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Link Resource Allocation for Maximizing the Rigidity of Multi-Agent Formations

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Abstract— **In this paper, the problem of optimizing the rigidity of multi-agent formations is formulated and solved using convex optimization methods. Two rigidity indices, the worst-case rigidity index (WRI) and the mean rigidity index (MRI), are proposed to measure the rigidity of formations of multiple agents connected by links with adjustable strengths. Under the assumption of limited total link resources, we develop efficient algorithms that can find the optimal allocation of link resources for maximizing the rigidity indices. Furthermore, through a sensitivity analysis of the optimization problems, the significance (priorities) of the different links are also characterized. Some simulations results are presented.**

Index Terms— **optimization, resource allocation, formation control, wireless sensor network,**

I. Introduction

Tasks arising in applications such as environmental monitoring, aerospace, hazard detection, etc., often necessitates the deployment of a large number of agents (sensors, robots, vehicles) in an uncertain environment. These agents coordinate with one another through communication links whose strengths depend on distances, transmission powers, and ambient noise level. The geographical locations of the agents together with their communication links define a *multi-agent formation*. Some previous studies of multi-agent formations can be found for applications such as unmanned air vehicle (UAV) [1], sensor networks [2], underwater vehicles [3], and other cooperative multi-vehicle systems [4], [5].

A fundamental problem in the study of multi-agent systems is to decide how their underlying formations affect the performance of distributed algorithms carrying out coordination tasks. Instances of such tasks include, e.g., the localization of all the sensor nodes in a sensor network using local, relative distance measurements; or maintaining the formation shape of a group of UAVs under persistent external perturbations by decentralized controllers. To answer qualitative questions on whether these coordination tasks can be accomplished at all, the concept of *rigid graphs* from graph theory has been shown to be relevant [4], [6]. To answer quantitative questions on how efficient these tasks can be completed, we proposed the numerical measures called the *rigidity indices* in [7]. These indices enable us to compare the rigidity of various formations and identify the most suitable one for a certain coordination task. Two specific application examples are given in this paper to demonstrate the usefulness of these indices.

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With the proposed rigidity indices, a natural question is how to design or adjust multi-agent formations so that the performance of distributed coordination algorithms as measured by the rigidity indices can be optimized. Since rigidity indices depend on both the agent positions and their interconnections, classical procedures for constructing rigid graphs without considering node locations (such as the Henneberg construction [8] and Laman's Theorem [9]) are not sufficient. In [10], an optimization method that alternatively adjust the positions and the connections of the agents is proposed.

This paper is an extension of our previous work in [10] on optimizing formation rigidity. We assume that, rather than being either on and off, the strength of each link between a pair of agent could vary continuously depending on the amount of resources allocated to it. The resources to be allocated could include, e.g., node transmission power, communication bandwidth, agent separation, and accuracy of distance-measuring sensors. Given a fixed amount of total resources, we study how to allocate them to individual links so that the two rigidity indices proposed in [7], [10], namely, the *Worst-case Rigidity Index (WRI)* and the *Mean Rigidity Index (MRI)*, are optimized. Using properties of the indices, we will formulate the rigidity optimization problems as convex optimization problems, and solve them efficiently. Furthermore, we will also use perturbational analysis to reveal how sensitive the overall formation rigidity is with respect to variations in the allocated resources. Such information can be used to prioritize the agents and links for guiding the allocation of additional resources; to save energy/resources by tuning down less significant ones; and to find balanced, robust formations whose performances are not vulnerable to a sudden drop of available resources anywhere in the system.

This paper is organized as follows. In Section II, we review the defitions of the stiffness matrix and the two rigidity indices as well as some of their notatble properties. Two application instances of the rigidity indices are illustrated in Section III. In Section IV, the optimal resource allocation problem for maximizing the rigidity of multi-agent formations is formulated as a convex optimization problem, and numerical results are presented. In Section V, sensitivity analysis of the overall performance with respect to individually allocated resources is carried out. Finally, some concluding remarks are given in Section VI.

A. Notations

We briefly outline the notations used in this paper.

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The notations $\mathbb{R}, \mathbb{R}_+, \mathbb{R}^n, \mathbb{R}^{n\times m}$ denote the sets of real scalars, positive scalars, n-dimensional column vectors, and n-by-m matrices, respectively. Italic lowercase letters such as k, p_{ij} represent scalar variables or constants. Bold letters represent column vectors. In particular, a *multi-quantity* is a vector obtained by stacking multiple vectors. For example, $\mathbf{p} = [\mathbf{p}_1^\top \quad \mathbf{p}_2^\top \quad \cdots \quad \mathbf{p}_n^\top]^\top$ with each $\mathbf{p}_i \in \mathbb{R}^2$ is a multi-quantity. Italic uppercase letters, such as R, K, S_{ij}, are matrices. For a matrix A, its transpose and trace are denoted by $A^\top$ and $\mathrm{tr}(A)$, respectively, while $A^\dagger$ denotes its Moore-Penrose pseudo inverse. For symmetric matrices A, B, we write $A \succeq 0$ if A is nonnegative definite and $A \succeq B$ if $A - B \succeq 0$. Sets are denoted by calligraphic letters, e.g., $\mathcal{I}$, whose cardinality are denoted by $\#\mathcal{I}$. In this paper, $\|\cdot\|$ by default is the Euclidean norm of vectors and matrices.

