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Robust stability properties of the ν-gap metric for time-varying systems

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***Abstract*— The stability of uncertain feedback interconnections of causal time-varying linear systems is studied in terms of a recently established generalisation of the ν-gap metric. In particular, a number of robustness results from the well-known linear time-invariant theory are extended. The time-varying generalisations include: sufficient conditions for robust stability; a bound on robust performance; and two-sided bounds on the induced norm of the variation in a closed-loop mapping as an open-loop component of the feedback interconnection is perturbed. Underlying assumptions are verified for causal systems that exhibit linear periodically time-varying behaviour. This includes a class of sampled-data systems as a special case. Within the periodic context considered, it can be shown that a robust stability condition is also necessary.**

***Index Terms*— Feedback, robust stability, ν-gap metric, time-varying systems, periodic systems**

I. Introduction

In [1], [2], a ν-gap metric and integral quadratic constraint (IQC) based robust stability framework, established for linear time-invariant (LTI) systems in [3], [4], is generalised to accommodate *causal* linear systems that are *time-varying* with *unbounded gain* over the space of finite-energy signals. In particular, a generalised ν-gap distance is defined, assuming the existence of certain normalised coprime representations of the system graphs, which is the case for various classes of linear systems. The main results in [1], [2] establish that the generalised ν-gap metric enjoys homotopy-type robustness properties when combined with IQC conditions. It remained unclear whether the metric could be used to quantify feedback robustness non-conservatively, as is the case for LTI systems [5], [6]. In this paper, aspects of this issue are addressed. The following are established: sufficient conditions for robust stability; properties of the topology induced by the ν-gap metric; and, for a class of linear periodically time-varying (LPTV) systems, the so-called *strong necessity* robustness condition.

By contrast with [1], [2], the definition of the generalised ν-gap metric is motivated here via a necessary and sufficient Fredholm index condition for the stability of an uncertain feedback interconnection, contingent on the robust stability margin of the nominal closed-loop being sufficiently large. Using this condition, we derive a lower bound on the robust stability margin of the perturbed feedback systems. We also consider the variation of a closed-loop mapping, used to gauge performance and robustness, as an open-loop component of the feedback interconnection is perturbed. Uniform upper and lower bounds on the induced norm of the difference are established in terms of the ν-gap distance between the perturbed and nominal open-loop systems. From these bounds it follows that the ν-gap metric induces the weakest topology with respect to which closed-loop stability is maintained in small neighbourhoods *and* closed-loop performance varies continuously.

Towards addressing the issue of conservatism, a class of causal LPTV systems, assumed to have transfer function representations with finite-dimensional realisations, is considered via the the well-known time-lifting and discrete-time Fourier transform isomorphisms. All underlying assumptions made in the preceding development are verified for this class of systems. A necessary condition for robust stability, analogous to the LTI results in [5], is then derived. This leads to a quantitative measure of the maximal ν-gap ball of causal LPTV perturbations a given feedback system can tolerate in terms of retaining the property of internal stability. Along with the results described above, it affirms that the generalised ν-gap metric is a natural dual of the robust performance/stability margin.

The paper is organised as follows. In Section II we specify the notation and some basic material, including the notions of Fredholm, Wiener-Hopf and Hankel operators. In Section III a definition of feedback stability is introduced and then characterised in terms of system graph symbols. Section IV contains the definition of the ν-gap metric, sufficient conditions for robust stability, and bilateral bounds on closed-loop errors. Finally, we consider the aforementioned class of LPTV systems and derive a necessary robust stability condition in Section V.

II. Basic notation and operator theory

The real numbers are denoted $\mathbb{R}$. The transpose of a matrix $M \in \mathbb{R}^{p\times m}$ is denoted M^T. For a linear operator $\mathbf{X} : \mathrm{dom}(\mathbf{X}) \subset \mathcal{H}_1 \to \mathcal{H}_2$, we define its kernel $\ker(\mathbf{X}) := \{x \in \mathrm{dom}(\mathbf{X}) \,|\, \mathbf{X}x = 0\}$ and its image $\mathrm{img}(\mathbf{X}) := \{y \in \mathcal{H}_2 \,|\, y = \mathbf{X}x \text{ for some } x \in \mathrm{dom}(\mathbf{X})\}$. We denote by $\mathscr{L}(\mathcal{H}_1, \mathcal{H}_2)$ the Banach space of all bounded linear operators mapping between the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. An operator $\mathbf{X} \in \mathscr{L}(\mathcal{H}_1, \mathcal{H}_2)$ is said to be compact if for any bounded sequence $\{x_k\}$ in $\mathcal{H}_1$, $\{\mathbf{X}x_k\}$ has a convergent subsequence in $\mathcal{H}_2$. The unique Hilbert adjoint of $\mathbf{X} \in \mathscr{L}(\mathcal{H}_1, \mathcal{H}_2)$ is denoted $\mathbf{X}^* \in \mathscr{L}(\mathcal{H}_2, \mathcal{H}_1)$, and satisfies $\langle \mathbf{X}w, v\rangle_{\mathcal{H}_2} = \langle w, \mathbf{X}^*v\rangle_{\mathcal{H}_2} \,\forall w \in \mathcal{H}_1, v \in \mathcal{H}_2$. We define

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respectively the upper and lower gains of $\mathbf{X}$ as $\bar{\gamma}(\mathbf{X}) := \sup_{\|w\|_{\mathcal{H}_1}=1} \|\mathbf{X}w\|_{\mathcal{H}_2}$ and $\underline{\gamma}(\mathbf{X}) := \inf_{\|w\|_{\mathcal{H}_1}=1} \|\mathbf{X}w\|_{\mathcal{H}_2}$.

Definition 2.1: An $\mathbf{X} \in \mathscr{L}(\mathcal{H}_1, \mathcal{H}_2)$ is said to be of Fredholm type if both $\dim\ker(\mathbf{X})$ and $\dim\ker(\mathbf{X}^*)$ are finite, where $\dim$ denotes the dimension of a subspace. The Fredholm index of $\mathbf{X}$ is defined to be $\mathrm{ind}(\mathbf{X}) := \dim\ker\mathbf{X} - \dim\ker(\mathbf{X}^*)$.

This paper is concerned with systems mapping between *finite-energy* time-domain signals. Define the Hilbert space

$$\mathbf{L}_2^m(\mathbb{R}) := \left\{ \phi : \mathbb{R} \to \mathbb{R}^m \;\middle|\; \|\phi\|_2 := \langle \phi, \phi \rangle_2^{\frac{1}{2}} < \infty \right\},$$

where $\langle u, v \rangle_2 := \int_{-\infty}^{\infty} u(t)^T v(t)\, dt$. In the sequel, we will suppress the spatial dimension m for notational simplicity but note that m is allowed to vary whenever $\mathbf{L}_2(\mathbb{R})$ is invoked and compatibility between the dimensions of the input-output spaces of operator mappings is always assumed for compositions. Define the following two subsets of $\mathbf{L}_2(\mathbb{R})$:

$$\mathbf{L}_2(\mathbb{I}) := \{\phi \in \mathbf{L}_2(\mathbb{R}) \mid \phi(t) = 0\, \forall t \notin \mathbb{I} \subset \mathbb{R}\};$$
$$\mathbf{L}_{2+} := \textstyle\bigcup_{\tau \in \mathbb{R}} \mathbf{L}_2[\tau, \infty).$$

For a linear operator $\mathbf{X} : \mathrm{dom}(\mathbf{X}) \subset \mathbf{L}_2(\mathbb{R}) \to \mathbf{L}_2(\mathbb{R})$, we define its *graph* as

$$\mathscr{G}_{\mathbf{X}} := \{ \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] : u \in \mathrm{dom}(\mathbf{X}) \text{ and } y = \mathbf{X}u \};$$
$$\mathscr{G}_{\mathbf{X}}^{\tau} := \mathscr{G}_{\mathbf{X}} \cap \mathbf{L}_2[\tau, \infty),$$

and its inverse graph as

$$\mathscr{G}'_{\mathbf{X}} := \{ \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] : y \in \mathrm{dom}(\mathbf{X}) \text{ and } u = \mathbf{X}y \};$$
$$\mathscr{G}'^{\tau}_{\mathbf{X}} := \mathscr{G}'_{\mathbf{X}} \cap \mathbf{L}_2[\tau, \infty).$$

