2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

On the Zeros of Consensus Networks

Benjamin Briegel, Daniel Zelazo, Mathias Bürger, and Frank Allgöwer

***Abstract*— In this paper, we focus on the signal transfer between any two agents in a consensus network and provide an analysis of the resulting transfer function and transmission zeros. Our contributions include relating the distance between the input and output node to the relative degree of the system and showing that the gain factor of the transfer function is equal to the number of shortest paths between the two nodes. We show that for certain input-output configurations, the zeros interlace with the poles. Bounds on the zero locations and minimum phase properties of certain graph and input-output configurations are provided.**

I. Introduction

A multi-agent system consists of autonomous and dynamic units that interact over a specific network topology. Distributed systems of this type arise in numerous fields in science and engineering and have applications including formation control of multiple vehicles [1], [5], sensor networks [17], and flocking [18]. Consensus scenarios, where the agents agree upon a common objective, have been the subject of extensive research in recent years [12], [19]. These systems provide the most direct and elegant connection between networks (or graphs) and dynamical control systems.

Due to this interesting property, the agreement scenario has been widely studied in the past. While much attention has been given to autonomous set-ups for consensus, an important variation studies input-output properties of the protocol. In [5], conditions for the stability of the networked system when a decentralized consensus-based controller is applied are examined. A variation with a leader-follower setup, where some of the agents in the network do not abide by the consensus algorithm, has also been considered in several works [4], [11]. The controllability of this setup is directly associated with the symmetry structure of the underlying graph [21]. Network sensitivity functions of a more general setup, with the leader-follower setup as a special case, are the focus of [22]. Bounds on the performance of consensus based systems have been examined by [23] from the perspective of the edge Laplacian. An infiltration scenario studied in [2] considers the effectiveness and cost of network infiltration.

We consider a setup where all agents in the network apply the consensus protocol. One of the nodes is presumed manipulable through an additional input. We study the transfer of a signal from the manipulable agent to a single other agent in terms of graph properties. By studying the corresponding input-output system, we are able to relate properties of the underlying graph to transfer function properties.

The authors thank the German Research Foundation (DFG) for financial support within the Cluster of Excellence in Simulation Technology (EXC 310/1) at the University of Stuttgart and within the Priority Program 1305 "Control Theory of Digitally Networked Dynamical Systems". The authors are with the Institute for Systems Theory and Automatic Control, University of Stuttgart, Pfaffenwaldring 9, 70550 Stuttgart, Germany, {firstname.lastname}@ist.uni-stuttgart.de.

The results contributed in this paper show that (i) the relative degree of the open loop system is equal to one plus the distance between the influenced node and the observed node, (ii) the gain factor of the transfer function is equal to the number of shortest paths between these two nodes, and (iii) the impulse response of the system is inversely proportional to the number of agents. For certain graph structures and controller/observer pairs, we state that the associated system is minimum phase. Following this analysis, we study the impact of these results on the possibilities of an infiltrator to manipulate a consensus-based multi-agent system by only accessing two nodes of the network.

The remainder of the paper is structured as follows. In §II we provide a short discourse on the basics of graph theory and introduce our system setup. The main results of the paper are given in §III, where we describe some properties of the open-loop SISO system that results from our setup, including statements on the system zeros. In §IV, we apply our results to a network infiltration scenario. Finally, some concluding remarks are offered in §V.

II. Preliminaries and Problem Setup

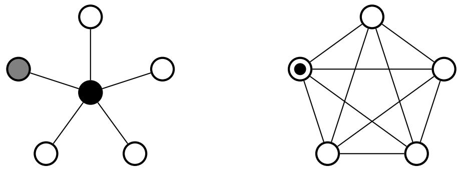
We recall some basic concepts in algebraic graph theory [6], and provide a model of our setup based on the well-known consensus algorithm.

A. Graph Theory

An undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is defined by the vertex (or node) set $\mathcal{V}$ with $n = |\mathcal{V}|$ and the edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. Two vertices i and j are adjacent if $\{i, j\} \in \mathcal{E}$; this is also indicated by $i \sim j$. The neighborhood of a vertex is the set of adjacent vertices, i.e. $\mathcal{N}(i) = \{j \in \mathcal{V} \,|\, i \sim j\}$, and $d_i = |\mathcal{N}(i)|$ is the degree of vertex i.

A path of length l is defined as a sequence of $l + 1$ distinct vertices, where successive vertices of the sequence are adjacent. The distance between two vertices i and j, $\mathrm{dist}(i, j)$, is the length of the shortest path between them. When the vertices of an adjacency sequence are distinct except for its end vertices, we refer to the sequence as a cycle. A connected graph without cycles is a tree. In this paper we will assume connected graphs in all cases.