II. Rigidity of Multi-Agent Formations

We study an n-agent system in the plane $\mathbb{R}^2$. The positions of the agents are denoted by $\mathbf{p}_i \in \mathbb{R}^2$, $i \in \mathcal{I} = \{1, \ldots, n\}$. Between each pair of agents i and j there may exist a communication link whose strength is modeled by a scalar constant $k_{ij} \geq 0$, with the assumptions that (i) $k_{ii} = 0$; (ii) $k_{ij} = k_{ji}$; and (iii) $k_{ij} = 0$ if agents i and j do not communicate directly. In this formulation, the n agents form a weighted graph with a *connectivity matrix* $K = [k_{ij}]$. The set of all valid connectivity matrices for the n-agent system is denoted by $\mathcal{K}_n$. The formation $(\mathcal{I}, \mathbf{p}, K)$ of the n-agent system is determined by both the connectivity matrix K and the agent positions $\mathbf{p} = [\mathbf{p}_1^\top \ \cdots \ \mathbf{p}_n^\top]^\top$, and thus is called a *$KP$-formation* in this paper.

A. Stiffness Matrix

To measure the rigidity of a KP-formation $(\mathcal{I}, \mathbf{p}, K)$, we invoke a spring-mass analogy to study its robustness under perturbations. Each agent is modeled by a unit mass, and the link between agents i and j is modeled by a spring with spring constant k_{ij} and natural length $\|\mathbf{p}_i - \mathbf{p}_j\|$. When the agents are in their given positions $\mathbf{p}$, all the springs are relaxed. Now assume that the position of each agent i is perturbed by an infinitesimal displacement $\Delta\mathbf{p}_i \in \mathbb{R}^2$, so that the new agent positions become $\mathbf{p} + \Delta\mathbf{p}$, where $\Delta\mathbf{p} \triangleq \left[\Delta\mathbf{p}_1^\top \quad \cdots \quad \Delta\mathbf{p}_n^\top\right]^\top$. Then, due to length changes of all the springs connecting to it, each agent i is subject to a net force $\mathbf{f}_i$. The total force applied on the n-agent system, $\mathbf{f} \triangleq \left[\mathbf{f}_1^\top \quad \cdots \quad \mathbf{f}_n^\top\right]^\top$, can be shown to be [7]

$$\mathbf{f} = -S\Delta\mathbf{p} + o\left(\|\Delta\mathbf{p}\|\right), \tag{1}$$

where $S = [S_{ij}]_{1\leq i,j\leq n} \in \mathbb{R}^{2n\times 2n}$, with each 2-by-2 block S_{ij} defined by

$$S_{ij} = \begin{cases} \sum_{j\in I} k_{ij}P_{ij} & \text{if } i = j \\ -k_{ij}P_{ij} & \text{if } i \neq j. \end{cases} \tag{2}$$

Here, $P_{ij} \in \mathbb{R}^2$ is the projection matrix defined by

$$P_{ij} \triangleq \mathbf{e}_{ij}\mathbf{e}_{ij}^\top, \quad \mathbf{e}_{ij} \triangleq \frac{\mathbf{p}_j - \mathbf{p}_i}{\|\mathbf{p}_j - \mathbf{p}_i\|}. \tag{3}$$

The relation (1) can be viewed as a generalization of the Hooke's Law in elastic mechanics. The matrix S is called the *stiffness matrix* of the given KP-formation $(\mathcal{I}, \mathbf{p}, K)$. Intuitively, S establishes the first-order linear relation between the infinitesimal displacement $\Delta\mathbf{p}$ and the resulting resistance force $\mathbf{f}$. In the following, we may write $S(K, \mathbf{p})$ to show its dependence on both the agents configuration $\mathbf{p}$ and the connectivity matrix K.

In the particular case where all entries in K are zero except k_{ij} and k_{ji} for some i, j, we can see from (2) that the stiffness matrix S has exactly four nonzero 2-by-2 blocks, namely, $S_{ii} = S_{jj} = k_{ij}P_{ij}, S_{ij} = S_{ji} = -k_{ij}P_{ij}$. Such an S is of rank one and can be factorized as $S = k_{ij}\mathbf{q}_{ij}\mathbf{q}_{ij}^\top$, where $\mathbf{q}_{ij} \triangleq \left[\mathbf{q}_{ij}^{(1)\top} \quad \cdots \quad \mathbf{q}_{ij}^{(n)\top}\right]^\top \in \mathbb{R}^{2n}$ is defined by

$$\mathbf{q}_{ij}^{(k)} = \begin{cases} \mathbf{e}_{ij}, & k = i \\ -\mathbf{e}_{ij}, & k = j \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

Proposition 1: (**Nonnegative Definiteness**) The stiffness matrix S of a KP-formation is nonnegative definite.

