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A Polynomial LPV Approach for Flexible Robot End-Effector Position Controller Analysis

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Abstract— **This paper presents an application of polynomial linear parameter varying (LPV) methods based on matrix sum-of-squares (SOS) relaxations for the end-effector position controller analysis of flexible-link manipulators. The proposed approach exploits an effective way for solving polynomial parameter-dependent linear matrix inequalities (PD-LMIs) and allows to consider more general admissible sets than hyper-rectangles or convex polytopes. This leads to less conservative results when considering an H_∞ output feedback controlled system. In particular, some performance analysis results are presented. A practical case study shows the effectiveness of the proposed methodology.**

I. INTRODUCTION

The control of robotic manipulators is a challenging research area that has benefited from an extensive effort since several decades [1]. In many applications, the mechanical structure of the robot is supposed to be completely rigid and the synthesis of control laws is made based on this assumption. The rigidity can be reinforced by using appropriately chosen materials or by performing a posteriori treatment of the existing structure. However, when large control torques are involved or when the control bandwidth is high, the flexibility effects become significant and they must be taken into account in the control algorithm.

One can distinguish two main classes of flexibility: joint flexibility and link flexibility. In the former, the elasticity is concentrated in the joints of the robot whereas in the latter, the elastic deformation is distributed along the whole mechanical structure. Typical examples of flexible link robots are the *lightweight robots* that can be found in aerospace [2] and medical [3] applications. When the flexible robot has more than one link, its dynamic model is nonlinear. Moreover, the presence of lightly damped flexible modes as well as the underactuated character of the control system (the deformation variables are neither measured nor actuated) make the problem of accurate tracking of a reference signal become very difficult. Solutions for trajectory tracking at the joint level have been proposed in the litterature (see for instance [4]), whereas direct position control of the end-effector of a flexible robot remains a difficult problem [5].

Our research objective is the development of a complete methodology for the linear parameter varying (LPV) identification [6] and control [7] of flexible robot manipulators. Our contribution herein is concerned with modeling and control.

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In our paper, we propose the use of a polynomial LPV approach for the end-effector control of a flexible manipulator. Based on an identified polynomial LPV model, we show how to carry out a performance analysis of the closed-loop system obtained using a linear H_∞ controller. The proposed controller analysis conditions are based on a matrix sum-of-squares (SOS) formulation [8] of some classical linear matrix inequality (LMI) results [9]. The interested reader can find in [10] a recent overview on the use of polynomial methods, in particular the SOS-based ones, in control applications. A key feature of the proposed approach lies in the fact that it offers the possibility to describe the admissibile domain of the varying parameters (and therefore the workspace of the robot), by using a collection of some (polynomial) semi-algebraic sets. In fact, a tighter approximation of the real operating conditions can lead to a significant reduction of conservatism in the analysis and synthesis of control laws, and as a consequence, to obtain better performance indices. We focus herein on the performance analysis problem as a first step towards the LPV controller synthesis problem.

Our paper is organized as follows. In the second section, we present the considered application that consists of a robotic manipulator with two flexible links. The proposed LPV modeling of the system is detailed in the third section. Section four is devoted to the development of LPV analysis conditions using matrix SOS relaxations. Simulation results are presented in section five, whereas the sixth section concludes our paper. Notations: $A \succ (\succeq) 0$ and $A \prec (\preceq) 0$ denote a positive (semi-)definite matrix and a negative (semi-)definite matrix respectively. A^T is the transpose of A. I_n stands for the identity matrix of dimension n. $*$ is used for the blocks induced by symmetry and $He\{A\}$ means $A + A^T$. $\otimes$ represents the Kronecker product and *diag* is used for block diagonal concatenation of matrices.

II. CONSIDERED FLEXIBLE ROBOT

A. Kinematics

A schematic view of the flexible robot that is considered as an application for our study is displayed in Figure 1. This case study is inspired by the second and third degrees-of-freedom (DOF) of a 6 DOF medical robot prototype [11].

In addition to the joint angular positions θ_1 and θ_2 that are measured using encoders, a direct measurement of the cartesian coordinates $F = [X\ Y]^T$ of the end-effector is provided by a video camera. The workspace of the robot is horizontal and only the transverse deformations $w_1(x_1,t), x_1 \in [0, l_1]$ and $w_2(x_2,t), x_2 \in [0, l_2]$ are considered.