The degree matrix $\mathcal{D}(\mathcal{G})$ is the diagonal $n \times n$ matrix with $[\mathcal{D}(\mathcal{G})]_{ii} = d_i$. The adjacency matrix $\mathcal{A}(\mathcal{G})$ is the symmetric $n \times n$ matrix defined by $[\mathcal{A}(\mathcal{G})]_{ij} = 1$ if $i \sim j$, and zero otherwise. Combining these matrices leads to the (combinatorial) graph Laplacian $\mathcal{L}(\mathcal{G}) = \mathcal{D}(\mathcal{G}) - \mathcal{A}(\mathcal{G})$. This is a positive semi-definite and symmetric matrix with the additional property that all rows sum to zero since $[\mathcal{D}(\mathcal{G})]_{ii} = \sum_j [\mathcal{A}(\mathcal{G})]_{ij}$. The multiplicity of the zero eigenvalue is equal to the number of connected components in the graph [6]. We assume the (real) eigenvalues of $\mathcal{L}(\mathcal{G})$ to be ordered as $0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$, and also refer to them as the eigenvalues of $\mathcal{G}$.



(a) The star graph S_6 with $v_{ctr} \neq v_{obs}$.

(b) The complete graph K_5 with $v_{ctr} = v_{obs}$.

Fig. 1: System setups where the control and observation vertices have been marked.

B. Infiltration Model

Our system model is based on the first-order linear consensus model [12]. Each agent has a scalar state $x_i(t) \in \mathbb{R}$ with integrator dynamics $\dot{x}_i(t) = u_i(t)$. The consensus algorithm is the n-th order linear system where every agent receives the state-delta from each of its neighbors as an input, i.e. $x_i(t) = \sum_{j \sim i} \big(x_j(t) - x_i(t)\big)$. This can be compactly expressed using the graph Laplacian as $\dot{x}(t) = -\mathcal{L}(\mathcal{G})x(t)$.

We now propose a single-input single-output setup based on the consensus algorithm. The input takes the form of a bias on one vertex - the control vertex, denoted v_{ctr} - that acts concurrently with the consensus algorithm. The output is taken to be the state of a single vertex that we term the observation vertex, denoted v_{obs}. We refer to the ordered pair (v_{ctr}, v_{obs}) as the transmission pair of the setup. This leads to the following SISO linear system,

$$\begin{aligned} \dot{x}(t) &= -\mathcal{L}(\mathcal{G})x(t) + bu(t) \\ y(t) &= c^T x(t), \end{aligned} \tag{1}$$

where b is a vector in which the component corresponding to the control vertex is 1 and all other components are zero. Equivalently, c is a vector in which the component corresponding to the observation vertex is 1 and all other components are zero. The control and observation vertex can be identical, in which case $b = c$. We graphically depict our setup by identifying the control vertex with a black node and the observation vertex with a gray node, as shown in Fig. 1-a. For the case when they are identical, we will mark the corresponding vertex with a dot (Fig. 1-b).

C. Transfer Function Representation

The input-output dynamics of the system (1) can be described by the transfer function, which is equivalently written in pole-zero-gain form,

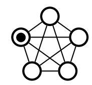
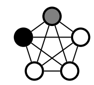
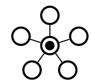
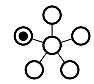
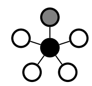
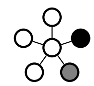
$$G(s) = c^T(sI + \mathcal{L})^{-1}b = k\,\frac{\prod_{i=0}^{m}(s - z_i)}{\prod_{i=0}^{n}(s - p_i)}. \tag{2}$$

The order n of the system is equal to the number of agents, and the poles correspond to the negative eigenvalues of the Laplacian $\mathcal{L}(\mathcal{G})$, which have been thoroughly studied [14]. Our main analytic results will be concerned with the remaining properties, i.e. the gain factor k, the number of zeros m, and their locations z_i.

We write $G_{ij}(s)$ for the transfer function which results when we have the transmission pair $(v_{ctr}, v_{obs}) = (i, j)$, and refer to the setup as $\mathcal{G}(i, j)$. Observe that due to the symmetry of $\mathcal{L}$, one has $G_{ij}(s) = G_{ji}(s)$.

For certain graph structures explicit analytic descriptions of the transfer functions are obtainable. The motivation for presenting these transfer-functions is to develop an initial intuition on the zero locations of the system. In particular, as presented in Table I, we observe that the transfer functions for the star graph, S_n, and the complete graph, K_n, are minimum phase. Table I shows these transfer functions for all possible transmission pairs. An interesting observation in these examples is that all the zeros are in the open left-half of the complex plane. A natural conjecture to consider, therefore, is whether this property extends to arbitrary graphs and transmission pairs. We examine this conjecture and some variations in the sequel.

TABLE I: Transfer Functions for K_n and S_n

	$\frac{s+1}{s(s+n)}$		$\frac{1}{s(s+n)}$
	$\frac{s+1}{s(s+n)}$		$\frac{(s+\frac{1}{2}(n+\sqrt{n^2-4}))(s+\frac{1}{2}(n-\sqrt{n^2-4}))}{s(s+1)(s+n)}$
	$\frac{1}{s(s+n)}$		$\frac{1}{s(s+1)(s+n)}$

III. Signal Transfer within a Network

In this section we establish connections between the graph topology, the location of the control and observation vertex in the graph, and the properties of the resulting transfer function. A study of input-output properties in terms of graph properties gives insights into the consensus problem beyond the classical studies.