Proof: Each connectivity matrix $K \in \mathcal{K}_n$ can be decomposed as $K = \sum_{i,j\in\mathcal{I},i<j} K^{(ij)}$, where $K^{(ij)}$ is the connectivity matrix obtained by setting all entries in K to zero except k_{ij} and k_{ji}. As we have shown, each $S(K^{(ij)}, \mathbf{p})$ can be factorized as $S(K^{(ij)}, \mathbf{p}) = k_{ij}\mathbf{q}_{ij}\mathbf{q}_{ij}^\top \succeq 0$. Thus, by linearity [7], $S(K, \mathbf{p}) = \sum_{i,j\in\mathcal{I},i<j} S(K^{(ij)}, \mathbf{p}) \succeq 0$. ■

More specifically, for $\mathbf{v} \in \mathbb{R}^{2n}$, $\mathbf{v}^\top S(K, \mathbf{p})\mathbf{v}$ can be expanded as follows,

$$\begin{aligned} \mathbf{v}^\top S(K, \mathbf{p})\mathbf{v} &= \sum_{\substack{i,j\in\mathcal{I} \\ i<j}} k_{ij}\mathbf{v}^\top\mathbf{q}_{ij}^\top\mathbf{q}_{ij}\mathbf{v} \\ &= \sum_{\substack{i,j\in\mathcal{I} \\ i<j}} k_{ij}\left|(\mathbf{v}_i - \mathbf{v}_j)^\top\mathbf{e}_{ij}\right|^2 \geq 0. \end{aligned} \tag{4}$$

In (4), let $\mathbf{v} = \Delta\mathbf{p}$ be the perturbation in the configuration $\mathbf{p}$ and define the *energy function* J as below,

$$J(\Delta\mathbf{p}) = \frac{1}{2}\Delta\mathbf{p}^\top S(K, \mathbf{p})\Delta\mathbf{p}. \tag{5}$$

From the right hand side of (4), we can see that $J(\Delta\mathbf{p})$ is the infinitesimal total potential energy stored in all the springs due to the perturbation. Hence, the stiffness matrix is exactly the Hessian matrix of the energy function J at its global minimum, namely, the unperturbed configuration.

Since $\mathbf{e}_{ij} = (\mathbf{p}_j - \mathbf{p}_i)/\|\mathbf{p}_j - \mathbf{p}_i\|$, it can also be seen from (4) that $\mathbf{v} \in \mathrm{null}(S(K, \mathbf{p}))$ if and only if $(\mathbf{v}_i - \mathbf{v}_j)^\top(\mathbf{p}_j - \mathbf{p}_i) = 0$ for all i, j such that $k_{ij} \neq 0$, or equivalently, if and only if $\mathbf{v}^\top R = 0$ for some matrix $R \in \mathbb{R}^{2n\times m}$ where m is the number of nonzero k_{ij} $(i < j)$. The coefficient matrix R is called the *rigidity matrix* [11]. The KP-formation $(\mathcal{I}, \mathbf{p}, K)$ is called *(infinitesimally) rigid* if $\mathrm{rank}(R) = 2n-3$ (see [12]). Since $\mathbf{v}^\top R = 0$ implies $\mathbf{v} \in \mathrm{null}(S(K, \mathbf{p}))$ and vice versa, we conclude that the KP-formation is rigid if and only if $\mathrm{rank}(S) = 2n - 3$. It is shown in [13] that the null space of S always contains a three-dimensional subspace, denoted by $\mathrm{iso}_n(\mathbf{p})$, which consists of all the infinitesimal displacements $\Delta \mathbf{p}$ resulted from simultaneous translations and rotations of all the agents that do not change the shape of the formation.

B. Rigidity Indices

A rigidity index $r(K, \mathbf{p})$ is a scalar derived from the stiffness matrix $S(K, \mathbf{p})$ which measures the rigidity of a KP-formation. In this paper, we will use the two rigidity indices proposed in [10], namely, the *worst-case rigidity index (WRI)* $r_{\rm w}$ and the *mean rigidity index (MRI)* $r_{\rm m}$, defined as follows,

$$r_{\rm w}(K,\mathbf{p}) \triangleq \lambda_4(S(K,\mathbf{p})) = \min_{\mathbf{u}\in \mathrm{iso}_n(\mathbf{p})^\perp} \frac{\mathbf{u}^\top S(K,\mathbf{p})\mathbf{u}}{\mathbf{u}^\top\mathbf{u}} \quad (6)$$

$$r_{\rm m}(K,\mathbf{p}) \triangleq \begin{cases} 0 & \text{if } \mathrm{rank}(S) < 2n-3 \\ \dfrac{2n-3}{\mathrm{tr}(S(K,\mathbf{p})^\dagger)} & \text{if } \mathrm{rank}(S) = 2n-3 \end{cases} \quad (7)$$

where $\lambda_k(S)$ denotes the k-th smallest eigenvalue of the matrix S and $S^\dagger$ the Moore-Penrose pseudo inverse of S. When we study the common properties shared by $r_{\rm w}$ and $r_{\rm m}$, we will use $r(K, \mathbf{p})$ to denote both rigidity indices and their dependence on the connectivity matrix K and the agents configuration $\mathbf{p}$.

Proposition 2 ([10]): Both rigidity indices $r(K, \mathbf{p})$ possess the following properties:

1) (**Nonnegativeness**) $r(K, \mathbf{p}) \geq 0$; and $r(K, \mathbf{p}) = 0$ if and only if the KP-formation $(\mathcal{I}, \mathbf{p}, K)$ is not rigid.
2) (**Homogeneity**) $r(\alpha K, \mathbf{p}) = \alpha r(K, \mathbf{p}), \quad \forall\, \alpha \in \mathbb{R}_+$.
3) (**Monotonicity**) $r(K, \mathbf{p})$ is monotonically nondecreasing with respect to each entry k_{ij} of K, $i \neq j$.

