2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Finite time estimation and containment control of second order perturbed directed networks

Di Yu, Qinghe Wu, *member, IEEE,* Li Song

Abstract— **In this paper, an efficient architecture is proposed to achieve finite time containment control of second-order perturbed directed networks with the introduction of distributed estimators. Two cases of dynamic leaders with constant velocity and variable velocity are analyzed based on finite time stability theory. In particular, we propose homogeneous and sequential estimators to guarantee accurate desired position and velocity estimation of followers in finite time. Then the accurate estimations obtained are employed to achieve robust finite time containment control. Distributed control protocols are developed by applying homogeneity theory and sliding mode control so as to make followers converge and remain within the dynamic convex hull spanned by the leaders in finite time and suppress perturbation effectively. Finally, several simulation results are presented as a proof of theoretical analysis.**

I. INTRODUCTION

Recently, research on multi-agent cooperative control has drawn significant interest and has received comprehensive successful applications in many fields. Its main task is to design suitable control protocols to achieve overall behavior with local interaction. At present, most results only achieve asymptotic cooperation where the control laws at most achieve exponential convergence in infinite time. In contrast, finite time control offers many benefits including faster convergence, precise performance and robustness to uncertainties and disturbances [1],[2]. Approaches used in finite time control mainly include homogeneity theory, sliding mode control method, nonsmooth analysis approach [3],[4] and so on. Reference [5] established the basis of finite time stability theory for homogeneous system which demonstrated that the homogeneous system having negative degree was finite time stable if it was asymptotic stable. Based on [5], continuous control protocol was designed to achieve consensus of second undirected networks in [6]. However, the homogeneity approach can only deal with the finite time stable problem of taking into account homogeneous disturbance[7]. Sliding mode control was often used due to its attractive characteristics of finite-time convergence and robustness to perturbations. For instance, [8] discussed the robust finite time tracking of multi-robot system with single leader applying terminal sliding mode control. Nevertheless, its specific disadvantage is the chattering phenomenon. Hence, many researchers focused on reducing the chattering while preserving its advantages. It is worth concerning that [9] designed finite time strategy combined homogeneous theory with sliding mode approach for high order system with bounded uncertainty, which attenuated the chattering effectively and provided new idea for robust finite time control.

Different from existing leaderless synchronization or cooperative tracking with single leader, containment control exists multiple leaders and followers, where followers are driven to the minimum geometric space spanned by the leaders. Moreover, the attention on this aspect is increasing motivated by numerous potential applications. In [10], a stop-and-go strategy was proposed for containment control of first order fixed undirected network. Based on the notion of convexity, containment control for single-integrator and double-integrator kinematics without perturbation were studied in [11],[12], moreover, drew conclusion that asymptotical containment control could be obtained for double integrator networks with dynamic leaders having nonidentical velocities by requiring that the communication patterns among the followers were undirected. Meanwhile, containment control for multiple Lagrangian systems and multiple rigid body systems with model uncertainty were discussed in the presence of both stationary and dynamic leaders in [13],[14], motivated by [15],[16], where undirected connected network topology between followers were also required, moreover, homogeneous uncertainty could only been tackled in the case of stationary leaders.

In practical applications, the agents will be influenced by the nonlinear perturbation including model uncertainty and stochastic disturbance. Furthermore, information exchange between agents often occurs in one direction, for example, some members of heterogeneous vehicles may have transceivers while other less capable ones only have receivers. Consequently, it is significant and valuable to discuss the finite time containment control of perturbed directed networks. But to the best of our knowledge, this problem hasn't been addressed in the literature. Suppose that there are multiple dynamic leaders, the goal of the paper is to drive the followers to the convex hull spanned by leaders in finite time for second order directed network with followers influenced by nonlinear perturbation. The main contributions of the paper are introducing the distributed control configuration to decouple the task into two sequential procedure of finite time estimation and containment control, then proposing two different finite time estimators and designing distributed control protocols to achieve robust finite time containment control in two cases of dynamic leaders with constant velocity and variable velocity.

This work was supported by by the National Natural Science Foundation of P. R. China under Grant 61074031.