Let the truncation operator at time $\tau \in \mathbb{R}$ be

$$\mathbf{\Pi}_\tau : \mathbf{L}_2(\mathbb{R}) \to \mathbf{L}_2(-\infty, \tau);\;\; \mathbf{\Pi}_\tau x(t) := \begin{cases} x(t) & t < \tau \\ 0 & t \geq \tau. \end{cases}$$

Definition 2.2: Given a linear operator $\mathbf{X} : \mathrm{dom}(\mathbf{X}) \subset \mathbf{L}_2(\mathbb{R}) \to \mathbf{L}_2(\mathbb{R})$, we say that it is causal if for all $\tau \in \mathbb{R}$, $\mathbf{\Pi}_\tau \mathscr{G}_{\mathbf{X}}$ is the graph of a linear operator, i.e.,

$$\forall \tau \in \mathbb{R}, \forall \left[\begin{smallmatrix} y_\tau \\ u_\tau \end{smallmatrix}\right] \in \mathbf{\Pi}_\tau \mathscr{G}_{\mathbf{X}}, u_\tau = 0 \Longrightarrow y_\tau = 0.$$

Similar definition holds if the inverse graph is used.

Finally we recall the following definitions of generalised Wiener-Hopf and Hankel operators from [1], [2].

Definition 2.3: Given $\mathbf{X} \in \mathscr{L}(\mathbf{L}_2(\mathbb{R}), \mathbf{L}_2(\mathbb{R}))$, we define

1) the Wiener-Hopf operator, relative to the 'initial' time $\tau \in \mathbb{R}$, by $\mathbf{T}_{\mathbf{X},\tau} := (\mathbf{I} - \mathbf{\Pi}_\tau)\mathbf{X}|_{\mathbf{L}_2[\tau,\infty)}$;
2) the forward Hankel operator, relative to 'initial' time $\tau \in \mathbb{R}$, by $\mathbf{H}^{+-}_{\mathbf{X},\tau} := (\mathbf{I} - \mathbf{\Pi}_\tau)\mathbf{X}|_{\mathbf{L}_2(-\infty,\tau)}$,

where $|_{\mathcal{X}}$ denotes the domain restriction to $\mathcal{X}$. Note that $(\mathbf{T}_{\mathbf{X},\tau})^* = \mathbf{T}_{\mathbf{X}^*,\tau}$.

III. Stability Criteria for Feedback Systems

A. Feedback interconnection

The main object of study here is the feedback interconnection illustrated in Fig. 1, denoted $[\mathbf{P}, \mathbf{C}]$, where

$$d_y = y_c + y_p,\; d_u = u_p + u_c,\; y_p = \mathbf{P}u_p,\; u_c = \mathbf{C}y_c, \quad (1)$$

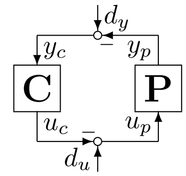

Fig. 1. Standard feedback configuration

and $\mathbf{P} : \mathrm{dom}(\mathbf{P}) \subset \mathbf{L}_2(\mathbb{R}) \to \mathbf{L}_2(\mathbb{R})$ and $\mathbf{C} : \mathrm{dom}(\mathbf{C}) \subset \mathbf{L}_2(\mathbb{R}) \to \mathbf{L}_2(\mathbb{R})$ are two *causal* linear operators. Note that by causality, $\mathrm{img}(\mathbf{P}|_{\mathrm{dom}(\mathbf{P}) \cap \mathbf{L}_2[\tau,\infty)}) \subset \mathbf{L}_2[\tau, \infty)$ and $\mathrm{img}(\mathbf{C}|_{\mathrm{dom}(\mathbf{C}) \cap \mathbf{L}_2[\tau,\infty)}) \subset \mathbf{L}_2[\tau, \infty)$.

Definition 3.1: The feedback interconnection $[\mathbf{P}, \mathbf{C}]$ is said to be internally *stable* if for all $\tau \in \mathbb{R}$ the operator

$$\mathbf{F}_\tau := \begin{bmatrix} \mathbf{I} & \mathbf{P} \\ \mathbf{C} & \mathbf{I} \end{bmatrix}\Bigg|_{(\mathrm{dom}(\mathbf{C}) \times \mathrm{dom}(\mathbf{P})) \cap \mathbf{L}_2[\tau,\infty)}$$

has an inverse on $\mathbf{L}_2[\tau, \infty)$, with $\sup_{\tau \in \mathbb{R}} \bar{\gamma}(\mathbf{F}_\tau^{-1}) < \infty$.

Remark 3.2: In the above definition, invertibility is required on a *singly infinite* space $\mathbf{L}_2[\tau, \infty)$, for all possible initial times $\tau \in \mathbb{R}$. This imposes a positive arrow of time [7]; see Remark 3.3 and Lemma 3.4 below. For bounded invertibility $\mathscr{G}^{\tau}_{\mathbf{P}}$ and $\mathscr{G}'^{\tau}_{\mathbf{C}}$ must be closed subspaces of $\mathbf{L}_2[\tau, \infty)$ [8], which is the case for various classes of LTV systems of interest. Invertibility over the *doubly infinite* $\mathbf{L}_2(\mathbb{R})$ is not considered since the graphs of $\mathbf{P}$ and $\mathbf{C}$ may not be closed subspaces of $\mathbf{L}_2(\mathbb{R})$ in cases of interest [9].

Remark 3.3: Definition 3.1 is adapted from [1, Def. 4]. It differs from the latter in two aspects:

1) a uniform bound on $\bar{\gamma}(\mathbf{F}_\tau^{-1})$ is required here;
2) in [1, Def. 4], causality of $\mathbf{F}_\tau^{-1}$ is *explicitly* required for each $\tau \in \mathbb{R}$, which is not necessary since this property is a direct consequence of Definition 3.1.

Lemma 3.4: If $[\mathbf{P}, \mathbf{C}]$ is stable in the sense of Definition 3.1, then $\mathbf{F}_\tau^{-1}$ is necessarily causal for every $\tau \in \mathbb{R}$.

Proof: According to the definition of feedback stability, $\mathbf{F}_\tau : \mathrm{dom}(\mathbf{F}_\tau) \to \mathbf{L}_2[\tau, \infty)$, where $\mathrm{dom}(\mathbf{F}_\tau) := (\mathrm{dom}(\mathbf{C}) \times \mathrm{dom}(\mathbf{P})) \cap \mathbf{L}_2[\tau, \infty)$, is bijective for each $\tau \in \mathbb{R}$. Note that, for real $\tau_2 \geq \tau_1$,

$$\mathbf{F}_{\tau_2} = \mathbf{F}_{\tau_1}|_{\mathrm{dom}(\mathbf{F}_{\tau_2})}, \quad (2)$$

since $\mathrm{dom}(\mathbf{F}_{\tau_2}) \subset \mathrm{dom}(\mathbf{F}_{\tau_1})$. Moreover, $\mathrm{dom}(\mathbf{F}_{\tau_2}^{-1}) = \mathbf{L}_2[\tau_2, \infty) \subset \mathrm{dom}(\mathbf{F}_{\tau_1}^{-1}) = \mathbf{L}_2[\tau_1, \infty)$.