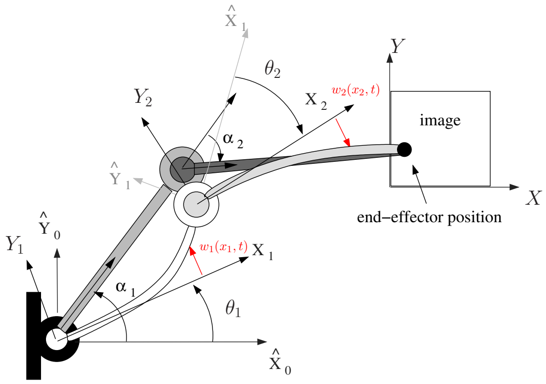

Fig. 1. Schematic view of the flexible robot

B. Nonlinear dynamic model

According to the assumed-modes approach [12], a dynamic model of the flexible manipulator can be obtained using Euler-Lagrange equations of motion. Such a model exhibits a second order behaviour:

$$M(q)\ddot{q}+D\dot{q}+Kq+C(q,\dot{q})+\Gamma(q)=G\tau \qquad (1)$$

where q is the vector of generalized coordinates. $M(q)$, D and K are the inertia, damping and stiffness matrices respectively, $C(q,\dot{q})$ and $\Gamma(q)$ are the Coriolis/centripetal and the gravitational torques vectors and G is an input matrix. The vector of torques τ is considered as the input of the system.

The deformation in any point of the link is considered as a sum of elementary deformations. Each of them represents a flexible mode that is characterized by a shaping function $\phi_{ki}(x_k)$ and a time-varying amplitude $\delta_{ki}(t)$:

$$w_k(x_k,t)=\sum_{i=1}^{n_d}\phi_{ki}(x_k)\delta_{ki}(t),\, x_k\in[0,l_k],\, k=1,2 \qquad (2)$$

Herein, we make the choice of monomial shaping functions $\phi_{ki}(x_k)=x_k^{i+1}$ of order greater than or equal to two in order to preserve the continuity and the smoothness of the bending deformation. Furthermore, we assume that $n_d=1$, i.e. only the first flexible mode is considered. The resulting vector of generalized coordinates is then $q=[\theta^T\,\delta^T]^T$, where $\theta=[\theta_1\,\theta_2]^T$ and $\delta=[\delta_{11}\,\delta_{21}]^T$. The gravitational torques vector $\Gamma(q)$ in (1) can be neglected because of the horizontal workspace of the robot. Using the kinematics of the robot, as depicted in Figure 1, the end-effector coordinates $F=[X\,Y]^T$ are given by:

$$\begin{aligned}X=&\cos(\theta_1)\left(l_1-2/3\,\delta_{11}^2 l_1^3\right)-\sin(\theta_1)\,\delta_{11}\,l_1^2\\&+\cos(\theta_1+\theta_2+2\,\delta_{11}\,l_1)\left(l_2-2/3\,\delta_{21}^2 l_2^3\right)\\&-\sin(\theta_1+\theta_2+2\,\delta_{11}\,l_1)\,\delta_{21}\,l_1^2\end{aligned} \qquad (3)$$

$$\begin{aligned}Y=&\sin(\theta_1)\left(l_1-2/3\,\delta_{11}^2 l_1^3\right)+\cos(\theta_1)\,\delta_{11}\,l_1^2\\&+\sin(\theta_1+\theta_2+2\,\delta_{11}\,l_1)\left(l_2-2/3\,\delta_{21}^2 l_2^3\right)\\&+\cos(\theta_1+\theta_2+2\,\delta_{11}\,l_1)\,\delta_{21}\,l_1^2.\end{aligned} \qquad (4)$$

An internal controller K_p with a proportional effect on the joint velocities is implemented:

$$\tau=K_p(\dot{\theta}^*(t)-\dot{\theta}(t)) \qquad (5)$$

This low-level control loop simplifies the control issue of the system and reduces the effects of some nonlinearities such as Coulomb friction. This leads to consider the velocity reference trajectory $\dot{\theta}^*(t)$ as the new input of the system.

III. LPV MODELING

A. State-space LPV model

Several techniques allow to obtain a reliable LPV model that accurately describes the behaviour of a dynamic system [13]. One can distinguish the analytical models that are based on the laws of physics from the identified models that are obtained by selecting an appropriate structure for the LPV system and using experimental data.

In our work, we consider the identified LPV model:

$$\Sigma(s,\rho):\begin{cases}\dot{x}(t)\,=A(\rho_1(t))x(t)+B(\rho_1(t))\dot{\theta}^*(t)\\ y(t)\,=C_0x(t)\end{cases} \qquad (6)$$

where $\rho_1(t)=\cos(\theta_2(t))$ is the time-varying parameter. The state matrices have the following expressions:

$$\begin{aligned}&A(\rho_1)=A_0+\rho_1A_1+\rho_1^2A_{11}\\&B(\rho_1)=B_0+\rho_1B_1+\rho_1^2B_{11}\\&C_0 \text{ is constant.}\end{aligned}$$