Lemma 1 ([14]): For any symmetric positive definite matrices $A, B \in \mathbb{R}^{m\times m}$, the following inequality holds,

$$\left(\mathrm{tr}\left((A+B)^{-1}\right)\right)^{-1} \geq \left(\mathrm{tr}\left(A^{-1}\right)\right)^{-1} + \left(\mathrm{tr}\left(B^{-1}\right)\right)^{-1}.$$

Proposition 3: (**Superadditivity**) For any $\mathbf{p} \in \mathbb{R}^{2n}$ and $K_1, K_2 \in \mathcal{K}_n$, $r(K_1 + K_2, \mathbf{p}) \geq r(K_1, \mathbf{p}) + r(K_2, \mathbf{p})$.

Proof: For simplicity, we drop the variable $\mathbf{p}$ and use the notation $S(K)$ for the stiffness matrix of the KP-formation $(\mathcal{I}, \mathbf{p}, K)$ in the following proof.

For WRI, since $S(K_1 + K_2) = S(K_1) + S(K_2)$, we have

$$\begin{aligned} r_{\rm w}(K_1+K_2) &= \min_{\mathbf{u}\in \mathrm{iso}_n(\mathbf{p})^\perp} \left(\frac{\mathbf{u}^\top S(K_1)\mathbf{u}}{\mathbf{u}^\top\mathbf{u}} + \frac{\mathbf{u}^\top S(K_2)\mathbf{u}}{\mathbf{u}^\top\mathbf{u}} \right) \\ &\geq r_{\rm w}(K_1) + r_{\rm w}(K_2). \end{aligned}$$

For MRI, if the MRI corresponding to either of K_1 and K_2, say, $r_{\rm m}(K_1)$, is zero, then the desired conclusion becomes $r_{\rm m}(K_1 + K_2) \geq r_{\rm m}(K_2)$, which immediately follows from the monotonicity property of $r_{\rm m}$. Now assume $r_{\rm m}(K_1) > 0$ and $r_{\rm m}(K_2) > 0$. This implies that $\mathrm{rank}(S(K_1)) = \mathrm{rank}(S(K_2)) = 2n - 3$. Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{2n-3}\}$ be an orthonormal basis of $\mathrm{iso}_n(\mathbf{p})^\perp$. Define $\hat{S}(K) = (2n - 3)V^\top S(K)V$ for $K \in \mathcal{K}_n$, where $V = [\mathbf{v}_1 \ \cdots \ \mathbf{v}_{2n-3}]$. Then $r_{\rm m}(K) = (2n - 3)\left(\mathrm{tr}(S(K)^\dagger)\right)^{-1} = \left(\mathrm{tr}(\hat{S}(K)^{-1})\right)^{-1}$ for all K such that $\mathrm{rank}\,(S(K)) = 2n - 3$, or equivalently $r_{\rm m}(K) > 0$. Note that $\hat{S}(K)$ is linear with respect to K, by Lemma 1,

$$\begin{aligned} r_{\rm m}(K_1+K_2) &= \left(\mathrm{tr}(\hat{S}(K_1+K_2)^{-1})\right)^{-1} \\ &\geq \left(\mathrm{tr}(\hat{S}(K_1)^{-1})\right)^{-1} + \left(\mathrm{tr}(\hat{S}(K_2)^{-1})\right)^{-1} \\ &= r_{\rm m}(K_1) + r_{\rm m}(K_2). \end{aligned}$$

This concludes the proof. ■

As a result of the superadditivity and the homogeneity of $r(K, \mathbf{p})$ with respect to K, we obtain the following property.

Corollary 1: (**Concavity**) $\forall\ K_1, K_2 \in \mathcal{K}_n,\ \alpha \in [0, 1]$, $r(\alpha K_1 + (1-\alpha)K_2, \mathbf{p}) \geq \alpha r(K_1, \mathbf{p}) + (1-\alpha) r(K_2, \mathbf{p})$.

Concavity is a vital property that makes the link resource allocation problem for maximizing the rigidity indices a convex optimization problem that is easy to solve.

III. Applications of Rigidity Indices

In this section, we will demonstrate how the concept of rigidity indices can be applied to assess the robustness and/or efficiency of various types of multi-agent systems.

A. Performance Evaluation of Formation Control

Consider an n-agent system with the connectivity matrix K. Let $\mathbf{p}$ and $\hat{\mathbf{p}}$ be the current and the desired agent configurations. Recall that the distance between agents i and j is constrained only when $k_{ij} > 0$. Inspired by [15], we design the following controller,

$$\dot{\mathbf{p}} = -\nabla F(\mathbf{p}), \quad (8)$$

where

$$F(\mathbf{p}) = \sum_{\substack{i,j\in\mathcal{I} \\ i>j}} k_{ij} \left(\|\mathbf{p}_j - \mathbf{p}_i\| - \|\hat{\mathbf{p}}_j - \hat{\mathbf{p}}_i\|\right)^2.$$

