvalues corresponding to each eigenvalue θ_i of M. The maximal one is $\lambda_{\max}(W^T W) = \lambda_{\max}^2(M)(1+\gamma^2)^2$, yielding $\|W\| = (1+\gamma^2)\lambda_{\max}(M) = (1+\gamma^2)\lambda_{\max}(L)$. ■

Finally by combining Lemma 6 and 7, we can bound $\dot{\tilde{V}}(y)$ in such a way that $\dot{\tilde{V}}(y) \leq -\lambda_{\min}(Q)|y|^2 + |y|\|W\||\mathcal{K}[q](y) - y| = -\lambda_{\min}(Q)|y|^2 + |y|(1+\gamma^2)\lambda_{\max}(L)\ |\mathcal{K}[q](y) - y|$.

In the case of uniform quantizers, $q = q_u$ and $|\mathcal{K}[q_u](y) - y| \leq \frac{\delta_u}{2}\sqrt{2m}$. Then we get $\dot{\tilde{V}}(y) \leq -\lambda_{\min}(Q)|y|(|y| - \frac{(1+\gamma^2)\lambda_{\max}(L)}{\lambda_{\min}(Q)} \frac{\delta_u}{2}\sqrt{2m})$ Based on the nonsmooth version of LaSalle's Invariance principle [6], all solutions of system (3) enter the ball

$$\{y \,|\, |y| \leq \frac{(1+\gamma^2)\lambda_{\max}(L)}{2\lambda_{\min}(Q)} \sqrt{2m}\,\delta_u\} \tag{4}$$

which is centered at the consensus point where $\bar{x} = \bar{v} = 0$.

In the case of logarithmic quantizers, we have $|\mathcal{K}[q_l](y) - y| \leq \delta_l |y|$ so that $\dot{\tilde{V}}(y) \leq -\lambda_{\min}(Q)|y|^2 + |y|(1+\gamma^2)\lambda_{\max}(L)\ \ |q(y) - y| \leq -|y|^2(\lambda_{\min}(Q) - (1+\gamma^2)\lambda_{\max}(L)\,\delta_l)$. If the logarithmic gain δ_l satisfies

$$0 < \delta_l < \frac{\lambda_{\min}(Q)}{(1+\gamma^2)\lambda_{\max}(L)} \tag{5}$$

we have $\dot{\tilde{V}}(y) \leq 0$ and equality holds when $|y| = 0$. This means that the logarithmic gain should be smaller than an upper bound to guarantee the asymptotic convergence.

Theorem 4: Assume that the undirected graph G is static and connected. If $\gamma > \sqrt{\frac{1}{\lambda_2(L)}}$, system (3) has the following convergence properties:

1) With uniform quantizers, the system converges to the consensus set (4), centered at the agreement point.
2) With logarithmic quantizers, the agents asymptotically converge to a consensus point and move with the same velocity, for all δ_l satisfying (5).

V. Simulations

We now provide computer simulations to support the presented results. In the first part, the communication sets of the four agents are chosen as $\mathcal{N}_1 = \{2\}, \mathcal{N}_2 = \{1,3\}, \mathcal{N}_3 = \{2,4\}, \mathcal{N}_4 = \{3\}$, which is a line graph. The first simulation involves a switching tree topology for both uniform and logarithmic quantizers. A relatively large logarithmic gain $\delta_l = 10$ is chosen, and $\delta_u = 0.01$ for uniform quantizers. The trajectories with uniform quantizers are depicted in Fig. 1 while the case with the logarithmic quantizers is shown in Fig. 2. Red circles denote the time instances when the communication topology switches. As expected, in the case of uniform quantizers all agents could reach the consensus set $\mathcal{I}_n$ while with logarithmic quantizers the average consensus is achieved asymptotically. The same system under more

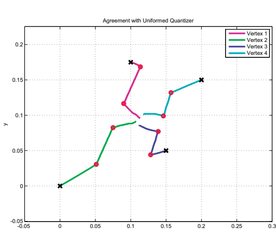

Fig. 1. Switching tree topologies and uniform quantizers.

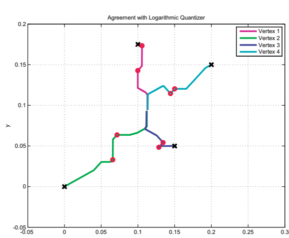

Fig. 2. Switching tree topologies and logarithmic quantizers.

general graphs is tested in the second part where the graph G is switching from disconnected graph to tree topology and finally to another disconnected graph. The simulation results

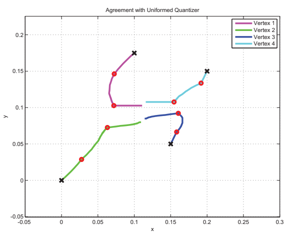

Fig. 3. General topology and uniform quantizers.

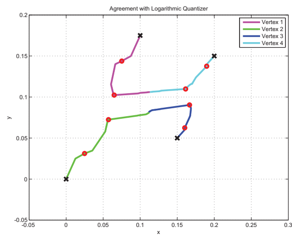

Fig. 4. General topology and logarithmic quantizers.

in Fig. 3 and 4 illustrate different convergence results with uniform and logarithmic quantizers.