The state vector is $x(t)=[\delta(t)^T\,\dot{\theta}(t)^T\,\dot{\delta}(t)^T]^T$. The time-dependence of the parameters is omitted from now on for simplicity. The LPV model (6) has been obtained under the hypothesis of relatively low joint velocities. For the sake of simplicity, the output $y(t)$ is chosen as the joint velocities of a fictitious rigid robot whose end-effector position is $F=[X\,Y]^T$. The corresponding virtual positions α can be obtained simply by setting to zero the deformation variables δ_{i1}, $i=1,2$ in the expression of $F=g(q)$ given in $(4)-(5)$. The resulting *rigid* kinematics expression is denoted $g_0(\theta)$. The rigid fictitious velocity $\dot{\alpha}$ is related to the end-effector velocity $\dot{F}$ by the formula: $\dot{\alpha}=J_0^{-1}(\theta)\dot{F}$ where $J_0(\theta)=\frac{d}{d\theta}(g_0(\theta))$ is the rigid Jacobian. The output of model (6) is simply $y(t)=\dot{\alpha}=C_0x(t)$ with $C_0=\begin{bmatrix}0&0&1&0&l_1&0\\0&0&0&1&l_1&l_2\end{bmatrix}$.

B. Model augmentation for end-effector position control

The LPV model (6) can be augmented in order to directly control the end-effector position of the robot, as depicted in Figure 2. By introducing the new varying parameters $\rho_2=\cos(\theta_1)$, $\rho_3=\sin(\theta_2)$ and $\rho_4=\sin(\theta_1)$, the augmented LPV model can be taken as:

$$\begin{bmatrix}\dot{x}\\ \dot{F}\end{bmatrix}=\begin{bmatrix}A(\rho)&0_{6\times2}\\C(\rho)&0_2\end{bmatrix}\begin{bmatrix}x\\F\end{bmatrix}+\begin{bmatrix}B(\rho)\\0_2\end{bmatrix}\dot{\theta}^*(t) \qquad (7)$$

The output of the augmented system is the end-effector position $\tilde{y}=F=\tilde{C}x$. Equation (7) can be rewritten in a compact way as: $\dot{\tilde{x}}=\tilde{A}(\rho)\tilde{x}+\tilde{B}(\rho)\dot{\theta}^*$ where $\tilde{x}=[x^T\,F^T]^T$. The vector of varying parameters is $\rho=[\rho_1\,\rho_2\,\rho_3\,\rho_4]^T$.

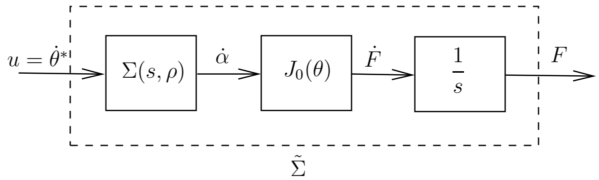

Fig. 2. Augmented LPV model

While matrices $A(\rho)$ and $B(\rho)$ are the same as in (6), the matrix $C(\rho)$ is obtained in the following way. The rigid kinematics of the robot gives the relation: $\dot{F} = J_0(\theta)\dot{\alpha} = J_0(\theta)C_0 x$. Actually, the rigid Jacobian of the 2-links planar manipulator has the expression:

$$J_0(\theta) = \begin{bmatrix} -l_1 \sin(\theta_1) - l_2 \sin(\theta_1+\theta_2) & -l_2 \sin(\theta_1+\theta_2) \\ l_1 \cos(\theta_1) + l_2 \cos(\theta_1+\theta_2) & l_2 \cos(\theta_1+\theta_2) \end{bmatrix} \quad (8)$$

Using the four varying parameters previously defined, equation (8) reads:

$$J_0(\rho) = \begin{bmatrix} -l_1\rho_4 - l_2(\rho_1\rho_4 + \rho_2\rho_3) & -l_2(\rho_1\rho_4 + \rho_2\rho_3) \\ l_1\rho_2 + l_2(\rho_1\rho_2 - \rho_3\rho_4) & l_2(\rho_1\rho_2 - \rho_3\rho_4) \end{bmatrix} \quad (9)$$

Therefore, $\dot{F} = C(\rho)x$, with $C(\rho) = J_0(\rho)C_0$.

As a result, we have obtained an LPV model $\tilde{\Sigma}(s,\rho)$: $\{\tilde{A}(\rho), \tilde{B}(\rho), \tilde{C}, 0\}$ where the state matrices $\tilde{A}(\rho)$ and $\tilde{B}(\rho)$ are polynomial with respect to the varying parameters and the output matrix $\tilde{C}$ is constant. This model may be used for the analysis and synthesis of controllers for the tracking of an operational space reference trajectory $F^*(t) = [X^*(t)\, Y^*(t)]^T$.