For a fixed $\tau_1 \in \mathbb{R}$, suppose to the contrapositive that there exist $x \in \mathbf{L}_2[\tau_1, \infty)$ and $\tau_2 > \tau_1$ for which $\mathbf{\Pi}_{\tau_2} x = 0$ (i.e. $x \in \mathbf{L}_2[\tau_2, \infty)$) and $\mathbf{\Pi}_{\tau_2} \mathbf{F}_{\tau_1}^{-1} x \neq 0$; in other words, suppose that $\mathbf{F}_{\tau_1}^{-1}$ is not causal. Let $\overline{z} := \mathbf{F}_{\tau_1}^{-1} x$ and $\underline{z} := \mathbf{F}_{\tau_2}^{-1} x$. Then

$$\mathbf{F}_{\tau_1} \overline{z} = x = \mathbf{F}_{\tau_2} \underline{z} = \mathbf{F}_{\tau_1} \underline{z},$$

where (2) has been used. As such, $\mathbf{F}_{\tau_1}(\overline{z} - \underline{z}) = 0$, which implies $\overline{z} = \underline{z} \in \mathbf{L}_2[\tau_2, \infty)$ since $\ker(\mathbf{F}_{\tau_1}) = \{0\}$. This contradicts the hypothesis that $\mathbf{\Pi}_{\tau_2} \overline{z} \neq 0$. Thus, $\mathbf{F}_{\tau_1}^{-1}$ must be causal as claimed. ∎

For causal linear operators $\mathbf{P}$ and $\mathbf{C}$ such that the feedback interconnection $[\mathbf{P}, \mathbf{C}]$ is stable, for all $\tau \in \mathbb{R}$ let

$$\begin{aligned}\mathbf{\Pi}_{\mathscr{G}_{\mathbf{P}}^{\tau}\|\mathscr{G}_{\mathbf{C}}'^{\tau}} &:= \left[\begin{smallmatrix} d_y \\ d_u \end{smallmatrix}\right] \in \mathbf{L}_2[\tau,\infty) \mapsto \left[\begin{smallmatrix} y_p \\ u_p \end{smallmatrix}\right] \in \mathscr{G}_{\mathbf{P}}^{\tau} \\ &= \left[\begin{smallmatrix} -\mathbf{I} & 0 \\ 0 & \mathbf{I} \end{smallmatrix}\right] \mathbf{F}_{\tau}^{-1} + \left[\begin{smallmatrix} \mathbf{I} & 0 \\ 0 & 0 \end{smallmatrix}\right] \text{ and} \\ \mathbf{\Pi}_{\mathscr{G}_{\mathbf{C}}'^{\tau}\|\mathscr{G}_{\mathbf{P}}^{\tau}} &:= \left[\begin{smallmatrix} d_y \\ d_u \end{smallmatrix}\right] \in \mathbf{L}_2[\tau,\infty) \mapsto \left[\begin{smallmatrix} y_c \\ u_c \end{smallmatrix}\right] \in \mathscr{G}_{\mathbf{C}}'^{\tau} \\ &= \left[\begin{smallmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{smallmatrix}\right] \mathbf{F}_{\tau}^{-1} + \left[\begin{smallmatrix} 0 & 0 \\ 0 & \mathbf{I} \end{smallmatrix}\right].\end{aligned} \tag{3}$$

The notation reflects that these are parallel projection operators onto and along the restricted graphs $\mathscr{G}_{\mathbf{P}}^{\tau}$ and $\mathscr{G}_{\mathbf{C}}'^{\tau}$, which are of importance in robust stability and performance analysis [8], [10], [11]. Note $\mathbf{\Pi}_{\mathscr{G}_{\mathbf{P}}^{\tau}\|\mathscr{G}_{\mathbf{C}}'^{\tau}} + \mathbf{\Pi}_{\mathscr{G}_{\mathbf{C}}'^{\tau}\|\mathscr{G}_{\mathbf{P}}^{\tau}} = \mathbf{I}$. Define $b_{\mathbf{P},\mathbf{C}}$ to be $(\sup_{\tau\in\mathbb{R}} \bar{\gamma}(\mathbf{\Pi}_{\mathscr{G}_{\mathbf{P}}^{\tau}\|\mathscr{G}_{\mathbf{C}}'^{\tau}}))^{-1}$ if $[\mathbf{P}, \mathbf{C}]$ is stable, and 0 otherwise. As seen later, this is a measure of robust stability, as well as nominal performance; $b_{\mathbf{P},\mathbf{C}} \leq 1$.

B. Characterising feedback stability via graph symbols

The stability of the feedback interconnection $[\mathbf{P}, \mathbf{C}]$ can be conveniently characterised in terms of right and left graph symbols for $\mathbf{P}$ and $\mathbf{C}$. As in [1], [2], we make use of the following three assumptions in our development.

Assumption 3.5: Given a *causal* operator $\mathbf{P} : \mathrm{dom}(\mathbf{P}) \subset \mathbf{L}_2(\mathbb{R}) \to \mathbf{L}_2(\mathbb{R})$, there exist *causal* operators $\mathbf{N}, \mathbf{M}, \tilde{\mathbf{N}}, \tilde{\mathbf{M}}, \mathbf{X}, \mathbf{Y}, \tilde{\mathbf{X}}, \tilde{\mathbf{Y}} \in \mathscr{L}(\mathbf{L}_2(\mathbb{R}), \mathbf{L}_2(\mathbb{R}))$ satisfying the following properties:

1) the double Bezout identity
$$\begin{bmatrix} \mathbf{Y} & \mathbf{X} \\ \tilde{\mathbf{M}} & -\tilde{\mathbf{N}} \end{bmatrix} \begin{bmatrix} \mathbf{N} & \tilde{\mathbf{X}} \\ \mathbf{M} & -\tilde{\mathbf{Y}} \end{bmatrix} = \mathbf{I};$$
2) $\mathrm{img}(\mathbf{G}) = \ker(\tilde{\mathbf{G}})$ and $\mathscr{G}_{\mathbf{P}}^{\tau} = \mathrm{img}(\mathbf{T}_{\mathbf{G},\tau}) = \ker(\mathbf{T}_{\tilde{\mathbf{G}},\tau})$ for all $\tau \in \mathbb{R}$, where
$$\mathbf{G} := \left[\begin{smallmatrix} \mathbf{N} \\ \mathbf{M} \end{smallmatrix}\right] \quad \text{and} \quad \tilde{\mathbf{G}} := \left[\begin{smallmatrix} -\tilde{\mathbf{M}} & \tilde{\mathbf{N}} \end{smallmatrix}\right]$$
are respectively called right and left graph symbols for the operator $\mathbf{P}$.

Note that right (resp. left) graph symbols are only unique up to right (resp. left) composition with a bounded causal operator which has a bounded and causal inverse. Furthermore, since for all $\tau \in \mathbb{R}$, $\mathbf{T}_{\mathbf{G},\tau}$ has a left causal inverse, whereby $\underline{\gamma}(\mathbf{T}_{\mathbf{G},\tau}) > 0$, it follows by [12, Thm. 5.2] that the subspace $\mathscr{G}_{\mathbf{P}}^{\tau} := \mathscr{G}_{\mathbf{P}} \cap \mathbf{L}_2[\tau,\infty) = \mathrm{img}(\mathbf{T}_{\mathbf{G},\tau})$ is closed, as is consistent with Remark 3.2.

Assumption 3.6: $\mathbf{G}^*\mathbf{G} = \mathbf{I}$ and $\tilde{\mathbf{G}}\tilde{\mathbf{G}}^* = \mathbf{I}$, i.e. the right and left graph symbols can be taken to be normalised.

Assumption 3.7: $\mathbf{H}_{\mathbf{G},\tau}^{+-}$ and $\mathbf{H}_{\tilde{\mathbf{G}},\tau}^{+-}$ are compact $\forall \tau \in \mathbb{R}$.

Througout, the notation $\mathbf{G}$ and $\tilde{\mathbf{G}}$ is used for normalised right and left graph symbols of $\mathbf{P}$. For $\mathbf{C}$, the notation $\mathbf{K} := \left[\begin{smallmatrix} \mathbf{V} \\ \mathbf{U} \end{smallmatrix}\right]$ and $\tilde{\mathbf{K}} := \left[\begin{smallmatrix} -\tilde{\mathbf{U}} & \tilde{\mathbf{V}} \end{smallmatrix}\right]$, is adopted for the right and left (inverse) graph symbols; i.e. $\mathscr{G}_{\mathbf{C}}'^{\tau} = \mathrm{img}(\mathbf{T}_{\mathbf{K},\tau}) = \ker(\mathbf{T}_{\tilde{\mathbf{K}},\tau})$ for every $\tau \in \mathbb{R}$.

Lemma 3.8: Given causal operators $\mathbf{P}$ and $\mathbf{C}$, suppose that Assumption 3.5 holds, then the following are equivalent:

1) $[\mathbf{P}, \mathbf{C}]$ is stable;
2) $\underline{\gamma}(\tilde{\mathbf{K}}\mathbf{G}) > 0$ and $\mathrm{ind}(\mathbf{T}_{\tilde{\mathbf{K}}\mathbf{G},\tau}) = 0 \, \forall \tau \in \mathbb{R}$;[1]
3) $\tilde{\mathbf{K}}\mathbf{G}$ has a bounded causal inverse.