A. Relative Degree and Transfer Gain

We begin with a study of the relative degree. The relative degree of a linear control system is the difference between the order of the denominator and numerator polynomials of a system transfer function. In particular, the number of transmission zeros of a SISO linear system can be derived from the system order and its relative degree. We can relate the relative degree of (1) to the length of the shortest path between the control and observation nodes of the system. First, we state a result concerning powers of the system matrix $-\mathcal{L}(\mathcal{G})$.

Lemma 3.1: Let $\mathcal{G}$ be a graph with the Laplacian matrix $\mathcal{L}(\mathcal{G})$ and the adjacency matrix $\mathcal{A}$. Then for $l \leq \operatorname{dist}(i, j)$,

$$[(-\mathcal{L}(\mathcal{G}))^l]_{ij} = [\mathcal{A}^l]_{ij} = \begin{cases} 0, & \text{for } l < \operatorname{dist}(i, j) \\ a, & \text{for } l = \operatorname{dist}(i, j), \end{cases} \tag{3}$$

where a is the number of shortest paths from node i to j.

Proof: Due to the symmetry of $\mathcal{A}$ and $\mathcal{D}$, we conclude that their product is commutative, i.e., $\mathcal{A}\mathcal{D} = \mathcal{D}\mathcal{A}$. The construction of the Laplacian allows for application of the binomial theorem [7], leading to $(\mathcal{A} - \mathcal{D})^l = \sum_{k=0}^{l}(-1)^k \binom{l}{k} \mathcal{A}^{l-k}\mathcal{D}^k$. As $\mathcal{D}$ and its powers are diagonal matrices, they only scale the elements of the respective powers of $\mathcal{A}$. Therefore, the zero elements of $\mathcal{A}^{l-k}$ and $\mathcal{A}^{l-k}\mathcal{D}^k$ coincide. Applying the "number of walks" Lemma 8.1.2 from [6] completes the proof. ■

We are now ready to state a result on the relative degree of the system (1).

Theorem 3.2: The relative degree of the system (1) is

$$r = \Delta + 1, \tag{4}$$

where $\Delta = \mathrm{dist}(v_{ctr}, v_{obs})$ is the length of the shortest path between the control vertex and the observation vertex.

Proof: The relative degree r of a SISO system is equal to the number of times one must differentiate the output $y(t)$ until the input $u(t)$ explicitly appears [10]. For $i \leq r$ the consecutive derivatives of the output y take the form

$$y^{(i)} = c^T(-\mathcal{L})^i x + c^T(-\mathcal{L})^{i-1} bu. \tag{5}$$

Application of Lemma 3.1 reveals that the factor $c^T(-\mathcal{L})^{i-1}b$ which scales the input u is zero until $i-1 = \Delta$. This term becomes non-zero precisely when $i = r$, and the result follows. ■

Corollary 3.3: The number of zeros in the system (1) is equal to $m = n - 1 - \Delta$.

We now relate the number of shortest paths between the control and observation node to the gain factor of (2).

Theorem 3.4: Consider a transmission pair (v_{ctr}, v_{obs}) on the graph $\mathcal{G}$ and its corresponding transfer function in zero-pole-gain form (2). Then the gain factor k is equal to the number of shortest paths from v_{ctr} to v_{obs}.

Proof: Assume (2) has relative degree r. We can express the system as a rational polynomial function, with k as the leading coefficient of the numerator polynomial,

$$G(s) = \frac{ks^{n-r} + b_{n-r-1}s^{n-r-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}. \tag{6}$$

The transfer function can also be derived from the state space representation (1) as follows

$$G(s) = c^T(sI + \mathcal{L})^{-1}b = \frac{c^T \,\mathrm{adj}(sI + \mathcal{L})b}{\det(sI + \mathcal{L})} \tag{7}$$

where $\det(sI + \mathcal{L}) = s^n + a_{n-1}s^{n-1} + \cdots + a_0$ is the characteristic polynomial of $-\mathcal{L}(\mathcal{G})$, and $\mathrm{adj}(sI + \mathcal{L})$ is the adjugate matrix. Combining equations (6) and (7) yields

$$ks^{n-r} + \cdots + b_0 = c^T \,\mathrm{adj}(sI + \mathcal{L})b. \tag{8}$$

According to the Leverrier-Faddeev algorithm [9], we can express an adjugate matrix of the form $\mathrm{adj}(sI - A)$ by using coefficient matrices B_i, so that $\mathrm{adj}(sI - A) = \sum_{i=0}^{n-1} B_i s^i$. In our case, these matrices take the form $B_{n-j} = (-\mathcal{L})^{j-1} - \sum_{\eta=1}^{j-1} \gamma_\eta (-\mathcal{L})^{j-1-\eta}$, with $\gamma_\eta \in \mathbb{R}$.