In the last part, we simulate the group of second-order agents moving only along x-coordinates in order to visualize both the trajectories of velocity and position. The same static tree graph is used as in the first order case. Figures on the right are zoomed details of the final configuration. Logarithmic quantizers with $\delta_l = 0.05 < \frac{\lambda_{\min}(Q)}{(1+\gamma^2)\lambda_N(L)} = 0.056$ are used to guarantee asymptotic convergence both in velocity and position as shown in Fig. 6.

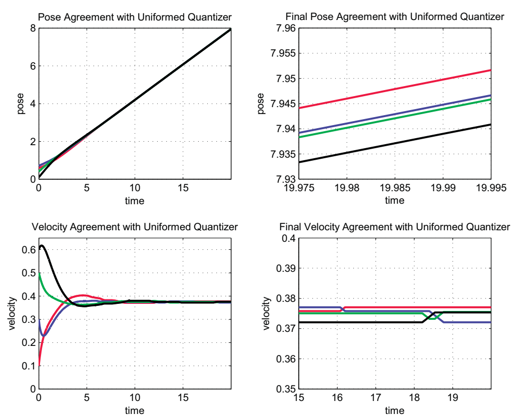

Fig. 5. Second order system with uniform quantizers.

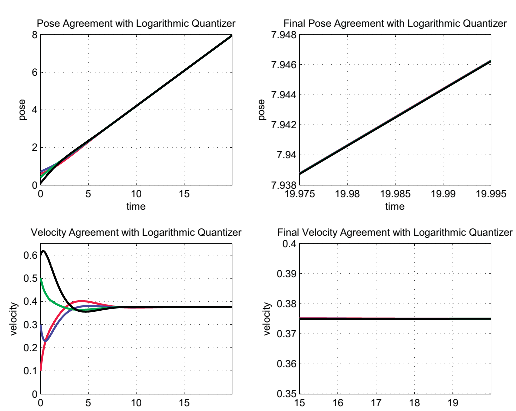

Fig. 6. Second order system with logarithmic quantizers.

VI. CONCLUSIONS

Stability of multi-agent systems under distributed control laws, composed of quantized value of relative states between neighboring agents, were considered. We distinguished between uniform and logarithmic quantizers as well as between static and time-varying communication topologies. The derived results are less conservative than our previous work with the same quantization constrains. It was established that a tree structure provides convergence guarantees in these cases. Similar conclusions were also shown to hold in the case of general undirected topologies. Second order dynamics were also taken into account and explicit convergence properties were obtained. Finally computer simulations supported the results.

REFERENCES

[1] R. Carli, F. Fagnani, and S. Zampieri. On the state agreement with quantized information. *17th Intern. Symp. Networks and Systems, pages 1500-1508, 2006*

[2] D. V. Dimarogonas and K. H. Johansson. Stability analysis for multi-agent systems using the incidence matrix: Quantized Communication and Formation Control, *Automatica*, Vol. 46, No. 4, pp. 695-700, April 2010.

[3] F. Fagnani, K. H. Johansson, A. Speranzon, and S. Zampieri. On multi-vehicle rendezvous under quantized communication. *16th Intern. Symp. Networks and Systems*, 2004

[4] A. Kashyap, T. Basar, and R. Srikant. Quantized consensus. *Automatica*, 43(7):1192-1203, 2007

[5] A. Nedic, A. Olshevsky, A. Ozdaglar, and J.N. Tsitsiklis. Distributed subgradient methods and quantization effects. *CDC*, 2008.

[6] D. Shevitz and B. Paden. Lyapunov Stability theory of nonsmooth Systems. *IEEE Transactions on Automatic Control*, 1994.

[7] B. E. Paden and S. S. Sastry. A calculus for computing Filippov's differential inclusion with application to the variable structure control of robot manipulators. *IEEE Transactions on Circuits and Systems*, 1987.

[8] F. Ceragioli, C. D. Persis, P. Frasca. Quantized average consensus: Discontinuities and hysteresis. *8th IFAC Symposium on Nonlinear Control Systems*, 2010

[9] D. Liberzon. Hybrid feedback stabilization of systems with quantized signals, *Automatica*, 39(9):15431554, 2003.

[10] D. V. Dimarogonas and K. J. Kyriakopoulos. On the rendezvous problem for multiple nonholonomic agents. *IEEE Transaction on Automatic Control*, 52(5):916-922, 2007

[11] C. Godsil and G. Royle. Algebraic Graph Theory. *Springer Graduate Texts in Mathematics*, 207, 2001

[12] R. Olfati-Saber and R. M. Murray. Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on Automatic Control*, 49(9):1520-1533, 2004

[13] J.A. Fax and R.M. Murray. Graph laplacian and stabilization of vehicle formations. *15th IFAC World Congress*, 2002

[14] W. Ren, E. Atkins. Distributed multi-vehicle coordinated control via local information exchange, *Int. J. Robust Nonlinear Control*, 2007;17:1002-1033

[15] R. A. Horn, C. R. Johnson. Matrix Analysis. *Cambridge University Press*, 1990

[16] F. Ceragioli, C. D. Persis and P. Frasca. Discontinuities and hysteresis in quantized average. *Automatica*, 2010.

[17] D. Libezon. *Switching in Systems and Control*. Birkhäuser, Boston, 2003

[18] A. Speranzon. Consensus and communication in multi-Robot control systems. *PhD Thesis*, KTH, 2006

[19] J. Lygeros, K. H. Johansson, S. N. Simic, J. Zhang, and S. S. Sastry. Dynamical properties of hybrid automata. *Transaction on Automatic Control*, 48(1), 2003 ,