Moreover, when $[\mathbf{P}, \mathbf{C}]$ is stable, we have for all $\tau \in \mathbb{R}$,
$$\mathbf{\Pi}_{\mathscr{G}_{\mathbf{P}}^{\tau}\|\mathscr{G}_{\mathbf{C}}'^{\tau}} = \mathbf{T}_{\mathbf{G},\tau}\mathbf{T}_{\tilde{\mathbf{K}}\mathbf{G},\tau}^{-1}\mathbf{T}_{\tilde{\mathbf{K}},\tau}.$$
Suppose further that Assumption 3.6 holds, then the robust performance margin
$$b_{\mathbf{P},\mathbf{C}} = \underline{\gamma}(\tilde{\mathbf{K}}\mathbf{G}) = \underline{\gamma}(\tilde{\mathbf{G}}\mathbf{K}) > 0.$$

Proof: This lemma is a time-varying generalisation of [6, Prop. 1.9 and 3.5]. Indeed, the equivalence of 1) and 2) and the expression for $\mathbf{\Pi}_{\mathscr{G}_{\mathbf{P}}^{\tau}\|\mathscr{G}_{\mathbf{C}}'^{\tau}}$ can be obtained using the tools developed in establishing the main results of [1, Section III]. The remainder of the proof can be established by exploiting the properties of graph symbols. For complete details, see [13]. ■

Remark 3.9: In [1], [2], an explicit hypothesis that $\tilde{\mathbf{K}}\mathbf{G}$ have non-singular instantaneous gain is made to ensure that $\mathbf{\Pi}_{\mathscr{G}_{\mathbf{P}}^{\tau}\|\mathscr{G}_{\mathbf{C}}'^{\tau}}$ is a *causal*. This is a redundant requirement; see Remark 3.3. The hypothesis is not needed here since if $[\mathbf{P}, \mathbf{C}]$ is stable as per Definition 3.1, $\mathbf{\Pi}_{\mathscr{G}_{\mathbf{P}}^{\tau}\|\mathscr{G}_{\mathbf{C}}'^{\tau}}$ is necessarily causal by Lemma 3.4 and (3).

Remark 3.10: Lemma 3.8 can also be established in terms of the following equivalent statements:

1) $[\mathbf{P}, \mathbf{C}]$ is stable;
2) $\underline{\gamma}(\tilde{\mathbf{G}}\mathbf{K}) > 0$ and $\mathrm{ind}(\mathbf{T}_{\tilde{\mathbf{G}}\mathbf{K},\tau}) = 0 \, \forall \tau \in \mathbb{R}$;
3) $\tilde{\mathbf{G}}\mathbf{K}$ has a bounded causal inverse.

Moreover, when $[\mathbf{P}, \mathbf{C}]$ is stable, we have for all $\tau \in \mathbb{R}$,
$$\mathbf{\Pi}_{\mathscr{G}_{\mathbf{C}}'^{\tau}\|\mathscr{G}_{\mathbf{P}}^{\tau}} = \mathbf{T}_{\mathbf{K},\tau}\mathbf{T}_{\tilde{\mathbf{G}}\mathbf{K},\tau}^{-1}\mathbf{T}_{\tilde{\mathbf{G}},\tau}.$$

IV. Robust stability analysis via the ν-gap

We present in this section sufficient conditions for robust feedback stability and topological properties of the ν-gap metric. Because all these results have their roots in the well-known time-invariant theory [5], [6], we only provide the ideas/directions of proofs and refer to [13] for full details. Throughout, the set of causal operators which satisfy all of Assumptions 3.5, 3.6, and 3.7 is denoted by $\mathbb{S}$.

Theorem 4.1: Given $\mathbf{P}_1, \mathbf{P}_2, \mathbf{C} \in \mathbb{S}$, suppose that $\bar{\gamma}(\tilde{\mathbf{G}}_2\mathbf{G}_1) < b_{\mathbf{P}_1,\mathbf{C}} = \underline{\gamma}(\tilde{\mathbf{G}}_1\mathbf{K})$. Then $\mathbf{T}_{\mathbf{G}_2^*\mathbf{G}_1,\tau}$ is Fredholm for all $\tau \in \mathbb{R}$, and
$$[\mathbf{P}_2, \mathbf{C}] \text{ is stable } \Leftrightarrow \mathrm{ind}(\mathbf{T}_{\mathbf{G}_2^*\mathbf{G}_1,\tau}) = 0 \, \forall \tau \in \mathbb{R}.$$

Proof: This result is a generalisation of [5, Prop. 4.1]. It can be established using the arguments in the second half of the proof for [1, Thm. 1], in conjunction with [1, Lem. 9 and Prop. 1]. In particular, Assumption 3.7 and Lemma 3.8 are required. ■

Motivated by Theorem 4.1, the following is in order.

Definition 4.2: The ν-gap metric is defined on $\mathbb{S}$ as
$$\delta_\nu(\mathbf{P}_1, \mathbf{P}_2) := \begin{cases} \bar{\gamma}(\tilde{\mathbf{G}}_2\mathbf{G}_1) & \text{if for all } \tau \in \mathbb{R}, \\ & \mathbf{T}_{\mathbf{G}_2^*\mathbf{G}_1,\tau} \text{ is Fredholm} \\ & \text{and } \mathrm{ind}(\mathbf{T}_{\mathbf{G}_2^*\mathbf{G}_1,\tau}) = 0 \\ & \\ 1 & \text{otherwise.} \end{cases}$$
Note, $\mathbf{T}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau}$ and $\mathbf{T}_{\mathbf{G}_2^*\mathbf{G}_1,\tau}$ are Fredholm for all $\tau \in \mathbb{R}$ if $\underline{\gamma}(\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*) = \underline{\gamma}(\mathbf{G}_2^*\mathbf{G}_1) > 0$ [1, Lem. 10]. That $\delta_\nu(\cdot,\cdot)$ is

[1]That the Wiener-Hopf operator must be Fredholm here is taken to be implicitly part of in the index condition.

a metric on $\mathbb{S}$ is established in [13]. We call the topology generated by the ν-gap metric the *graph topology*.

The following corollary of Theorem 4.1 provides a bound on robust performance; it encompasses the robust stability result $\delta_\nu(\mathbf{P}_1, \mathbf{P}_2) < b_{\mathbf{P}_1,\mathbf{C}} \implies [\mathbf{P}_2, \mathbf{C}]$ is stable.

Corollary 4.3: For any $\mathbf{P}_1, \mathbf{P}_2, \mathbf{C} \in \mathbb{S}$,

$$\arcsin b_{\mathbf{P}_2,\mathbf{C}} \geq \arcsin b_{\mathbf{P}_1,\mathbf{C}} - \arcsin \delta_\nu(\mathbf{P}_1, \mathbf{P}_2).$$

Proof: The result can be proved as in [5, Thm. 4.2] by making use of Proposition 4.1. ■

Proposition 4.4: For any $\mathbf{P}_1, \mathbf{P}_2, \mathbf{C} \in \mathbb{S}$ such that $[\mathbf{P}_1, \mathbf{C}]$ and $[\mathbf{P}_2, \mathbf{C}]$ are stable,

$$\delta_\nu(\mathbf{P}_1, \mathbf{P}_2) \leq \sup_{\tau\in\mathbb{R}} \bar{\gamma}\left(\mathbf{\Delta}_\tau\right) \leq \frac{\delta_\nu(\mathbf{P}_1, \mathbf{P}_2)}{b_{\mathbf{P}_1,\mathbf{C}} b_{\mathbf{P}_2,\mathbf{C}}}, \qquad (4)$$

where $\mathbf{\Delta}_\tau := \mathbf{\Pi}_{\mathscr{G}^\tau_{\mathbf{P}_2} \| \mathscr{G}'^\tau_{\mathbf{C}}} - \mathbf{\Pi}_{\mathscr{G}^\tau_{\mathbf{P}_1} \| \mathscr{G}'^\tau_{\mathbf{C}}}$.