The above function $F(\cdot)$, called the *formation constraint function* in [15], is exactly the total elastic energy $J(\cdot)$ defined in (5) in the spring-mass analogy of the KP-formation. The controller in (8) tries to reduce the value of $F(\cdot)$, and will only stop when the distance between agents i and j whenever $k_{ij} > 0$ is exactly at its desired value. Now if we let $\mathbf{q} = \mathbf{p} - \hat{\mathbf{p}}$ and assume that the agents are perturbed very slightly, (8) can be approximated using (5) as

$$\dot{\mathbf{q}} = \dot{\mathbf{p}} = -\nabla F(\mathbf{p}) \approx -\nabla J(\mathbf{q}) \ \Rightarrow\ \dot{\mathbf{q}} \approx -S\mathbf{q}, \quad (9)$$

where S is the stiffness matrix of the KP-formation $(\mathcal{I}, \hat{\mathbf{p}}, K)$ and the Hessian of $F(\mathbf{p})$ at $\mathbf{p} = \hat{\mathbf{p}}$. Let t_0 denote the initial time. Although the matrix S is only marginally stable because $\mathrm{rank}(S) \leq 2n - 3$, we can decompose $\mathbf{q}(t_0)$ into $\mathbf{q}_1(t_0) + \mathbf{q}_2(t_0)$ such that $\mathbf{q}_1(t_0) \in \mathrm{iso}_n(\mathbf{p}), \mathbf{q}_2(t_0) \in \mathrm{iso}_n(\mathbf{p})^\perp$. Denote by $\mathbf{q}_1(t)$ the state dynamics $\mathbf{q}(t)$ from intial state $\mathbf{q}(t_0) = \mathbf{q}_1(t_0)$ and $\mathbf{q}_2(t)$ from $\mathbf{q}(t_0) = \mathbf{q}_2(t_0)$. It can be seen that $\mathbf{q}_1(t) \in \mathrm{iso}_n(\mathbf{p}), \mathbf{q}_2(t) \in \mathrm{iso}_n(\mathbf{p})^\perp$ for all $t \geq t_0$ because $\mathrm{iso}_n(\mathbf{p})$ and $\mathrm{iso}_n(\mathbf{p})^\perp$ are both invariant spaces of S. We can disregard $\mathbf{q}_1(t)$ because the rigid body motions do not lead to shape deformation. Consequently, $\|\mathbf{q}_2(t)\| \leq ce^{-(t-t_0)r_{\rm w}}\|\mathbf{q}_2(t_0)\|$ where $r_{\rm w} = \lambda_4(S)$ is the WRI of the KP-formation and c is some constant.

The above result shows that the shape of the formation can be recovered exponentially fast if and only if $r_{\mathrm{w}} > 0$, and the larger r_{w} the faster the recovering process. Therefore, WRI can be effectively used as a quantitative measure of the degree of stabilizability of a formation in the context of formation control.

B. Bound of Estimation Errors in Network Localization

For a given KP-formation $(\mathcal{I}, \mathbf{p}, K)$ that is rigid, suppose the measured distance between two connected agents i and j (i.e., with $k_{ij} > 0$) is $\hat{d}_{ij}$, which is not necessarily equal to their true distance $d_{ij} \triangleq \|\mathbf{p}_j - \mathbf{p}_i\|$ due to a measurement error $\Delta d_{ij} \triangleq \hat{d}_{ij} - d_{ij}$. Based on all such measurements $\hat{d}_{ij}$, an estimate of the actual configuration can be obtained through the following minimization,

$$\mathcal{P} = \arg\min_{\hat{\mathbf{p}}} \sum_{i,j \in \mathcal{I}} k_{ij} (\|\hat{\mathbf{p}}_j - \hat{\mathbf{p}}_i\| - \hat{d}_{ij})^2. \tag{10}$$

Again in the spring-mass analogy, the objective function is the total elastic energy stored in the springs. Note that the result $\mathcal{P}$ is a collection of configurations rather than a singleton due to the freedom given by the rigid body motions. Therefore, to assess the performance of the localization algorithm (10), the error between the estimation $\mathcal{P}$ and the true configuration $\mathbf{p}$ is measured as follows,

$$e(\mathcal{P}) = \inf_{\hat{\mathbf{p}} \in \mathcal{P}} \|\hat{\mathbf{p}} - \mathbf{p}\|^2.$$

It is reasonable to assume[1] that if Δd_{ij} is infinitesimal, the best-fitting configuration $\hat{\mathbf{p}}$ can be expresses as $\hat{\mathbf{p}} = \mathbf{p} + \Delta\mathbf{p}$ where $\|\Delta\mathbf{p}\|$ is also infinitesimal. By the principle of minimum energy, such $\hat{\mathbf{p}}$ is an equilibrium configuration of the spring-mass system deduced from (10), which implies that the resultant force on each agent should be zero,

$$\sum_{j \in \mathcal{I}\setminus\{i\}} k_{ij} \mathbf{e}_{ij} (\|\hat{\mathbf{p}}_j - \hat{\mathbf{p}}_i\| - \hat{d}_{ij}) = 0, \quad \forall\, i \in \mathcal{I}. \tag{11}$$