Di Yu, Qinghe Wu and Li Song are with Faculty of Automation, Beijing Institute of Technology, Beijing, 100081, P. R. China yudizlg@yahoo.cn, qinghew@bit.edu.cn

II. Preliminaries

A. Algebraic graph theory and matrix theory

For a network of n agents, the interaction for all agents can be naturally illustrated by a directed graph $G(\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = \{\nu_1, \ldots, \nu_n\}$ denotes the set of nodes involving a finite nonempty set denoting the agents, $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ denotes the set of edges involving a set of ordered pairs denoting the direction of information flow and $\mathcal{A} = [a_{ij}] \in R^{n\times n}$ represents nonnegative weighted adjacency matrix. Each edge denoted as ε_{ij} means that ith agent can access the state information of jth agent if $\varepsilon_{ij} \in \mathcal{E}$, meanwhile, $a_{ij} = 1$ and $a_{ij} = 0$, otherwise. We use $\mathcal{N}_i$ to denote the neighbor net of node v_i. The Laplacian matrix associated with $\mathcal{A}$ is defined as $\mathcal{L} = [l_{ij}] \in R^{n\times n}$ with $l_{ii} = \sum_{j\neq i} a_{ij}, l_{ij} = -a_{ij}$. Moreover, its row sums are all one and zero is an eigenvalue of $\mathcal{L}$ with an associated right eigenvector 1_n, where 1_n is the $n \times 1$ vector of all ones. A directed path is a sequence of edges of the form $(\nu_1, \nu_2), (\nu_2, \nu_3), \ldots, \nu_j \in \mathcal{V}$. The directed graph is strongly connected if there is a directed path between all distinct nodes.

Let $M_n(R)$ denote $n \times n$ real matrix set. Let matrix $C = [c_{ij}] \in M_n(R)$. It is defined as nonnegative matrix and denoted as $C \geq 0$ if its entries are all nonnegative. If $|c_{ii}| > \sum_{j\neq i} |c_{ij}|$, $i = 1, 2, \cdots, n$, it is said to be strongly diagonally dominant. It is defined as M-matrix if its off-diagonal elements are nonpositive and all principal minors are nonnegative. Moreover, it is said to be reducible if it can be transformed to lower triangular matrix with permutation matrix $P \in M_n(R)$ and irreducible or else.

B. Definations and lemmas

Definition 2.1: Let X be a set in a real vector space $V \subseteq R^n$, the convex hull $Co(X)$ is defined as

$$Co(X) = \{\sum_{i=1}^{k} \alpha_i x_i | x_i \in X, \alpha_i \in R, \alpha_i \geq 0, \sum_{i=1}^{k} \alpha_i = 1, k = 1, 2, \cdots\}$$

Definition 2.2: For simplicity, let $sig^\alpha(x) = |x|^\alpha sign(x)$, where $\alpha > 0$ and $sign(\cdot)$ is the signum function. And let $sig^\alpha(X) = [sig^\alpha(x_1), sig^\alpha(x_2), \cdots, sig^\alpha(x_n)]^T$ for $X = [x_1, x_2, \cdots, x_n]^T$.

Definition 2.3: Consider the following system

$$\dot{x} = f(x), f(0) = 0, x \in R^n \tag{1}$$

let $(r_1, \cdots, r_n) \in R^n$, $r_i > 0, i = 1, \cdots, n$. And let $f(x) = [f_1(x), \cdots, f_n(x)]^T$ be a continuous vector field, $f(x)$ is said to be homogeneous of degree $k \in R$ with respect to $(r_1, \ldots, r_n)$ if $f_i(\varepsilon^{r_1} x_1, \cdots, \varepsilon^{r_n} x_n) = \varepsilon^{k+r_i} f_i(x)$, $i = 1, \ldots, n$, $\forall x \in R^n$, for any given $\varepsilon > 0$. System (1) is said to be homogeneous if $f(x)$ is homogeneous.

Lemma 2.1: [5] Suppose that system (1) is homogeneous of degree $k \in R$, then the system is finite-time stable if it is asymptotically stable and $k < 0$.

Lemma 2.2: [13] Consider the following system

$$\dot{x} = f(x, t), f(0, t) = 0, x \in U_0 \subset R^n \tag{2}$$

Where $f : U_0 \times \mathrm{R}^+ \to \mathrm{R}^n$ is continuous with respect to x on an open neighborhood U_0 of the origin $x = 0$. Suppose that there are a C^1 positive-definite function $V(x, t)$(defined on $\hat{V} \times R^+$ where $\hat{V} \subset U_0 \subset R^n$ is a neighborhood of the origin), real number $c > 0, 0 < \alpha < 1$, such that $\dot{V}(x, t) + cV^\alpha(x, t)$ is negative semi-definite on $\hat{V}$,then $V(x, t)$ approaches to 0 in finite time, in addition, the finite settling time satisfies that $T \leq \dfrac{V(x(t_0), t_0)^{1-\alpha}}{c(1-\alpha)}$.

Lemma 2.3: [17] Suppose the directed topology is strongly connected and its Laplacian matrix is denoted as $\mathcal{L}$. Let diagonal matrix $B = diag(b_1, \cdots, b_n)$. If $b_i > 0$ for at least one i, then $\mathcal{L} + B$ is nonsingular.