Proof: It can be established as in the proof for [11, Thm. III.2] that $\mathbf{\Delta}_\tau = \mathbf{\Pi}_{\mathscr{G}'^\tau_{\mathbf{C}} \| \mathscr{G}^\tau_{\mathbf{P}_1}} \mathbf{\Pi}_{\mathscr{G}^\tau_{\mathbf{P}_2} \| \mathscr{G}'^\tau_{\mathbf{C}}} \; \forall \tau \in \mathbb{R}$. The rest of the proof then follows by applying Lemma 3.8 and Remark 3.10 to the expressions for $\mathbf{\Delta}_\tau$ and the arguments employed in [5, Cor. 6.5]. ■

As mentioned in the Section I, the bounds in Proposition 4.4 facilitate simple and direct proofs that the graph topology is the weakest topology with respect to which both feedback stability and performance are robust properties; see [6, Cor. 7.9] and [11, Prop. V.2].

V. Quantitative robustness analysis for periodic systems

The previous section presented sufficient conditions for robust feedback stability in terms of the ν-gap metric. Here we derive the so-called strong necessity condition which is analogous to the first part of [6, Thm. 3.10]. Since the proof of this result relies on explicit construction of systems having certain properties, we depart from the purely abstract setting and focus in this section on a class of linear periodically time-varying (LPTV) systems having 'rational' transfer function realisations with causal feedthrough terms.

A. Preliminaries on periodic systems

We denote respectively by $\mathbb{C}$, $\mathbb{Z}$, $\mathbb{T}$ and $\mathbb{D}$ the complex numbers, the integers, the unit circle and the open unit disc in the complex plane. Two normed spaces $\mathcal{V}_1$ and $\mathcal{V}_2$ are said to be isometrically isomorphic if there exists a bijective bounded linear operator $\mathbf{\Phi} : \mathcal{V}_1 \to \mathcal{V}_2$ such that $\|\mathbf{\Phi} v_1\|_{\mathcal{V}_2} = \|v_1\|_{\mathcal{V}_1}, \forall v_1 \in \mathcal{V}_1$. When this is the case, we denote the isomorphic relationship between $\mathcal{V}_1$ and $\mathcal{V}_2$ via the isomorphism $\mathbf{\Phi}$ by $\mathcal{V}_1 \overset{\mathbf{\Phi}}{\approx} \mathcal{V}_2$.

The following signal spaces, with $h > 0$ as a parameter, play a central role in our study of periodic systems:

$$\boldsymbol{\ell}^2_{\mathbb{Z}} := \left\{ f : \mathbb{Z} \to \mathbf{L}_2[0, h) \;\middle|\; \|f\|^2_{\boldsymbol{\ell}^2_{\mathbb{Z}}} := \textstyle\sum_{i=-\infty}^{\infty} \|f_i\|^2_2 < \infty \right\}$$
$$\boldsymbol{\ell}^{2+}_{\mathbb{Z}} := \left\{ f \in \boldsymbol{\ell}^2_{\mathbb{Z}} \,|\, f_i = 0, \forall i < 0 \right\}.$$

We define $\mathbf{L}^2_{\mathbb{T}}$ (resp. $\mathbf{H}^2_{\mathbb{D}}$) to comprise of the discrete-time Fourier transform $\mathbf{Z}$ of the signals in $\boldsymbol{\ell}^2_{\mathbb{Z}}$ (resp. $\boldsymbol{\ell}^{2+}_{\mathbb{Z}}$) so that $\boldsymbol{\ell}^2_{\mathbb{Z}} \overset{\mathbf{Z}}{\approx} \mathbf{L}^2_{\mathbb{T}}$ and $\boldsymbol{\ell}^{2+}_{\mathbb{Z}} \overset{\mathbf{Z}}{\approx} \mathbf{H}^2_{\mathbb{D}}$, where the isomorphism $(\mathbf{Z}f)(z) := \sum_{i\in\mathbb{Z}} z^i f_i$; see [14, Chapter 5]. The norm on $\mathbf{L}^2_{\mathbb{T}}$ is given by $\|f\|^2_{\mathbf{L}^2_{\mathbb{T}}} := \int_{z\in\mathbb{T}} \|f(z)\|^2_2 \, dz$. Note also $\mathbf{L}_2(\mathbb{R}) \overset{\mathbf{W}_h}{\approx} \boldsymbol{\ell}^2_{\mathbb{Z}}$, where $\mathbf{W}_h$ denotes the *time-lifting* isomorphism [15], and is defined by $(\mathbf{W}_h f)_i(t) = f(hi+t), t \in [0, h)$. Together, it holds that $\mathbf{L}_2(\mathbb{R}) \overset{\mathbf{Z}\mathbf{W}_h}{\approx} \mathbf{L}^2_{\mathbb{T}}$ and $\mathbf{L}_2[0, \infty) \overset{\mathbf{Z}\mathbf{W}_h}{\approx} \mathbf{H}^2_{\mathbb{D}}$.

In line with the input-output approach adopted in this paper, we study the relationships between operators by way of the graphs.

Definition 5.1: Two linear operators $\mathbf{X} : \mathrm{dom}(\mathbf{X}) \subset \mathcal{X}_1 \to \mathcal{X}_2$ and $\mathbf{Y} : \mathrm{dom}(\mathbf{Y}) \subset \mathcal{Y}_1 \to \mathcal{Y}_2$ are said to be equivalent if there exists an isometric isomorphism $\mathbf{\Phi} : \left[\begin{smallmatrix} \mathcal{X}_2 \\ \mathcal{X}_1 \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{Y}_2 \\ \mathcal{Y}_1 \end{smallmatrix}\right]$ such that $\mathscr{G}_{\mathbf{X}} \overset{\mathbf{\Phi}}{\approx} \mathscr{G}_{\mathbf{Y}}$. When this is the case, we denote it by $\mathbf{X} \overset{\mathbf{\Phi}}{\approx} \mathbf{Y}$. We will also use the notation $\mathbf{X} \overset{\mathbf{\Phi}}{\hookleftarrow} \mathbf{Y}$ to denote $\mathbf{X}$ is defined by $\mathbf{Y}$ via the isomorphism $\mathbf{\Phi}$ such that $\mathbf{X} \overset{\mathbf{\Phi}}{\approx} \mathbf{Y}$.

Now define $\boldsymbol{\mathcal{L}}$ to be the set of rational transfer functions

$$P := z \in \mathbb{C} \mapsto zC(\mathbf{I} - zA)^{-1}B + D \in \mathscr{L}(\mathbf{L}_2[0, h), \mathbf{L}_2[0, h)),$$

for which $A \in \mathbb{R}^{n\times n}$; $B \in \mathscr{L}(\mathbf{L}_2[0, h), \mathbb{R}^n)$; $C \in \mathscr{L}(\mathbb{R}^n, \mathbf{L}_2[0, h))$; $\mathbf{\Pi}_\tau D(\mathbf{I} - \mathbf{\Pi}_\tau) = 0 \; \forall \tau \in [0, h)$, i.e. D is causal on $[0, h)$; and D^*D, DD^*, B^*XB, and CYC^* are all Hilbert-Schmidt operators [16, Chapter VIII], where $X = X^T$ and $Y = Y^T$ are arbitrary positive semidefinite matrices in $\mathbb{R}^{n\times n}$. Henceforth, we write $P = (A, B, C, D)$ for simplicity. Both finite-dimensional LTI systems and sampled-data systems can be *equivalently* represented by a transfer function in $\boldsymbol{\mathcal{L}}$ [17]; the Hilbert-Schmidt condition stated above is satisfied as the required terms can be written as integral operators with square integrable kernel functions [16, Pg. 142]. Define the *stable* subclass of $\boldsymbol{\mathcal{L}}$ as $\boldsymbol{\mathcal{L}}\mathbf{H}^\infty_{\mathbb{D}} := \{P = (A, B, C, D) \in \boldsymbol{\mathcal{L}} \mid \mathrm{spec}(A) \subset \mathbb{D}\}$, where $\mathrm{spec}(\cdot)$ denotes the spectrum of a matrix. Note that any $P \in \boldsymbol{\mathcal{L}}\mathbf{H}^\infty_{\mathbb{D}}$ is analytic in $\mathbb{D}$ and has finite norm $\|P\|_\infty := \sup_{z\in\mathbb{D}} \bar{\gamma}(P(z))$. Also, a $P = (A, B, C, D) \in \boldsymbol{\mathcal{L}}$ is invertible in $\boldsymbol{\mathcal{L}}$, i.e. $P^{-1} \in \boldsymbol{\mathcal{L}}$, if, and only if, its feedthrough term D has a bounded causal inverse. Given $P(z) = zC(I - zA)^{-1}B + D \in \boldsymbol{\mathcal{L}}$, the conjugate transfer function, denoted $P^* \in \boldsymbol{\mathcal{L}}$, is given by $P^*(z) = B^*(zI - A^T)^{-1}C^* + D^*$.