Using the following approximation,

$$\begin{aligned} \|\hat{\mathbf{p}}_j - \hat{\mathbf{p}}_i\| - \hat{d}_{ij} &= \|\hat{\mathbf{p}}_j - \hat{\mathbf{p}}_i\| - \|\mathbf{p}_j - \mathbf{p}_i\| + d_{ij} - \hat{d}_{ij} \\ &= \mathbf{e}_{ij}^\top (\Delta\mathbf{p}_j - \Delta\mathbf{p}_i) + o(\|\Delta\mathbf{p}_j - \Delta\mathbf{p}_i\|) \\ &\quad - \Delta d_{ij}, \end{aligned} \tag{12}$$

we can rewrite equation (11) as below,

$$\sum_{j \in \mathcal{I}\setminus\{i\}} k_{ij} \mathbf{e}_{ij} \mathbf{e}_{ij}^\top (\Delta\mathbf{p}_j - \Delta\mathbf{p}_i) = \sum_{j \in \mathcal{I}\setminus\{i\}} k_{ij} \mathbf{e}_{ij} \Delta d_{ij}, \forall\, i \in \mathcal{I}. \tag{13}$$

Let $\Delta\mathbf{d}$ be a vector with Δd_{ij} as components for all $i > j$ whose dimension is denoted by m. Then equation (13) for all i can be arranged in a matrix form as

$$S\ \Delta\mathbf{p} = HD\ \Delta\mathbf{d},$$

where D is an $m \times m$ diagonal matrix with the diagonal entries $\sqrt{k_{ij}}$ and H is a $2n \times m$ matrix with columns $\mathbf{h}_{ij} \in \mathbb{R}^{2n}$ for some ordering of the connected agent pairs (i, j). More specifically, $\mathbf{h}_{ij}$ is a stacked vector of n two-dimensional subvectors with $\sqrt{k_{ij}}\ \mathbf{e}_{ij}$ as the ith subvector, $\sqrt{k_{ji}}\ \mathbf{e}_{ji}$ as the jth subvector and zero elsewhere. It can be verified that $HH^\top = S$.

From the above derivation, the best-fitting infinitesimal solution is $\Delta\mathbf{p} = S^\dagger HD\ \Delta\mathbf{d}$. Suppose that the measurement error $\Delta\mathbf{d}$ is a random vector with covariance $E\left[\Delta\mathbf{d}\ \Delta\mathbf{d}^\top\right] = \Sigma = \mathrm{diag}(\sigma_{ij}^2)$. Then,

$$\begin{aligned} E\left[e(\mathcal{P})\right] &= E\left[\|\Delta\mathbf{p}\|^2\right] = E\left[\Delta\mathbf{d}^\top D H^\top S^\dagger S^\dagger HD\ \Delta\mathbf{d}\right] \\ &= \mathrm{tr}\left(H^\top S^\dagger S^\dagger HD\Sigma D\right) \end{aligned}$$

If we choose $k_{ij} = \sigma_{ij}^{-2}$ in the KP-formation model, $D\Sigma D$ becomes an identity matrix. Consequently,

$$E\left[e(\mathcal{P})\right] = \mathrm{tr}\left(S^\dagger \left(HH^\top\right) S^\dagger\right) = \mathrm{tr}(S^\dagger) = \frac{1}{r_{\mathrm{m}}(S)}.$$

That is, the mean square error is equal to the reciprocol of the MRI r_{m}. In fact, the same value has been shown to be the Cramér-Rao lower bound (CRLB) of the localization estimator in [16]. As a result, a formation with a larger MRI leads to a smaller mean square error in its localization outcome under random measurement noises. This example demonstrates the relevance of the MRI in the context of network localization problems.

IV. Optimal Link Resource Allocation

In this section, we will use the proposed rigidity indices to optimize link resource allocation for problems arising in formation control and localization applications.

A. Problem Formulation

Given a KP-formation with fixed agent positions $\mathbf{p}$, finding the optimal link resource allocation in the sense of the largest formation rigidity as measured by the rigidity indices can be formulated as the following optimization problem,

$$\max_{K \in \mathcal{K}_n} \quad r(K, \mathbf{p}) \tag{14}$$

$$\text{subject to} \quad k_{ij} = 0 \qquad \text{if } \{i,j\} \notin \mathcal{L} \tag{15}$$

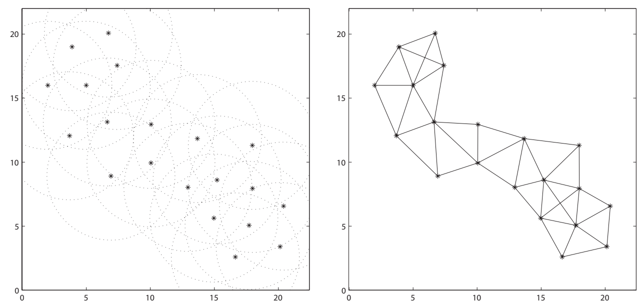
$$\sum_{i<j} k_{ij} \le c \tag{16}$$

where r is the rigidity index r_{w} or r_{m}.

In the constraint (15), $\mathcal{L}$ denotes the set of active links, where each link is denoted by an unordered pair $\{i, j\}$ $(i, j \in \mathcal{I})$. This constraint specifies that K should lie within a convex cone in $\mathcal{K}_n$. For instance, $\mathcal{L}$ could be chosen so that $\{i, j\} \in \mathcal{L}$ if and only if agent i is within a disc of radius R (called an *R-disc*) of agent j, where $R > 0$ is the communication range. See Fig. 1 for an example.