This yields $ks^{n-r} + \cdots + b_0 = \sum_{i=0}^{n-1} c^T B_i b s^i$. From a comparison of the coefficients of s, it follows that $c^T B_i b = 0$ for $i > n - r$, and in particular $k = c^T B_{n-r} b$. Expressing B_{n-r} as a polynomial of the form shown above leads to

$$\begin{aligned} k = c^T B_{n-r} b &= c^T \left((-\mathcal{L})^{r-1} - \sum_{\eta=1}^{r-1} \gamma_\eta (-\mathcal{L})^{r-1-\eta} \right) b \\ &= c^T(-\mathcal{L})^{r-1}b - \sum_{\eta=1}^{r-1} \gamma_\eta c^T (-\mathcal{L})^{r-1-\eta} b \end{aligned} \tag{9}$$

With $\Delta = \mathrm{dist}(v_{ctr}, v_{obs})$, and applying Theorem 3.2,

$$k = c^T(-\mathcal{L})^{\Delta}b - \sum_{\eta=1}^{\Delta} \gamma_\eta c^T (-\mathcal{L})^{\Delta-\eta} b. \tag{10}$$

Examining the powers of the system matrix $-\mathcal{L}(\mathcal{G})$, the vectors c^T and b select the component in the row corresponding to the observation vertex and the column corresponding to the control vertex. This allows us to apply Lemma 3.1 for each of the matrix powers. The sum in (10) becomes zero and we are left with $k = a$, where a is the number of shortest paths from the control to the observation node. ■

Corollary 3.5: The gain factor k for an arbitrary transmission pair (v_{ctr}, v_{obs}) when the underlying graph is a spanning tree is unity.

Studying the input-output behavior of the system, a natural question that arises concerns the impulse response of the system. Let $M_{(ij)}$ denote the submatrix of an arbitrary matrix M obtained by removing the i-th row and the j-th column. $M_{(ii)}$ is abbreviated to $M_{(i)}$.

Lemma 3.6: The impulse response $g(t)$ of the system given by the transfer function (2) has the limit

$$\lim_{t\to\infty} g(t) = \lim_{s\to 0} s\,G(s) = \frac{1}{n}, \tag{11}$$

where n is the number of agents.

Proof: Consider the transfer function

$$\breve{G}(s) = s\,G(s) = s\,\frac{c^T \,\mathrm{adj}(sI + \mathcal{L})b}{\det(sI + \mathcal{L})}. \tag{12}$$

The Laplace expansion for the determinant is used to conclude that its coefficient a_{n-k} is the sum of the determinants of the principal $k \times k$ submatrices of $\mathcal{L}$ [6, p. 284]. It follows that $a_0 = \det \mathcal{L} = 0$ and $a_1 = \sum_{v \in \mathcal{V}(\mathcal{G})} \det \mathcal{L}_{(v)}$. Since all the principal minors of $\mathcal{L}$ are the same and correspond to the number of spanning trees $t(\mathcal{G})$ in the graph (Matrix-Tree Theorem [6]), the coefficient of the linear term is $a_1 = n\,t(\mathcal{G})$. Using this fact and Lemma 13.2.3 in [6] that states $\mathrm{adj}(\mathcal{L}) = \mathbf{1}\mathbf{1}^T t(\mathcal{G})$, leads to

$$\lim_{s\to 0} \breve{G}(s) = \breve{G}(0) = \frac{c^T t(\mathcal{G}) \mathbf{1}\mathbf{1}^T b}{n\,t(\mathcal{G})} = \frac{1}{n}, \tag{13}$$

concluding the proof. ■

The results presented up to now indicate that by accessing only one pair of agents, significant information about the network can be obtained. The manipulated agents can estimate their distance in the graph and the number of shortest connections. Additionally, they can estimate the number of other agents in the network by letting one agent apply an impulse signal.

B. Transmission Zeros

All the presented results have directly related properties of the transfer functions to graph properties. However, the transfer behavior is determined to a large extent by the transmission zeros of the system. It is in general difficult to relate the location to the zeros explicitly to the graph properties (in the same way as it is hard to relate the poles of the system explicitly to the graph). Fortunately, some statements concerning the location of the transmission zeros can be made.

In general, the transmission zeros of a control system provide information about the frequency response and behavior of the system. Frequencies corresponding to zeros of the system do not appear at the output; furthermore, for every zero there exists a nontrivial input function $u_0(t)$ and corresponding initial condition $x(0)$, for which the output $y(t)$ remains zero for all time. A system is *minimum phase* if there are no zeros in the open right-half plane and no double zeros at the origin.

Theorem 3.7: The system (1) has no transmission zeros at the origin.

Proof: The numerator polynomial of the transfer function, as shown in (7), is $n(s) = c^T \operatorname{adj}(sI + \mathcal{L})b$. For $s = 0$ the numerator is $n(0) = c^T \operatorname{adj}(\mathcal{L})b = c^T \mathbf{1}\mathbf{1}^T t(\mathcal{G})b \neq 0$. Therefore 0 cannot be a root of $n(s)$. ■

We now consider the zeros of (1) when the control and observation vertex are the same.

Theorem 3.8: Assume that the control and observation vertex are identical. Then the poles p_i and zeros z_i of (1) interlace as

$$p_n \leq z_{n-1} \leq p_{n-1} \leq \cdots \leq z_1 < p_1 = 0, \tag{14}$$

with a strict inequality for the largest zero z_1 with respect to the pole at the origin. As such, the system is minimum phase.