Proposition 5.2: Given any $P = (A, B, C, D) \in \boldsymbol{\mathcal{L}}$, there exist $N, M, \tilde{N}, \tilde{M}, X, Y, \tilde{X}, \tilde{Y} \in \boldsymbol{\mathcal{L}}\mathbf{H}^\infty_{\mathbb{D}}$ such that

$$\begin{bmatrix} Y & X \\ \tilde{M} & -\tilde{N} \end{bmatrix} \begin{bmatrix} N & \tilde{X} \\ M & -\tilde{Y} \end{bmatrix} = \mathbf{I} \quad NM^{-1} = \tilde{M}^{-1}\tilde{N} = P$$
$$M^*M + N^*N = \mathbf{I} \quad \tilde{M}\tilde{M}^* + \tilde{N}\tilde{N}^* = \mathbf{I}.$$

Proof: In [18, Lem. 5.4], it is shown that normalised coprime factors with all of the properties claimed, except 'causality' of the corresponding direct feedthrough 'D'-terms, can be constructed from a minimal realisation (A, B, C, D) of P. That the 'D'-terms of the factors can be taken to be causal here, as required to reside in $\boldsymbol{\mathcal{L}}$, follows by the so-called triangular spectral factorisation results in [19]. Consider, for instance, the 'D'-term of the factor N as constructed in [18, Lem. 5.4]. This takes the form DV, where $V := (\mathbf{I} + D^*D + BX^*B)^{-1/2}$ and $0 \leq X = X^T \in \mathbb{R}^{n\times n}$ is a stabilising solution to a certain discrete-time finite-dimensional algebraic Riccati equation that depends on (A, B, C, D). By exploiting the Hilbert-Schmidt conditions assumed above, it follows that the square-root factor V exists as a causal mapping and that this is unique within the class of operators in $\mathscr{L}(\mathbf{L}_2[0,h), \mathbf{L}_2[0,h))$ with 'diagonal' equal to the identity [19]. As such, DV can be taken to be causal, since D is causal; i.e. $N \in \boldsymbol{\mathcal{L}}$. The arguments for the other factors M, $\tilde{M}$ and $\tilde{N}$ follow in a similar fashion. For complete details, see [13]. ∎

With each $P \in \boldsymbol{\mathcal{L}}$, we associate a multiplication operator denoted by $\mathbf{M}_P : \operatorname{dom}(\mathbf{M}_P) \subset \mathbf{L}^2_{\mathbb{T}} \to \mathbf{L}^2_{\mathbb{T}}$ and defined by $(\mathbf{M}_P u)(z) := P(z)u(z)$ for

$$u \in \operatorname{dom}(\mathbf{M}_P) := \left\{ \mathbf{M}_M u \;\middle|\; u \in z^k \mathbf{H}^2_{\mathbb{D}};\, k \in \mathbb{Z} \right\},$$

where $M \in \boldsymbol{\mathcal{L}}\mathbf{H}^\infty_{\mathbb{D}}$ is as defined in Proposition 5.2. Note that $P \in \boldsymbol{\mathcal{L}}\mathbf{H}^\infty_{\mathbb{D}}$ if, and only if, $\mathbf{M}_P \mathbf{H}^2_{\mathbb{D}} \subset \mathbf{H}^2_{\mathbb{D}}$ and

$$\bar{\gamma}(\mathbf{M}_P) = \sup_{u \in \operatorname{dom}(\mathbf{M}_P) : \|u\|=1} \|\mathbf{M}_P u\|_{\mathbf{L}^2_{\mathbb{T}}} = \|P\|_\infty < \infty,$$

in which case we say P is a stable transfer function [14, Chapter 5]. Using the properties of coprime factors for P in Proposition 5.2, it can be shown via a standard argument (see [6, Prop. 1.33], for instance) that

$$\mathscr{G}_{\mathbf{M}_P} \cap z^k \mathbf{H}^2_{\mathbb{D}} = \operatorname{img}(\mathbf{M}_G|_{z^k\mathbf{H}^2_{\mathbb{D}}}) = \ker(\mathbf{M}_{\tilde{G}}|_{z^k\mathbf{H}^2_{\mathbb{D}}}) \,\forall k \in \mathbb{Z},$$

where $G := \left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right] \in \boldsymbol{\mathcal{L}}\mathbf{H}^\infty_{\mathbb{D}}$ and $\tilde{G} := \left[\begin{smallmatrix} -\tilde{M} & \tilde{N} \end{smallmatrix}\right] \in \boldsymbol{\mathcal{L}}\mathbf{H}^\infty_{\mathbb{D}}$. Consider now the discrete-time operator $\mathbf{P}_d \overset{\mathbf{Z}}{\leftrightarrow} \mathbf{M}_P$. $\mathbf{P}_d$ is a shift-invariant operator in the sense that $\mathbf{D}_k \mathscr{G}_{\mathbf{P}_d} \subset \mathscr{G}_{\mathbf{P}_d} \,\forall k \in \mathbb{Z}$, where the discrete-time shift operator $\mathbf{D}_i : \boldsymbol{\ell}^2_{\mathbb{Z}} \to \boldsymbol{\ell}^2_{\mathbb{Z}}$ is defined as $\mathbf{D}_i u(k) := u(k-i)$ for any $i \in \mathbb{Z}$; see [17, Lem. II.11] for a proof which exploits the properties of the graph symbol G. By the fact that $\mathbf{W}_h^{-1}\mathbf{D}_k = \mathbf{S}_{kh}\mathbf{W}_h^{-1} \,\forall k \in \mathbb{Z}$, where $\mathbf{S}_\tau : \mathbf{L}_2(\mathbb{R}) \to \mathbf{L}_2(\mathbb{R})$ is the continuous-time shift operator defined by $\mathbf{S}_\tau u(t) := u(t-\tau)$ for any $\tau \in \mathbb{R}$, it follows that with $\mathbf{P} \overset{\mathbf{W}_h}{\leftrightarrow} \mathbf{P}_d$, we have

$$\mathbf{S}_{kh} \mathscr{G}_{\mathbf{P}} \subset \mathscr{G}_{\mathbf{P}} \,\forall k \in \mathbb{Z}.$$

That is, $\mathbf{P} \overset{\mathbf{ZW}_h}{\leftrightarrow} \mathbf{M}_P$ is a continuous-time LPTV operator with period h. With $\mathbf{N} \overset{\mathbf{ZW}_h}{\leftrightarrow} \mathbf{M}_N$, $\mathbf{M} \overset{\mathbf{ZW}_h}{\leftrightarrow} \mathbf{M}_M$, $\tilde{\mathbf{N}} \overset{\mathbf{ZW}_h}{\leftrightarrow} \mathbf{M}_{\tilde{N}}$, and $\tilde{\mathbf{M}} \overset{\mathbf{ZW}_h}{\leftrightarrow} \mathbf{M}_{\tilde{M}}$ it follows that $\mathbf{G} := \left[\begin{smallmatrix} \mathbf{N} \\ \mathbf{M} \end{smallmatrix}\right]$ and $\tilde{\mathbf{G}} := \left[\begin{smallmatrix} -\tilde{\mathbf{M}} & \tilde{\mathbf{N}} \end{smallmatrix}\right]$ satisfy

$$\mathscr{G}_{\mathbf{P}} \cap \mathbf{L}_2[\tau,\infty) = \operatorname{img}(\mathbf{G}|_{\mathbf{L}_2[\tau,\infty)}) = \ker(\tilde{\mathbf{G}}|_{\mathbf{L}_2[kh,\infty)})$$

for all $\tau = kh$ with $k \in \mathbb{Z}$. That this continues to hold

We shall see below that all of Assumptions 3.5, 3.6, and 3.7 are satisfied by the class of operators $\mathbb{P} := \{\mathbf{P} \overset{\mathbf{ZW}_h}{\leftrightarrow} \mathbf{M}_P : P \in \boldsymbol{\mathcal{L}}\}$, i.e. $\mathbb{P} \subset \mathbb{S}$.