C. Modeling of the robot workspace

The workspace of the robot can be modeled by the admissible sets of the varying parameters of the LPV system $\tilde{\Sigma}(s,\rho)$. Clearly, these varying parameters are not independent. They are interrelated pairwise by *exact* trigonometric relations:

$$G_1 = \{(\rho_1,\rho_3) \in \mathbb{R}^2 : g_1(\rho) = \rho_1^2 + \rho_3^2 - 1 = 0\} \quad (10)$$

$$G_2 = \{(\rho_2,\rho_4) \in \mathbb{R}^2 : g_2(\rho) = \rho_2^2 + \rho_4^2 - 1 = 0\} \quad (11)$$

Given the maximal joint velocities V_{M_1} and V_{M_2}, the admissible set of the parameter derivatives $\mathscr{S}_{\dot{\rho}}$ can be described by the following semi-algebraic sets:

$$G_3 = \{(\dot{\rho}_1,\dot{\rho}_3) \in \mathbb{R}^2 : g_3(\rho) = \dot{\rho}_1^2 + \dot{\rho}_3^2 - V_{M_2}^2 \leq 0\} \quad (12)$$

$$G_4 = \{(\dot{\rho}_2,\dot{\rho}_4) \in \mathbb{R}^2 : g_4(\rho) = \dot{\rho}_2^2 + \dot{\rho}_4^2 - V_{M_1}^2 \leq 0\} \quad (13)$$

In brief, the admissible domain of the parameters and their time-derivatives are defined as: $\mathscr{S}_\rho = G_1 \cup G_2$ and $\mathscr{S}_{\dot{\rho}} = G_3 \cup G_4$.

The sets (10) and (11) can be modified in order to model some robustness properties. For instance, inexact measurement of the varying parameters can be considered. For this purpose, the unit circle within which evolve the parameters in G_1 and G_2 is replaced by a circular band of width $\varepsilon \in [0\,1]$ (tolerance on the parameters measurement) and of mean radius 1. Therefore, the resulting admissible sets are defined as the intersection of some newly defined semi-algebraic sets. $G_1 = G_{11} \cap G_{12}$ and $G_2 = G_{21} \cap G_{22}$, where:

$$G_{11} = \{(\rho_1,\rho_3) \in \mathbb{R}^2 : g_{11}(\rho) = \rho_1^2 + \rho_3^2 - R_M^2 \leq 0\} \quad (14)$$

$$G_{12} = \{(\rho_1,\rho_3) \in \mathbb{R}^2 : g_{12}(\rho) = -\rho_1^2 - \rho_3^2 + R_m^2 \leq 0\} \quad (15)$$

$$G_{21} = \{(\rho_2,\rho_4) \in \mathbb{R}^2 : g_{21}(\rho) = \rho_2^2 + \rho_4^2 - R_M^2 \leq 0\} \quad (16)$$

$$G_{22} = \{(\rho_2,\rho_4) \in \mathbb{R}^2 : g_{22}(\rho) = -\rho_2^2 - \rho_4^2 + R_m^2 \leq 0\}. \quad (17)$$

$R_m = 1 - \frac{\varepsilon}{2}$ and $R_M = 1 + \frac{\varepsilon}{2}$ are the minimal and the maximal radius of the circular band respectively.

It is well known that the LPV analysis and controller synthesis conditions that are based on the resolution of parameter-dependent LMIs generally lead to an infinite dimensional LMI feasibility problem. Classical control approaches mostly consider hyperrectangles or more general convex polytopes for the admissible sets, which allows to solve the problem on the vertices based on some convexity or multiconvexity properties [14]. More recently, some specific methods for polynomial LPV systems have been proposed [15], [16]. These methods allow to consider admissibility regions described by semi-algebraic sets that are similar to those given in (12)-(17). The approach considered herein is based on appropriate matrix SOS relaxations of the original analysis or synthesis semi-definite program (SDP).

IV. Analysis using matrix SOS relaxations

A. Matrix sum-of-squares

1) Scalar positivity: A sum-of-squares decomposition [17] of a scalar multivariate polynomial $f(\vartheta) = f(\vartheta_1, ..., \vartheta_n)$, $\vartheta_i \in \mathbb{R}$ is given by:

$$f(\vartheta) = \sum_{k=1}^{m} (f_k(\vartheta))^2 \quad (18)$$

where $f_k(\vartheta)$, $k = 1, .., m$ are given polynomials. Obviously, the existence of such a decomposition implies the global nonnegativity of the scalar polynomial $f(\vartheta)$.

2) Matrix positivity: Let us consider the following *spectral factorization* of a symmetric matrix $S(\vartheta)$:

$$S(\vartheta) = H^T(\vartheta) Q H(\vartheta). \quad (19)$$

Given the monomial matrix $H(\vartheta)$, the existence of a symmetric matrix $Q \succeq 0$ satisfying (19) is a necessary and sufficient condition for (19) to be an SOS decomposition, which in turn, guarantees the global positive semi-definiteness of the matrix $S(\vartheta)$. The matrix SOS formulation (19) is very interesting in that it transforms the problem of proving the global positive semi-definiteness of a parameter-dependent matrix $S(\vartheta)$ into a problem of proving the positive semi-definiteness of a constant matrix Q. This transformation is referred to as *matrix SOS relaxation*.