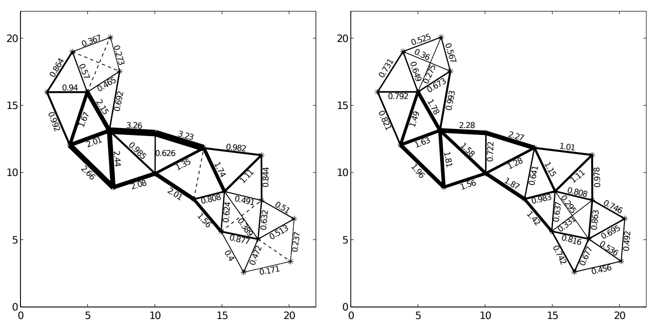
In the constraint (16), $c > 0$ is the maximum amount of resources to be allocated over all active links. This quantity could represent the total power available to maintain or localize the formation. For instance, wireless localization systems using Time of Arrival (TOA) estimation [17] can be modeled by KP-formations with k_{ij} being the inverse of the Cramer-Rao lower bound (CRB) of distance sensing

[1] Strictly speaking, the validity of this assumption requires that the formation be globally rigid [2]. Since we are studying the performance of a presumed localization algorithm rather than the localizability of formations, the concept of global rigidity is not emphasized here.



(a) (b)

Fig. 1. R-discs of agents ($R = 5$) and the generated formation



(a) WRI (b) MRI

Fig. 2. Optimal link resource allocation schemes

errors, which is inversely proportional to the energy of the sensing signals [16]. As a result, c may be interpreted as the total power of pulse signals used for distance measurements, and k_{ij} indicates the portion of the power used to measure the distance between agents i and j within certain accuracy.

Therefore, solving the optimization problem (14) with the constraints (15) and (16) will actually reveal how the signal power for distance sensing should be allocated over all active links so that the agents will be most efficiently localized. Since both rigidity indices $r_{\mathrm{w}}(K, \mathbf{p})$ and $r_{\mathrm{m}}(K, \mathbf{p})$ are concave functions of K and the constraints define a compact convex feasible set, this problem is a convex optimization problem whose optimal solution is guaranteed to exist and can be effectively solved by existing numerical methods.

B. Examples

We optimize the link resources for the formation depicted in Fig. 1 with respect to both the WRI r_{w} and the MRI r_{m}, where the resource limit c is set to $\#\mathcal{L}$, that is, each link can be allocated unit amount of resources on average. The convex optimization problem (14) is then solved using the CVX toolbox in MATLAB [18]. The obtained optimal resource allocations with respect to r_{w} and r_{m} are displayed in Fig. 2(a) and Fig. 2(b), respectively. A thicker line segment indicates that a larger portion of resources should be allocated to that link; whereas a dashed line indicates that the link has been eliminated (k_{ij} drops to zero) by the optimization process.

Several observations can be made by comparing Fig. 2(a) and Fig. 2(b). Both results show that the "bridge" links connecting the two clusters of agents need significantly more resources than links that are close to the fringe of the formation. However, the optimal allocation scheme in Fig. 2(a) is further polarized than that in Fig. 2(b), with several links in the original formation eliminated (as shown by the dashed lines) after the optimization, thus leaving the peripheral agents connected very weakly. In general, the allocations obtained by optimizing the MRI r_{m} tend to be more smoothly distributed and thus preferable in practice. On the other hand, by tracking the links eliminated in the optimization process using the WRI r_{w}, dispensable links that contribute little to the overall formation rigidity can be identified. This is especially useful when the number of active links needs to be reduced.

V. Sensitivity of Rigidity Indices

In this section, we will investigate how sensitive the rigidity indices are to changes in link connectivity. To this purpose, we compute the partial derivatives of both r_{w} and r_{m} with respect to k_{ij} for all links $\{i, j\} \in \mathcal{L}$. Throughout the section, the agents configuration $\mathbf{p}$ is assumed to be fixed and hence can be dropped in the notations for simplicity.

A. Worst-Case Rigidity Index Sensitivity

By definition, r_{w} is the fourth smallest eigenvalue λ_4 of S. If λ_4 is a distinct eigenvalue of $S(K)$, the partial derivative of r_{w} with respect to k_{ij} is given by [19] as

$$\frac{\partial r_{\mathrm{w}}}{\partial k_{ij}} = \mathbf{v}_4^\top \left(\frac{\partial S}{\partial k_{ij}} \right) \mathbf{v}_4,$$

where $\mathbf{v}_4$ is the unit eigenvector corresponding to the eigenvalue $\lambda_4(S)$. Noting that $\frac{\partial S}{\partial k_{ij}} = \mathbf{q}_{ij}\mathbf{q}_{ij}^\top$ where $\mathbf{q}_{ij}$ is defined in Section II, the *WRI sensitivity* with respect to link $\{i, j\}$ at K can be expressed as

$$\frac{\partial r_{\mathrm{w}}}{\partial k_{ij}} = \left| \mathbf{v}_4^\top \mathbf{q}_{ij} \right|^2, \tag{17}$$

where $\mathbf{v}_4$ depends on K. The fact that $\frac{\partial r_{\mathrm{w}}}{\partial k_{ij}} \geq 0$ for all k_{ij} conforms with the monotonicity property of the WRI r_{w}.