Suppose we are given a (stable) $\Phi = (A,B,C,D) \in \boldsymbol{\mathcal{L}}\mathbf{H}^\infty_{\mathbb{D}}$, then any $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathscr{G}_{\mathbf{\Phi}_d}$, where $\mathbf{\Phi}_d \overset{\mathbf{Z}}{\leftrightarrow} \mathbf{M}_\Phi$, can be described by the convolution operation [20, Thm. 2.6.1]:

$$y_k = \textstyle\sum_{i=-\infty}^{k-1} CA^{k-i-1}Bu_i + Du_k, \forall k \in \mathbb{Z}.$$

Since D is causal by definition, it follows immediately that $\mathbf{\Phi} \overset{\mathbf{W}_h}{\leftrightarrow} \mathbf{\Phi}_d$ is a bounded causal operator on $\mathbf{L}_2(\mathbb{R})$. Indeed, $\mathbf{\Phi}$ is a bounded causal operator on $\mathbf{L}_2(\mathbb{R})$ if, and only if, the corresponding $\Phi \in \boldsymbol{\mathcal{L}}\mathbf{H}^\infty_{\mathbb{D}}$. Moreover, for any $\tau \in \mathbb{R}$, the Hankel factorisation $\mathbf{H}^{+-}_{\mathbf{\Phi},\tau} = \mathbf{L}_{O,\tau}\mathbf{L}_{C,\tau}$ holds, where the observability operator $\mathbf{L}_{O,\tau} \in \mathscr{L}(\mathbb{R}^n, \mathbf{L}_2[\tau,\infty))$ and the controllability operator $\mathbf{L}_{C,\tau} \in \mathscr{L}(\mathbf{L}_2(-\infty,\tau), \mathbb{R}^n)$ are

$$(\mathbf{L}_{O,\tau}x)(t) := \left(CA^{j(t)}x\right)\left(t - \big(k + j(t)\big)h\right), t \geq \tau \quad \text{and}$$
$$\mathbf{L}_{C,\tau}u := \textstyle\sum_{i=-\infty}^{k-1} A^{k-i-1}B(\mathbf{W}_h u)_i,$$

in which $k := \lfloor \tau/h \rfloor$, $j(t) := \lfloor (t-kh)/h \rfloor$, and $\lfloor\cdot\rfloor$ denotes the floor function. Since $\mathbf{L}_{C,\tau}$ has finite-dimensional image and $\mathbf{L}_{O,\tau}$ has finite-dimensional domain, both operators are compact [21, Thm. 8.1-4]. Hence, $\mathbf{H}^{+-}_{\mathbf{\Phi},\tau}$ is compact.

Given a $P \in \boldsymbol{\mathcal{L}}$ normalised coprime factors for the LPTV operator $\mathbf{P} \overset{\mathbf{ZW}_h}{\leftrightarrow} \mathbf{M}_P$ are given by $\mathbf{N} \overset{\mathbf{ZW}_h}{\leftrightarrow} \mathbf{M}_N$, $\mathbf{M} \overset{\mathbf{ZW}_h}{\leftrightarrow} \mathbf{M}_M$, $\tilde{\mathbf{N}} \overset{\mathbf{ZW}_h}{\leftrightarrow} \mathbf{M}_{\tilde{N}}$, and $\tilde{\mathbf{M}} \overset{\mathbf{ZW}_h}{\leftrightarrow} \mathbf{M}_{\tilde{M}}$, where $N, M, \tilde{M}, \tilde{N} \in \boldsymbol{\mathcal{L}}\mathbf{H}^\infty_{\mathbb{D}}$ are as defined in Proposition 5.2. Since M and $\tilde{M}$ are invertible in $\boldsymbol{\mathcal{L}}$, both $\mathbf{M}$ and $\tilde{\mathbf{M}}$ have non-singular instantaneous gains, whereby $\mathbf{P}$ is causal [1, Rem. 3]. Using the properties of coprime factors in Proposition 5.2 and the preceding development, it is straightforward to verify that all of Assumptions 3.5, 3.6 and 3.7 hold; i.e.

$$\mathbb{P} := \{\mathbf{P} \overset{\mathbf{ZW}_h}{\leftrightarrow} \mathbf{M}_P : P \in \boldsymbol{\mathcal{L}}\} \subset \mathbb{S}.$$

B. A necessary condition for robust stability

The main result in this section characterises the maximal ν-gap metric ball of systems in $\mathbb{P}$ a nominal system, which stabilises in feedback the centre of the ball, is guaranteed to stabilise. It is analogous to the time-invariant case of [6, Rem 3.11(i)], whose proof relies on frequency-domain interpolation methods that do not generalise to the class of systems considered here. Our development borrows the ideas from [22, Thm 4.2], in which the standard gap metric is studied within a setting largely different to ours.

Theorem 5.3: Given $\mathbf{P}_1, \mathbf{C} \in \mathbb{P}$ for which $[\mathbf{P}_1, \mathbf{C}]$ is stable, then $[\mathbf{P}_2, \mathbf{C}]$ is stable for all $\mathbf{P}_2 \in \mathbb{P}$ satisfying $\delta_\nu(\mathbf{P}_1, \mathbf{P}_2) < \beta$ if, and only if, $b_{\mathbf{P}_1,\mathbf{C}} \geq \beta$.

Proof: Sufficiency is immediate from Definition 4.2 and Theorem 4.1. For the necessity proof, suppose $b_{\mathbf{P}_1,\mathbf{C}} < \beta$, we establish below that it is possible to construct a system $\mathbf{P}_2 \in \mathbb{P}$ such that $\delta_\nu(\mathbf{P}_1, \mathbf{P}_2) < \beta$ and $[\mathbf{P}_2, \mathbf{C}]$ is unstable.

First note that the stability of $[\mathbf{P}_1, \mathbf{C}]$ is equivalent to $\tilde{\mathbf{K}}\mathbf{G}_1$ having a bounded causal inverse by Lemma 3.8, which in turn is equivalent to $(\tilde{K}G_1)^{-1} \in \boldsymbol{\mathcal{L}}\mathbf{H}^\infty_{\mathbb{D}}$. In addition,

$$b^{-1}_{\mathbf{P}_1,\mathbf{C}} = \bar{\gamma}((\tilde{\mathbf{K}}\mathbf{G}_1)^{-1}) = \|(\tilde{K}G_1)^{-1}\|_\infty = \|(\tilde{K}G_1)^{-1}\tilde{K}\|_\infty,$$

where the last equality holds since $\tilde{K}$ is normalised, i.e. $\tilde{K}\tilde{K}^* = \mathbf{I}$. Let

$$\Gamma := (\tilde{K}G_1)^{-1}\tilde{K} \in \boldsymbol{\mathcal{L}}\mathbf{H}^\infty_{\mathbb{D}}.$$

Since Γ is analytic in $\mathbb{D}$, for any $\epsilon_0 > 0$ and $\epsilon_1 > 0$, there exists by the maximum modulus principle a complex number $z_0 \in \{z : (1-\epsilon_0) \leq |z| < 1\}$ such that

$$(b_{\mathbf{P}_1,\mathbf{C}} + \epsilon_1)^{-1} < \bar{\gamma}(\Gamma(z_0)) \leq b^{-1}_{\mathbf{P}_1,\mathbf{C}}.$$

It follows by the definition of the induced norm that there exists a normalised signal $u \in \mathbf{L}_2[0,h)$ such that

$$\bar{\gamma}(\Gamma(z_0)) \geq \|\Gamma(z_0)u\|_2 \geq (b_{\mathbf{P}_1,\mathbf{C}} + \epsilon_1)^{-1}.$$