3) Positivity over a specified domain: Matrix SOS relaxation (19) considers the global positivity of a parameter-dependent matrix $S(\vartheta)$, where ϑ typically stands for the parameters and their time-derivatives $\vartheta = (\rho, \dot{\rho})$. In order to restrict the positive semi-definiteness condition to a domain described by the semi-algebraic sets given in (12)-(17), it is possible to use the following conditions that are obtained using the weak Lagrange duality [15]. Let us introduce:

$$S'(\vartheta) = S(\vartheta) + \sum_{j=1}^{N} Z_j g_j(\vartheta) \qquad (20)$$

If matrix $S'(\vartheta)$ is SOS (i.e. if it admits an SOS decomposition of the form (19)), where the Z_j, $j = 1,..,N$ are positive semi-definite symmetric matrices, then $S(\vartheta) \succeq 0$ for all the values of ϑ that satisfy $g_j(\vartheta) \leq 0$, $\forall j = 1,..,N$.

B. Application to H_∞ controller analysis

In several application fields such as medical robotics, because of the small workspace of the robot, a common control approach is the synthesis of an H_∞ controller for a nominal operating point [11]. In such a case, the obtained stability guaranty and performance index are valid in the vicinity of this operating point only. It may be interesting, however, to assess the effectiveness (stability, performance, robustness) of the control design over a larger operating space. It is the objective of an *a posteriori* H_∞ analysis.

1) Controller synthesis around an operating point: Let us consider an operating point described by a nominal value of the vector of varying parameters $\rho^0 = [\rho_1^0\ \rho_2^0\ \rho_3^0\ \rho_4^0]^T$. Based on the corresponding linear model $\tilde{\Sigma}_0(s) : \{\tilde{A}_0, \tilde{B}_0, \tilde{C}_0, 0\}$, a linear time-invariant (LTI) controller $K_0(s)$ can be synthesized using well-known control methods [18] [19]. We use the following generalized plant description of the system (Figure 3) containing a performance channel whose input is $w(t) \in \mathbb{R}^{n_w}$ and whose output is $z(t) \in \mathbb{R}^{n_z}$. The augmented state vector is $x_1(t) = [x^T(t)\ x_w^T(t)]^T$, where $x_w(t)$ contains the states of the weighting filter $W_1(s)$.

$$\Sigma_1(s) : \begin{cases} \dot{x}(t) &= \tilde{A}_0 x(t) + B_w w(t) + \tilde{B}_0 \dot{\theta}^*(t) \\ z(t) &= C_z x(t) + D_w w(t) \\ e_z(t) &= F^*(t) - F(t) \end{cases} \qquad (21)$$

By an adequate tuning of the filter $W_1(s)$, this scheme allows to impose the following performance specifications (see Section V): a minimum modulus margin M_{mod}, a minimum bandwidth ω_c and a maximum relative position error E_p [20].

2) Analysis of the closed-loop over a larger domain: When an LTI controller $K_0(s) : \{A_{K_0}, B_{K_0}, C_{K_0}, 0\}$ is used in interconnection with the LPV system $\tilde{\Sigma}(s,\rho)$, the weighted closed-loop system is given by $\Sigma'_{Cl}(s,\rho)$: $\{A'_{Cl}(\rho), B'_{Cl}, C'_{Cl}, D'_{Cl}\}$ where the state vector is $x'_{Cl}(t) = [x_1^T(t)\ x_K^T(t)]^T$ of dimension n'. $x_K(t)$ contains the states of the controller. The analysis scheme is obtained by replacing $\tilde{\Sigma}_0(s)$ by $\tilde{\Sigma}(s,\rho)$ in Figure 3. The proposed approach relies on the LPV version of the well-known bounded real lemma that is recalled in the following theorem.

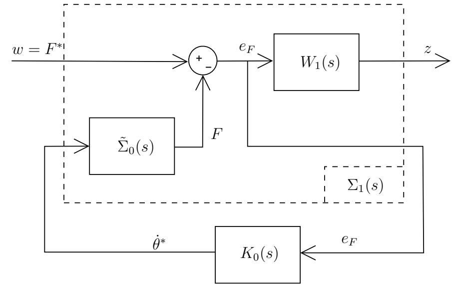

Fig. 3. H_∞ nominal controller synthesis scheme

Theorem 4.1 *(Stability and induced $\mathscr{L}_2$ performance [21]):* The weighted closed-loop system $\Sigma'_{Cl}(s,\rho)$ is stable and has an induced $\mathscr{L}_2$ performance index less than a positive scalar γ if there exists a matrix $X(\rho) = X^T(\rho) \succ 0$ that satisfies the following parameter-dependent LMI, $\forall(\rho,\dot{\rho}) \in \mathscr{S}_\rho \times \mathscr{S}_{\dot{\rho}}$:

$$\begin{bmatrix} He\{X(\rho)A'_{Cl}(\rho)\} + \sum_k \dot{\rho}_k \frac{\partial X(\rho)}{\partial \rho_k} & * & * \\ {B'_{Cl}}^T X(\rho) & -\gamma I & * \\ C'_{Cl} & D'_{Cl} & -\gamma I \end{bmatrix} \prec 0 \qquad (22)$$