Proof: Let the transmission pair of the system be $(v_{ctr}, v_{obs}) = (i, i)$. By Corollary 3.3, we have $n - 1$ zeros. The condition for zeros z_k simplifies to $\det(z_k I_{n-1} + \mathcal{L}_{(i)}) = 0$. The zeros are therefore equivalent to the eigenvalues of the matrix $-\mathcal{L}_{(i)}$. We will denote the eigenvalues of $\mathcal{L}_{(i)}$ in ascending order by $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{n-1}$. Applying the Interlacing Eigenvalues Theorem [8] it holds that

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \cdots \leq \mu_{n-1} \leq \lambda_n \tag{15}$$

Multiplying with -1 provides the non-strict inequalities in (14). The strict bound with respect to the origin follows directly from Theorem 3.7. ■

We now present results on the zeros for different control and observation node configurations and graph structures.

Theorem 3.9: Let the control and observation nodes be distinct. Then the system (1) has no transmission zeros on the open positive real line, $\mathbb{R}_{>0}$.

Proof: Without loss of generality, we assume that the control node is at the vertex 1, and the observation node at the vertex n.[1] The numerator polynomial of (7) is given as $n(s) = c^T \operatorname{adj}(sI + \mathcal{L})b$, which corresponds to the $(1n)$ cofactor of the matrix $sI + \mathcal{L}$. Thus, a necessary and sufficient condition for system zeros s is $\det\big((sI + \mathcal{L})_{(1n)}\big) = 0$.

The sub-matrix $(sI + \mathcal{L})_{(1n)}$ has the following structure,

$$\begin{aligned}(sI+\mathcal{L})_{(1n)} &= \left[\begin{array}{c|ccc} \{0,-1\} & d_2+s & \cdots & \{0,-1\} \\ \vdots & \vdots & \ddots & \vdots \\ \{0,-1\} & \{0,-1\} & \cdots & d_{n-1}+s \\ \hline \{0,-1\} & \{0,-1\} & \cdots & \{0,-1\} \end{array}\right] \\ &= \left[\begin{array}{c|c} q & Q(s) \\ \hline d & r^T \end{array}\right] = H(s).\end{aligned}$$

Here, the notation $\{0, -1\}$ denotes that the corresponding entry can only take the value 0 or -1. Note also that $d \in \{0, -1\}$ is a scalar, $Q(s)$ is a $(n-2) \times (n-2)$ matrix, and q, r are $(n-2)$ element vectors.

The submatrix components q, r, and d have a direct interpretation with the underlying graph of the system. In particular, we note that $d = -1$ only if there exists an edge between the control and observation node (i.e. $\{v_{ctr}, v_{obs}\} \in \mathcal{E}$), and 0 otherwise. The vectors q and r give information on direct connections between the control and observation node with the remaining nodes in the graph. For example, the kth element of q is -1 only if there is an edge between the corresponding node and the control node, and is 0 otherwise. Similarly, the kth element of r is -1 only if there is an edge between the corresponding node and the observation node. Consequently, we note that the inner-product $r^T q$ is non-zero whenever there is a node connected to *both* the control and observation node, and 0 otherwise. In fact, this quantity gives the number of nodes connected to both the control and observation node. This further implies that when $d = 0$ and $r^T q \neq 0$, the length of a shortest path must be 2, and the number of these paths is $r^T q$.

We now show by contradiction, that there can be no zeros on the open positive real line $\mathbb{R}_{>0}$. Assume that $s \in \mathbb{R}_{>0}$ is a root of $n(s)$. This implies that there exists a non-zero vector x, such that $H(s)x = 0$. Using our notation from above we have

$$\begin{bmatrix} q & Q(s) \\ d & r^T \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{16}$$

Note that, for $s \in \mathbb{R}_{>0}$, the matrix $Q(s)$ has full rank (i.e. $\operatorname{rank} Q(s) = n - 2$). Furthermore, all of its eigenvalues are contained in the open right half plane. This is a direct consequence of the Gershgorian circle theorem [8].

Since $d \in \{0, -1\}$ we have two cases for which (16) can have a nontrivial solution,

$$d = 0 \quad \Rightarrow \quad r^T Q(s)^{-1} q = 0 \tag{17}$$

$$d = -1 \quad \Rightarrow \quad r^T Q(s)^{-1} q = -1. \tag{18}$$

We will show in the following by contradiction that none of the two conditions can be true.

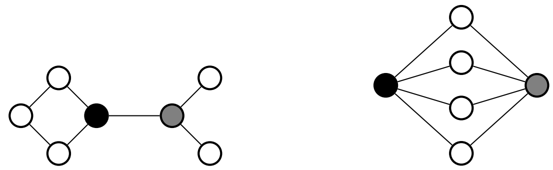
The matrix $Q(s)$ can be written as $Q(s) = sI + \Gamma + \mathcal{L}(\hat{\mathcal{G}})$, where $\hat{\mathcal{G}} \subset \mathcal{G}$ is the subgraph of $\mathcal{G}$ induced by the vertex set $\hat{\mathcal{V}} = \mathcal{V} - \{v_{ctr}, v_{obs}\}$, i.e. by all the vertices with the exception of the control vertex and the observation vertex. The diagonal matrix Γ contains the remaining degrees, with $\Gamma_{ii} \in \{0, 1, 2\}$, depending on how the corresponding vertex is attached to the control and observation vertices.