We define $\Delta_0 : \mathbf{L}_2[0,h) \to \mathbf{L}_2[0,h)$ to map $\alpha\Gamma(z_0)u$ to $-\alpha u$ for all $\alpha \in \mathbb{C}$ and every element in $\{x \in \mathbf{L}_2[0,h) : \langle x, \alpha\Gamma(z_0)u\rangle_2 = 0; \alpha \in \mathbb{C}\}$ to 0, so that

$$\bar{\gamma}(\Delta_0) \leq (b_{\mathbf{P}_1,\mathbf{C}} + \epsilon_1) \quad \text{and} \quad \Gamma(z_0)u \in \ker(\mathbf{I} + \Gamma(z_0)\Delta_0).$$

As such, $\mathbf{I} + \Gamma(z_0)\Delta_0$ is not invertible. Define

$$\Delta(z) := \frac{z}{z_0}\Delta_0 \in \boldsymbol{\mathcal{L}}\mathbf{H}_{\mathbb{D}}^{\infty},$$

so that $\Delta(z_0) = \Delta_0$, whereby it is clear that $\mathbf{I} + \Gamma\Delta$ is not invertible in $\boldsymbol{\mathcal{L}}\mathbf{H}_{\mathbb{D}}^{\infty}$. We set $\epsilon_1 := (\beta - b_{\mathbf{P}_1,\mathbf{C}})/2$ and $\epsilon_0 := \epsilon_1/2\beta$, so that

$$\|\Delta\|_\infty = \frac{1}{|z_0|}\bar{\gamma}(\Delta_0) \leq \frac{b_{\mathbf{P}_1,\mathbf{C}} + \epsilon_1}{1 - \epsilon_0} < \beta.$$

Now define $\hat{G}_2 := G_1 + \Delta \in \boldsymbol{\mathcal{L}}\mathbf{H}_{\mathbb{D}}^{\infty}$. As G_1 is normalised, i.e. $G_1^* G_1 = \mathbf{I}$, and $\|\Delta\|_\infty < \beta < 1$, we have $\underline{\gamma}(\mathbf{M}_{\hat{G}_2}) > 0$, by which $\ker(\mathbf{M}_{\hat{G}_2}) = \{0\}$. We partition conformably $G_1 = \left[\begin{smallmatrix} N_1 \\ M_1 \end{smallmatrix}\right]$ and $\hat{G}_2 = \left[\begin{smallmatrix} \hat{N}_2 \\ \hat{M}_2 \end{smallmatrix}\right]$. Given that Δ does not have a feedthrough 'D' term in its realisation, M_2 has the same feedthrough term as that of M_1. Consequently, M_2 is invertible in $\boldsymbol{\mathcal{L}}$ as M_1 is. This implies that $\bigcup_{k\in\mathbb{Z}} \operatorname{img}(\mathbf{M}_{\hat{G}_2}|_{z^k\mathbf{H}_{\mathbb{D}}^2})$ is the graph of a multiplication operator with symbol in $\boldsymbol{\mathcal{L}}$, which we shall call P_2. By Proposition 5.2, there exists a normalised right graph symbol G_2 for P_2. Since $\operatorname{img}(\mathbf{M}_{\hat{G}_2}|_{\mathbf{H}_{\mathbb{D}}^2}) = \operatorname{img}(\mathbf{M}_{G_2}|_{\mathbf{H}_{\mathbb{D}}^2}) = \mathscr{G}_{\mathbf{M}_{P_2}} \cap \mathbf{H}_{\mathbb{D}}^2$ and $\ker(\mathbf{M}_{\hat{G}_2}) = \{0\}$, there exists by the Beurling-Lax-Halmos theorem [23, Cor. IX.2.2] a $\hat{Q}, \hat{Q}^{-1} \in \boldsymbol{\mathcal{L}}\mathbf{H}_{\mathbb{D}}^{\infty}$ such that $G_2 = \hat{G}_2\hat{Q}$. Now observe that

$$(\tilde{K}G_1)^{-1}\tilde{K}G_2 = (\tilde{K}G_1)^{-1}\tilde{K}\hat{G}_2\hat{Q} = (\mathbf{I} + \Gamma\Delta)\hat{Q},$$

from which it follows that $\tilde{K}G_2$ is not invertible in $\boldsymbol{\mathcal{L}}\mathbf{H}_{\mathbb{D}}^{\infty}$. By Lemma 3.8, this implies that $[\mathbf{P}_2, \mathbf{C}]$, where $\mathbf{P}_2 \overset{\mathbf{ZW}_h}{\leftrightarrow} \mathbf{M}_{P_2}$, is not a stable feedback interconnection. To complete the proof, we establish below that $\delta_\nu(\mathbf{P}_1, \mathbf{P}_2) < \beta$.

First we see that $G_1^* G_2 = G_1^*(G_1 + \Delta)\hat{Q} = (\mathbf{I} + G_1^*\Delta)\hat{Q}$. Define $\mathbf{G}_1 \overset{\mathbf{ZW}_h}{\leftrightarrow} \mathbf{M}_{G_1}$, $\mathbf{G}_2 \overset{\mathbf{ZW}_h}{\leftrightarrow} \mathbf{M}_{G_2}$, $\boldsymbol{\Delta} \overset{\mathbf{ZW}_h}{\leftrightarrow} \mathbf{M}_{\Delta}$, and $\hat{\mathbf{Q}} \overset{\mathbf{ZW}_h}{\leftrightarrow} \mathbf{M}_{\hat{Q}}$. Noting that G_1 is normalised, we have

$$\bar{\gamma}(\mathbf{G}_1^*\boldsymbol{\Delta}) = \|G_1^*\Delta\|_\infty \leq \|\Delta\|_\infty < \beta < 1.$$

Now by [1, Lem. 1(i)–(iii)] in the order they are stated,

$$\begin{aligned} -\operatorname{ind}(\mathbf{T}_{\mathbf{G}_2^*\mathbf{G}_1,\tau}) &= \operatorname{ind}(\mathbf{T}_{\mathbf{G}_1^*\mathbf{G}_2,\tau}) \\ &= \operatorname{ind}(\mathbf{T}_{\mathbf{I}+\mathbf{G}_1^*\boldsymbol{\Delta},\tau}) + \operatorname{ind}(\mathbf{T}_{\hat{\mathbf{Q}},\tau}) \\ &= \operatorname{ind}(\mathbf{T}_{\hat{\mathbf{Q}},\tau}) = 0\, \forall \tau \in \mathbb{R}, \end{aligned}$$

where the last equality holds because $\hat{Q}^{-1} \in \boldsymbol{\mathcal{L}}\mathbf{H}_{\mathbb{D}}^{\infty}$. Finally, since $\mathbf{G}_2\mathbf{G}_2^* + \tilde{\mathbf{G}}_2^*\tilde{\mathbf{G}}_2 = \mathbf{I}$ (see the end of [1, Section III]), by defining $\mathbb{Q} := \{\mathbf{Q} \overset{\mathbf{ZW}_h}{\leftrightarrow} \mathbf{M}_Q : Q \in \boldsymbol{\mathcal{L}}\mathbf{H}_{\mathbb{D}}^{\infty}\}$, we have that

$$\begin{aligned} \delta_\nu(\mathbf{P}_1, \mathbf{P}_2) = \bar{\gamma}(\tilde{\mathbf{G}}_2\mathbf{G}_1) &\leq \inf_{\mathbf{Q}\in\mathbb{Q}} \bar{\gamma}\left(\begin{bmatrix} \mathbf{G}_2^*\mathbf{G}_1 - \mathbf{Q} \\ \tilde{\mathbf{G}}_2\mathbf{G}_1 \end{bmatrix}\right) \\ &= \inf_{\mathbf{Q}\in\mathbb{Q}} \bar{\gamma}\left(\begin{bmatrix} \mathbf{G}_2^* \\ \tilde{\mathbf{G}}_2 \end{bmatrix}(\mathbf{G}_1 - \mathbf{G}_2\mathbf{Q})\right) \\ &= \inf_{Q\in\boldsymbol{\mathcal{L}}\mathbf{H}_{\mathbb{D}}^{\infty}} \|G_1 - G_2 Q\|_\infty \\ &= \inf_{Q\in\boldsymbol{\mathcal{L}}\mathbf{H}_{\mathbb{D}}^{\infty}} \|G_1 - (G_1 + \Delta)\hat{Q}Q\|_\infty \\ &\leq \|\Delta\|_\infty < \beta. \end{aligned}$$

This concludes the proof. ■

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