LMI (22) is denoted $\mathscr{M}(\rho,\dot{\rho}) \prec 0$ in the sequel.

The H_∞ analysis problem is infinite dimensional because of the parametric dependence of $\mathscr{M}(\rho,\dot{\rho})$, $(\rho,\dot{\rho}) \in \mathscr{S}_\rho \times \mathscr{S}_{\dot{\rho}}$. Nevertheless, an SOS relaxation based on the form (20) can be used in order to express (22) as a finite dimensional SDP problem, as given in the following corollary.

Corollary 4.1: If the matrix $\mathscr{M}'(\rho,\dot{\rho})$ that is defined as:

$$\mathscr{M}'(\rho,\dot{\rho}) = -\mathscr{M}(\rho,\dot{\rho}) + \sum_{j=1}^{N} Z_j g_j(\rho,\dot{\rho}) - \lambda I \qquad (23)$$

where $Z_j = Z_j^T \succeq 0$ and $g_j(\rho,\dot{\rho})$ are negative semi-definite polynomials over $\mathscr{S}_\rho \times \mathscr{S}_{\dot{\rho}}$, admits an SOS decomposition of the form: $\mathscr{M}'(\rho,\dot{\rho}) = H^T(\rho,\dot{\rho})QH(\rho,\dot{\rho})$, with $Q = Q^T \succeq 0$ and $\lambda > 0$ is a scalar then $\mathscr{M}(\rho,\dot{\rho}) \prec 0$, $\forall(\rho,\dot{\rho}) \in \mathscr{S}_\rho \times \mathscr{S}_{\dot{\rho}}$.

Remark 4.1: An important issue of the SOS relaxation approach is the selection of an appropriate matrix $H(\rho,\dot{\rho})$ for the spectral factorization of $\mathscr{M}'(\rho,\dot{\rho})$. This choice influences the size of the final LMI condition $Q = Q^T \succeq 0$ to be implemented.

Remark 4.2: In order to reduce the *relaxation gap*, i.e. to get closer to the optimal value of the linear objective γ of the original parameter-dependent SDP problem (22), several extensions of the conditions of Corollary 4.1 have been proposed [22]. It is possible, for instance, to consider parameter-dependent multipliers $Z_j(\rho,\dot{\rho})$ in (23). Furthermore, the spectral factorization $\mathscr{M}'(\rho,\dot{\rho}) = H^T(\rho,\dot{\rho})(Q+N)H(\rho,\dot{\rho})$ can be used, where N is any symmetric matrix satisfying $H^T(\rho,\dot{\rho})NH(\rho,\dot{\rho}) = 0$. In that case, the LMI condition to be implemented would be $Q + N \succeq 0$.

3) Proposed implementation: In order to simplify the developments, let us consider a constant Lyapunov matrix $X = X^T \succ 0$. The proposed approach relies on the choice of $H(\rho) = \mathrm{diag}(H_1(\rho), I_2, I_2)$ for the spectral factorization, where $H_1^T(\rho) = [I_{n'} \;\; \rho_1 I_{n'} \;\; \rho_2 I_{n'} \;\; \rho_3 I_{n'} \;\; \rho_4 I_{n'}]$. Thus, we have:

$$\begin{aligned} \mathscr{M}(\rho) &= H^T(\rho) Q H(\rho) \\ &= \begin{bmatrix} H_1^T(\rho) Q_{11} H_1(\rho) & H_1^T(\rho) Q_{21}^T & H_1^T(\rho) Q_{31}^T \\ Q_{21} H_1(\rho) & Q_{22} & Q_{32}^T \\ Q_{31} H_1(\rho) & Q_{32} & Q_{33} \end{bmatrix} \end{aligned} \tag{24}$$

Numerical software is available for performing the spectral factorization, mainly involving scalarization [23]. In our our work, the decomposition is done manually as described in the following. Using (22), a monomial decomposition of each $(i,j)-$block $\mathscr{M}_{ij}$, $i, j = 1..3$ of the matrix $\mathscr{M}(\rho)$ is:
$\mathscr{M}_{11} = He\{XA'_{Cl}(\rho)\} = He\{XA'_0 + XA'_1\rho_1 + XA'_2\rho_2 + XA'_4\rho_4 + XA'_{11}\rho_1^2 + XA'_{12}\rho_1\rho_2 + XA'_{14}\rho_1\rho_4 + XA'_{23}\rho_2\rho_3 + XA'_{34}\rho_3\rho_4\}$, where the constant matrices $A'_0, A'_1, A'_2, A'_4, A'_{11}, A'_{12}, A'_{14}, A'_{23}$ and $A'_{34} \in \mathbb{R}^{n' \times n'}$ result from a monomial decomposition of $A'_{Cl}(\rho)$.