Note that the subgraph $\hat{\mathcal{G}}$ is not necessarily connected. However, there always exists a permutation matrix P such that $P^T \mathcal{L}(\hat{\mathcal{G}}) P = \operatorname{diag}(\mathcal{L}(\hat{\mathcal{G}}_1), \ldots, \mathcal{L}(\hat{\mathcal{G}}_p))$ for the case when $\hat{\mathcal{G}}$ has p connected components $\hat{\mathcal{G}}_i$. This also implies that, with the same permutation, $Q(s)$ can be partitioned accordingly as

$$P^T Q(s) P = \begin{bmatrix} Q_1(s) & & \\ & \ddots & \\ & & Q_p(s) \end{bmatrix}, \tag{19}$$

where $Q_i(s) = sI + \Gamma_i + \mathcal{L}(\hat{\mathcal{G}}_i)$. Consequently, we can also write the inverse of $P^T Q P$ as a block diagonal matrix. Applying the same permutation to the vectors q and r yields $P^T r = [r_1^T, \ldots, r_p^T]^T$ and $P^T q = [q_1^T, \ldots, q_p^T]^T$. We can use the permuted forms of r^T, q and $Q(s)^{-1}$ in (17) and (18), since $r^T P P^T Q(s)^{-1} P P^T q = r Q(s)^{-1} q$.

[1] This can always be made true via an appropriate permutation of the graph labeling.



(a) Control and observation nodes adjacent and not contained in any cycle.

(b) Control and observation nodes share the same neighborhood $\mathcal{N}$.

Fig. 2: Examples of setups discussed by Theorem 3.10.

Therefore, our conditions can be expressed as

$$d = 0 \quad \Rightarrow \quad \sum_i r_i^T Q_i^{-1} q_i = 0 \tag{20}$$

$$d = -1 \quad \Rightarrow \quad \sum_i r_i^T Q_i^{-1} q_i = -1. \tag{21}$$

All submatrices $Q_i(s)$ have (strictly) positive elements on the diagonal and non-positive off-diagonal elements. We know additionally that all eigenvalues of $Q_i(s)$ have positive real part. Therefore $Q_i(s)$ are symmetric *M-matrices*.[2] Since every $Q_i(s)$ corresponds to a connected component $\hat{\mathcal{G}}_i$ of $\hat{\mathcal{G}}$, it is also irreducible. It is known that the inverse of a symmetric, irreducible *M*-matrix is a symmetric positive matrix (all elements are strictly positive) [8], [13]. The matrix $Q(s)^{-1}$ is therefore a block matrix with strictly positive elements in the blocks $Q_i(s)^{-1}$ and zeros otherwise.

Now we can complete the proof. Note that the components of the vectors r and q are elements of $\{0, -1\}$. The product $r^T Q(s)^{-1} q$ is therefore the sum of elements from $Q(s)^{-1}$. This sum can never be negative. If the control and the observation node are directly connected ($d = -1$) the condition (21) cannot hold. Now if the two nodes are not directly connected ($d = 0$) there exists at least one connected component $\hat{\mathcal{G}}_j$, which is connected to both nodes. But this implies that $r_j^T Q_j(s)^{-1} q_j \neq 0$, since elements from the strictly positive block $Q_j^{-1}(s)$ are selected. Thus condition (20) can never hold either. This completes the proof and shows that no $s \in \mathbb{R}_{>0}$ can be a transmission zero of (1). ■

Theorem (3.9) makes a statement on the location of the transmission zeros for arbitrary graphs and any distinct pair of control and observation nodes. No system will have real zeros in the right half plane. Unfortunately, this result cannot be directly extended to exclude zeros in the complex open right half plane.

The previous theorems hold for setups on arbitrary graphs. If we add restrictions to the graph structure of our setup, we can further bound the location of the zeros.

Theorem 3.10: The system (1) has no transmission zeros in the open right half plane, i.e. $\mathrm{Re}(s) \leq 0$, when either of the following holds:

- The control and observation nodes are adjacent and the edge $\{v_{ctr}, v_{obs}\}$ is not part of a cycle.
- The control and observation nodes share the same neighborhood disregarding their own connection, that is $\mathcal{N}(v_{ctr}) = \mathcal{N}(v_{obs})$ when $\{v_{ctr}, v_{obs}\} \notin \mathcal{E}$ and $\mathcal{N}(v_{ctr}) \backslash \{v_{obs}\} = \mathcal{N}(v_{obs}) \backslash \{v_{ctr}\}$ if $\{v_{ctr}, v_{obs}\} \in \mathcal{E}$.