B. Mean Rigidity Index Sensitivity

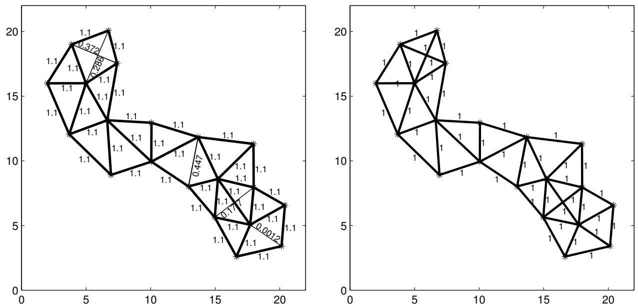
Suppose the KP-formation $(\mathcal{I}, \mathbf{p}, K)$ is rigid. Then the MRI r_{m} is given by $(2n-3)\left(\mathrm{tr}(S(K)^\dagger)\right)^{-1}$. Since both S and $S^\dagger$ are symmetric, we have

$$\begin{aligned} \frac{\partial r_{\mathrm{m}}(K)}{\partial k_{ij}} &= \frac{\partial}{\partial k_{ij}} \left(\frac{2n-3}{\mathrm{tr}(S^\dagger)} \right) = -\frac{2n-3}{\left(\mathrm{tr}(S^\dagger)\right)^2} \cdot \mathrm{tr}\left(\frac{\partial S^\dagger}{\partial k_{ij}} \right) \\ &= \frac{2n-3}{\left(\mathrm{tr}(S^\dagger)\right)^2} \cdot \mathrm{tr}\left(S^\dagger \frac{\partial S}{\partial k_{ij}} S^\dagger \right). \end{aligned}$$

By noting again that $\frac{\partial S}{\partial k_{ij}} = \mathbf{q}_{ij}\mathbf{q}_{ij}^\top$, the *MRI sensitivity* with respect to link $\{i, j\}$ at K can be expressed as

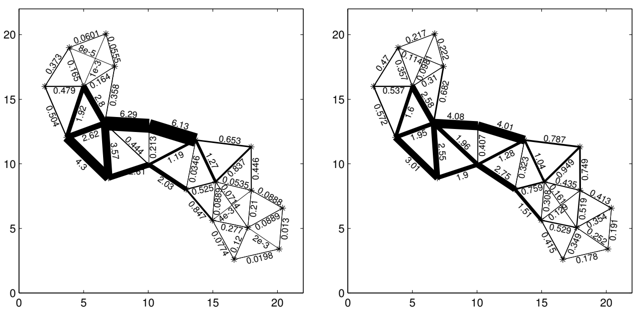
$$\frac{\partial r_{\mathrm{m}}}{\partial k_{ij}} = \frac{2n-3}{\left(\mathrm{tr}(S^\dagger)\right)^2} \cdot \left\| S^\dagger \mathbf{q}_{ij} \right\|^2, \tag{18}$$

where S, hence $S^\dagger$, depends on K. As in the WRI case, it also holds that $\frac{\partial r_{\mathrm{m}}}{\partial k_{ij}} \geq 0$ for all k_{ij}, which verifies the monotonicity property of the MRI r_{m}.



(a) WRI (b) MRI

Fig. 3. Sensitivity distributions at optimal allocation schemes K^*



(a) WRI (b) MRI

Fig. 4. Sensitivity distributions at equal allocation scheme

C. Examples

We compute both the WRI and the MRI sensitivities for all links $\{i, j\} \in \mathcal{L}$ to obtain the sensitivity distributions at different allocation schemes K. The numerical results are plotted in Fig. 3 and Fig. 4. The thickness of the links represent the sensitivity values, which are normalized so that the mean sensitivity value is equal to 1.

Fig. 3 illustrates the sensitivity distributions of both the WRI $r_{\rm w}$ and the MRI $r_{\rm m}$ at their respective optimal allocation schemes (as plotted in Fig. 2(a) and Fig. 2(b)). In both figures, it can be seen that the sensitivity values are mostly uniform across the whole formation except for only a few links in Fig. 3(a) whose allocated resources have already been reduced to zero in optimizing the WRI (as depicted by dashed lines in Fig. 2(a)). This is consistent with the optimality of both schemes, as the rigidity indices cannot be further increased by shifting resources from one link to another.

Next, we compute the sensitivity distributions of both rigidity indices at the equal allocation scheme, i.e., each link is allocated the same unit amount of resources. The computed sensitivity distributions are shown in Fig. 4. The thicker line segments indicate the more "sensitive" links. Shifting resources from other links to these more sensitive ones will result in an increase of the rigidity indices. As a result, it can be expected that these links will demand more resources in the optimal allocation schemes, as is verified by the plots in Fig. 2.

VI. CONCLUSION

In this paper, the definitions and properties of the stiffness matrix and the two rigidity indices, namely the WRI and the MRI, are presented. These concepts are then used in formulating the link resource allocation problem for maximizing the rigidity of multi-agent formations. This resource allocation problem is shown to be a convex optimization problem, and can be solved by well known solution algorithms. Perturbational analysis is also applied to study the sensitivity of rigidity to resources allocated to individual links. Numerical examples suggest that the proposed model and problem formulation lead to consistent results.

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