$H_1^T(\rho) Q_{11} H_1(\rho)$ is a spectral factorization of $\mathscr{M}_{11}$ where: $Q_{11} =$

$$\begin{bmatrix} He\{XA'_0 - \sigma_{13} - \sigma_{24}\} & XA'_1 & XA'_2 & 0 & XA'_4 \\ * & He\{XA'_{11} + \sigma_{13}\} & XA'_{12} & 0 & XA'_{14} \\ * & * & He\{\sigma_{24}\} & XA'_{23} & 0 \\ * & * & * & He\{\sigma_{13}\} & XA'_{34} \\ * & * & * & * & He\{\sigma_{24}\} \end{bmatrix} \tag{25}$$

Let us point out that additional matrix variables σ_{13} and $\sigma_{24} \in \mathbb{R}^{n' \times n'}$ have been introduced in order to avoid the occurence of zero diagonal terms in Q_{11} that would make the condition $Q \succeq 0$ become impossible to satisfy. Introducing these new variables has no effect on the value of $\mathscr{M}_{11}$ due to the algebraic relations between the varying parameters in (10)-(11). The other matrices of the decomposition are: $Q_{21} = \left[B'^{\,T}_{Cl} X \quad 0_{2\times 4n'}\right]$, $Q_{22} = -\gamma I_{n_w}$, $Q_{31} = \left[C'_{Cl}\; 0_{2\times 4n'}\right]$, $Q_{32} = D'_{Cl}$ and $Q_{33} = -\gamma I_{n_z}$.

The semi-algebraic sets G_{ij}, $i, j = 1..2$ in (14)-(17) admit the factorizations $H_1^T(\rho) \tilde{G}_{ij} H_1(\rho)$, where:

$$\begin{aligned} \tilde{G}_{11} &= \mathrm{diag}(-R_M^2 I_{n'}, I_{n'}, 0_{n'}, I_{n'}, 0_{n'}) \\ \tilde{G}_{12} &= \mathrm{diag}(R_m^2 I_{n'}, -I_{n'}, 0_{n'}, -I_{n'}, 0_{n'}) \\ \tilde{G}_{21} &= \mathrm{diag}(-R_M^2 I_{n'}, 0_{n'}, I_{n'}, 0_{n'}, I_{n'}) \\ \tilde{G}_{22} &= \mathrm{diag}(R_m^2 I_{n'}, 0_{n'}, -I_{n'}, 0_{n'}, -I_{n'}). \end{aligned} \tag{26}$$

Moreover, the terms $Z_{ij} g_{ij}(\rho)$ can be factorized as $H_1^T(\rho)(\tilde{G}_{ij} \otimes Z_{ij}) H_1(\rho)$. Finally, the implemented LMI condition is $-Q + \sum_{i,j=1}^{2} (\tilde{G}_{ij} \otimes Z_{ij}) - \lambda I \succeq 0$.

V. SIMULATIONS

The proposed methodology has been applied on an identified LPV model of the flexible robot of the form (6) where the length of each link is $l_1 = l_2 = 0.5$ m. A nominal H_∞ controller $K_0(s)$ is synthesized on the operating point $\rho^0 = \frac{\sqrt{2}}{2}[1\ 1\ 1\ 1]^T$ that corresponds to the robot configuration $(\theta_1, \theta_2) = (45°, 45°)$ and the end-effector position $F = [X\,Y]^T = [0.3536\,\mathrm{m}\ 0.8536\,\mathrm{m}]^T$. A one-block synthesis scheme is used (Figure 3), in which $W_1(s)$ ensures the closed-loop charactersitics: $M_{mod} = 0.65$, $\omega_c = 20\,\mathrm{rad.sec}^{-1}$ and $E_p = 10^{-2}$. When using the LMI synthesis method available in MATLAB, the obtained performance index is $\gamma_{syn} = 1.0581$. In Figure 4, the sensitivity function $S_y(s) = T_{r\to e}(s)$ is compared to the frequency template $\frac{\gamma}{W_1(s)}$. Figure 5 displays time simulations carried out using the LPV system $\tilde{\Sigma}(s,\rho)$ and the nominal controller $K_0(s)$.