Proof: The proof idea follows that of Theorem 3.9. First, note that $Q(s)$ also has full rank for $\mathrm{Re}(s) > 0$, so the steps up to equation (21) are identical. For the case when the control and observation node are directly connected by an edge, we show a contradiction for (21). The edge $\{v_{ctr}, v_{obs}\}$ is not part of a cycle, therefore the subgraphs induced by $\hat{\mathcal{V}}$ fall into two groups. Either they were connected to the rest of the graph only through the control vertex or only through the observation vertex but never through both. For every connected component $\hat{\mathcal{G}}_i$, it subsequently holds that $r_i^T Q_i^{-1} q_i = 0$, since either r_i or q_i is a zero vector. Thus the sum over all components is also zero, contradicting the condition in (21), and proving the first part of the statement.

The proof for the second part follows from the fact that when the control and observation nodes are connected to the same nodes, we have $r = q$. This results in a quadratic form $r^T Q^{-1} q = r^T Q^{-1} r$, and since Q^{-1} is positive definite, the quadratic form is also positive definite. Note that the vector r cannot be zero, as we are assuming a connected graph $\mathcal{G}$, and $r = q = 0$ would imply that the control and observation nodes are disconnected from the rest of the graph. Therefore $r^T Q^{-1} r > 0$, contradicting both conditions (17) and (18). ■

Figure 2 shows examples for the two setup categories discussed by Theorem 3.10. This result can be applied to certain graphs.

Corollary 3.11: The system (1) when the underlying graph is the complete graph or the star graph (Figure 1) is minimum phase for arbitrary transmission pair.

Note that this corollary is directly verified in Table I from the analytic expressions of the transfer functions. Observe that the star graph corresponds to the first case in Theorem 3.10, and the complete graph to the second. This suggests that strong statements on the location of the zeros can be made without an explicit description of the transfer-function. Implications of this are explored in the next section.

We have established explicit connections between properties of the graph of a consensus system and the transfer functions between any two of its nodes. Although we consider these theoretical results interesting for their own sake, they also have direct implications on the vulnerability of consensus networks when an infiltrator has access to one or more agents.

IV. Network Infiltration Scenario

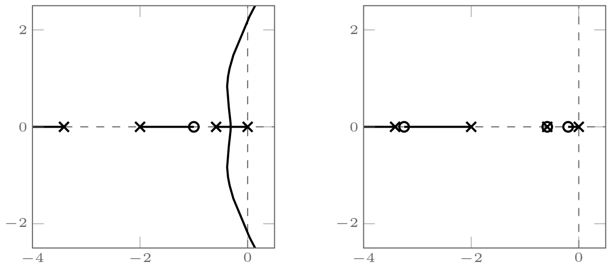
In this section we show some ramifications of the input-output characteristics provided in §III, by example of an infiltration scenario on the network. Specifically, we assume that the infiltrator only has limited access to the networks nodes and no knowledge of its structure.

Consider a scenario where an infiltrator can manipulate the state of one agent and observe the state of another. Using the results presented in §III, we show that the infiltrator can drive the states of all the agents to any position. Moreover, the infiltrator can always destabilize the system without knowledge of the precise network structure.

First we ask how an infiltrator could steer the consensus system to an arbitrary state. The simplest way for the infiltrator to do this is by applying a feedback between the nodes. In particular, we study the effect of a proportional control $K \in \mathbb{R}$,

$$u(t) = K\left(w(t) - y(t)\right). \tag{22}$$

[2] A symmetric *M*-matrix is also called a *Stieltjes* matrix.



(a) P_4(1,3). (b) P_4(1,1).

Fig. 3: Root locus analysis.

The dynamics for the closed-loop system are then given by

$$\begin{aligned} \dot{x}(t) &= -\left(\mathcal{L}(\mathcal{G}) + Kbc^T\right)x(t) + Kbw(t) \\ y(t) &= c^T x(t). \end{aligned} \tag{23}$$

Proposition 4.1: There exists a $K > 0$ so that the feedback given by (22) moves the poles of (1) into the open left-half plane.

Proof: Root locus analysis states that any portion of the real axis which lies to the left of an odd number of poles and zeros is part of the plot. All poles and real zeros lie on the negative real axis, with exception of the pole at the origin. Therefore, this pole must initially move to the left as the feedback gain K is increased. ■

The infiltrator can thus steer the system to any reference state w using an arbitrarily small feedback gain. A small positive feedback gain will suffice to steer the system, but it will not destabilize it. However, for large gain values K, the poles of the closed-loop system may be driven into the right-half plane even when there are no zeros with positive real part. This phenomenon is illustrated in Figure 3-a, where the underlying graph is a path graph on 4 nodes and $v_{ctr} = 1$, $v_{obs} = 3$ (denoted $P_4(1,3)$). Whether this happens or not depends on the system's relative degree. The relative degree, as given by Theorem 3.2, determines the number of asymptotes of the root locus plot. We have shown that it depends only on the length of the shortest path between the infiltrator and the observed node.

A particular robust situation is given when the infiltrator has control and observation access of a single agent.

Proposition 4.2: Assume that the control and observation vertex are identical. Then any feedback (22) with $K > 0$ results in a stable system.