Performance analysis of the previously described closed-loop system is performed following the procedure detailed in paragraph IV.B.3 and using the filter $W_1(s)$ employed for the synthesis. The LMI problem $Q \succeq 0$ that results from an SOS relaxation is solved using the numerical solver SeDuMi [24] associated with the YALMIP interface [25]. When applying the analysis algorithm to the whole admissible set $\mathscr{S}_\rho$ that is defined using (14)-(17), the LMI problem was found to be infeasible. This result is confirmed when carrying out time simulations in operating points that are different from the nominal one ρ^0. Indeed, the oscillatory modes have been observed to be very lightly damped and closed-loop instability may occur. This result points out the limitations of the nominal controller for the achievement of the desired performance requirements over a large operating range. In order to address this issue, we limit the admissible domain $\mathscr{S}_\rho$ to some arcs of circles that include the nominal operating point ρ^0 and its vicinity. In other words, we add the constraint $\rho \in [\underline{\rho} \quad \overline{\rho}]$ in the analysis conditions, where $\underline{\rho} = \rho^0(1-\nu)$ and $\overline{\rho} = \rho^0(1+\nu)$, $\nu > 0$. Figure 6 illustrates the considered analysis domain.

For instance, a choice of $\nu = 0.1$ corresponds to the workspace $\theta_1, \theta_2 \in [39°\ 51°]$. The performance indices γ_{ana} that are obtained by the analysis are reported on Table 1 for the values $\varepsilon = 0.01$ and $\varepsilon = 0.04$ of the measurement tolerance. Clearly, a degradation of the performance index that is guaranteed by the nominal synthesis is observed, due to the analysis over a larger domain. Another analysis is performed in the immediate vicinity of the nominal parameter ρ^0, by taking $\nu = 0.0001$. The obtained performance index γ_{ana}, that is reported in the last row of Table 1, indicates an upper bound of the optimal value of the linear objective γ that is slightly smaller than the γ_{syn} guaranteed by the nominal synthesis. The size of the LMI is 104×104 and the computaional time on an Intel Core 2 Duo processor is 264.17 seconds for the last test. These results demonstrate the effectiveness of the proposed analysis approach.

ν	ε	γ_{ana}
0.1	0.01	1.4209
0.1	0.04	1.4232
0.0001	0	1.0012

Table 1 - Performance analysis results: three different tests

VI. CONCLUSION

The main contribution of this paper is to propose a polynomial LPV methodology for the performance analysis of flexible robot controllers. This issue is addressed through the use an appropriate sum-of-squares relaxation. An important feature of the proposed approach is to consider semi-algebraic sets for the modeling of the parametric domain. Such a description provides a tighter approximation of the real workspace, and as a consequence, allows to obtain better upper bounds for the linear objective of the original semi-definite program. Future work will be devoted to the application of a similar approach for the LPV synthesis problem in order to a priori guarantee the performance requirements over the whole parametric domain.

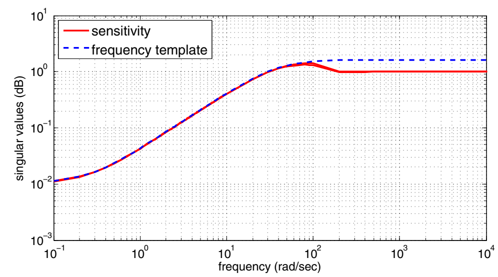

Fig. 4. Realized frequency transfers

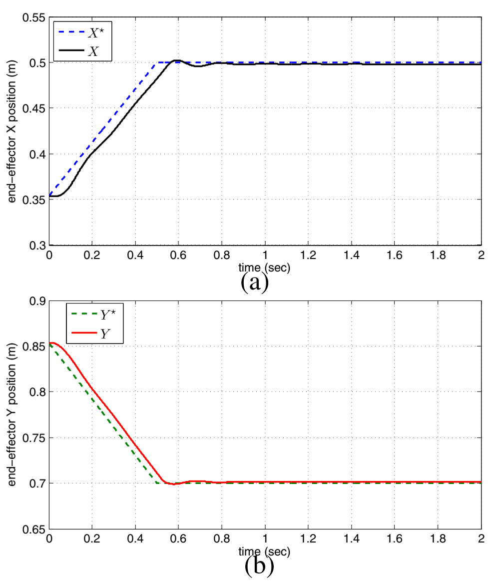

Fig. 5. Tracking of the reference trajectories

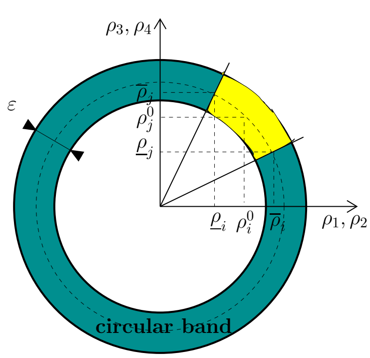

Fig. 6. Local analysis domain

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