Proof: The result follows directly from the interlacing property of the zeros and poles (Theorem 3.8). ■

The root locus for this case is illustrated in Figure 3-b on the setup $P_4(1,1)$. Even though there are in general some network robustness properties against positive feedback gains, we also have to observe that there is no robustness against negative gains.

Proposition 4.3: The system (1) can be destabilized by applying the feedback (22) with $K < 0$.

Proof: By Theorems 3.7 and 3.9, there are no real non-negative zeros. Therefore the pole at the origin moves to the right when a feedback with $K < 0$ is applied. ■

An infiltrator who manipulates a single node, while observing another node, has thus many possibilities to modify the network functionality. We have only highlighted some possibilities which are directly connected to the transfer function properties we have analyzed in the previous part of the paper.

V. CONCLUSIONS

In this paper, we have considered an input-output setup for a consensus network. We have established a connection between the input-output properties and the underlying graph structure. Our results concerned in particular the relative degree, the gain factor and the impulse response of the system. Additionally we provided conditions on the location of the transfer zeros. These results present an intimate connection between the zeros of a consensus system and properties of a graph, and provides an early characterization of their dependencies. The relevance of the presented theoretical results is highlighted by the study of an infiltration scenario for consensus networks. We discussed in particular the consequences of our results on the possibilities of a single infiltrator to manipulate a complete consensus network.

REFERENCES

[1] T. Balch and R. C. Arkin. Behavior-based formation control for multi-robot teams, *IEEE Transactions on Robotics and Automation*, 14: 926-939, 1998.
[2] A. Chapman, M. Nabi-Abdolyousefi, and M. Mesbahi. Identification and Infiltration in Consensus-type Networks, *IFAC Workshop on Estimation and Control of Networked Systems*, Venice, Italy, Sept. 2009.
[3] R. Diestel. *Graph Theory*, Springer, 2000.
[4] D. V. Dimarogonas, P. Tsiotras, and K. J. Kyriakopoulos. Leader-follower cooperative attitude control of multiple rigid bodies, *Systems & Control Letters*, (58) 6: 429-435, 2009.
[5] J. A. Fax and R. M. Murray. Information flow and cooperative control of vehicle formations, *IEEE Transactions on Automatic Control*, (49) 9: 1465- 1476, 2004.
[6] C. Godsil and G. Royle. *Algebraic Graph Theory*. Springer, 2001.
[7] R. Graham, D. Knuth, and O. Patashnik. *Concrete Mathematics*. Addison Wesley, 1994.
[8] R. Horn and C. Johnson. *Topics in Matrix Analysis*. Cambridge University Press, 1991.
[9] S. Hou. A simple proof of the Leverrier-Faddeev characteristic polynomial algorithm, *SIAM Review*, (40) 3: 706-709, 1998.
[10] A. Isidori. *Nonlinear Control Systems*. Springer, 1995.
[11] M. Mesbahi and F. Hadaegh. Graphs, matrix inequalities, and switching for the formation flying control of multiple spacecraft, *Journal of Guidance, Control, and Dynamics*, (24) 2: 369-377, 2001.
[12] M. Mesbahi and M. Egerstedt. *Graph Theoretic Methods in Multiagent Networks*. Princenton University Press, 2010.
[13] C. D. Meyer and M. W. Stadelmaier. Singular M-matrices and inverse positivity, *Linear Algebra and its Applications*, 22: 139-156, 1978.
[14] M. W. Newman. *The Laplacian Spectrum of Graphs*. Master's Thesis, University of Manitoba, 2000.
[15] K. Ogata. *Modern Control Engineering*. Prentice-Hall, 1997.
[16] R. Olfati-Saber and R. M. Murray. Consensus problems in networks of agents with switching topology and time-delays, *IEEE Transactions on Automatic Control*, (49) 9: 1520-1533, 2004.
[17] R. Olfati-Saber and J. S. Shamma. Consensus Filters for Sensor Networks and Distributed Sensor Fusion, *IEEE Conference on Decision and Control*, Seville, Spain, Dec. 2005.
[18] R. Olfati-Saber. Flocking for multi-agent dynamic systems: Algorithms and theory, *IEEE Transactions on Automatic Control*, (51) 3: 401-420, 2006.
[19] R. Olfati-Saber, J. Fax, and R. Murray. Consensus and Cooperation in Networked Multi-Agent Systems, *Proceedings of the IEEE*, (95) 1: 1-17, 2007.
[20] K. Petersen and M. Pedersen. *The Matrix Cookbook*. Technical University of Denmark, 2008.
[21] A. Rahmani, M. Ji, M. Mesbahi, and M. Egerstedt. Controllability of multi-agent systems from a graph-theoretic perspective, *SIAM Journal on Control and Optimization*, (48) 1: 162-186, 2009.
[22] S. Tonetti and R. M. Murray. Limits on the network sensitivity function for homogeneous multi-agent systems on a graph, *American Control Conference*, Baltimore, MD, 2010, pp. 3217-3222.
[23] D. Zelazo and M. Mesbahi. Edge agreement: Graph-theoretic performance bounds and passivity analysis, *IEEE Transactions on Automatic Control*, 56(3):554-555, 2011.