III. Problem statement

For a group of n agents, suppose that $F = \{1, \ldots, m\}$ and $L = \{m+1, \ldots, n\}$ denote the follower and the leader index sets, respectively. The goal of the paper is to design suitable estimators and control protocols to drive the followers to converge to the dynamic convex hull spanned by the leaders.

A. Network dynamics

The dynamics of followers and leaders are described as

$$\begin{aligned} &\dot{x}_i = v_i \ , \ \dot{v}_i = u_i + d_i \ , \ i \in F \\ &\dot{x}_i = v_i \ , \ \dot{v}_i = u_i \ , \ i \in L \end{aligned} \tag{3}$$

where $x_i \in R$, $v_i \in R$, $u_i \in R$, $i = 1, 2, \ldots, n$ are the position, the velocity and the control input associated with ith agent, respectively, in addition, $d_i \in R, i \in F$ is nonlinear perturbation associated with ith follower. Let $x_f = [x_1, \cdots, x_m]^T$, $v_f = [v_1, \cdots, v_m]^T, u_f = [u_1, \cdots, u_m]^T$ and $d = [d_1, \cdots, d_m]^T$, where $\|d\|_\infty < \sigma$. Similarly, Let $x_l = [x_{m+1}, \cdots, x_n]^T$, $v_l = [v_{m+1}, \cdots, v_n]^T, u_l = [u_{m+1}, \cdots, u_n]^T$, so $x = [x_f, x_l]^T$, $v = [v_f, v_l]^T, u = [u_f, u_l]^T$ are the position, velocity and control vector of the whole network, respectively.

B. Network topology

Suppose that the leaders have no communication between each other and the communication between a leader and a follower is unidirectional with the leader issuing the communication. So the topology between followers and one between followers and leaders determine the whole network communication. We have the following structure by proper decomposition of Laplacian matrix $\mathcal{L}$

$$\mathcal{L} \begin{bmatrix} x_f \\ x_l \end{bmatrix} = \begin{bmatrix} \mathcal{T} & \mathcal{T}_d \\ 0_{(n-m)\times m} & 0_{(n-m)\times(n-m)} \end{bmatrix} \begin{bmatrix} x_f \\ x_l \end{bmatrix} \tag{4}$$

where $\mathcal{T} = [\mathcal{T}_{ij}] \in R^{m\times m}$,$\mathcal{T}_d \in R^{m\times(n-m)}$.

C. Network error

Define the error functions as

$$e_i^1 = \sum_{j=1}^{n} a_{ij}(x_i - x_j), e_i^2 = \sum_{j=1}^{n} a_{ij}(v_i - v_j), i \in F$$

so the error dynamics of the whole network is described as

$$\begin{aligned} &\dot{E}_1 = E_2 \\ &\dot{E}_2 = \mathcal{T}(u_f + d) + \mathcal{T}_d u_l \end{aligned} \tag{5}$$

where$E_1 = [e_1^1, \cdots, e_m^1]^T$, $E_2 = [e_1^2, \cdots, e_m^2]^T$.

Assumption 3.1: Suppose that the topology between followers are strongly connected, and there exists at least one leader that has a path to each follower.

We develop the following lemma before proposing the main result at first.

Lemma 3.1: Consider the directed network given by the dynamics (3) and the error dynamics given by (5). Assume Assumption 3.1 holds, then the followers remain within the convex hull spanned by the leaders if $E_1 = 0$ and $E_2 = 0$, particularly, they have the desired positions and velocities.

Proof: The Laplacian matrix corresponding to the topology between followers, denoted as $\mathcal{T}_F$, is irreducible M-matrix, and each row sum of nonnegative matrix $-\mathcal{T}_d$ is positive, denoted as $b_i > 0, i \in F$ from assumption 1. Let $B = diag(b_1, \cdots, b_m)$, so $\mathcal{T} = \mathcal{T}_F + B$ is strongly diagonally dominant M-matrix, so its inverse $\mathcal{T}^{-1} = [T_{ij}] \in R^{m\times m}$ is positive according to Lemma 2.3 and $-\mathcal{T}^{-1}\mathcal{T}_d$ is nonnegative. Note that $\begin{bmatrix}\mathcal{T} & \mathcal{T}_d\end{bmatrix} 1_m = 0_n$, thus each row sum of $-\mathcal{T}^{-1}\mathcal{T}_d$ is equal to one. Let $x_d = -\mathcal{T}^{-1}\mathcal{T}_d x_l$, $v_d = -\mathcal{T}^{-1}\mathcal{T}_d v_l$, $x_d = [x_{d1}, \cdots, x_{dm}]^T$ and $v_d = [v_{d1}, \cdots, v_{dm}]^T$, where $x_{di} \in R$, $v_{di} \in R$, $i \in F$, are the desired position and velocity of followers, which are the convex weighted average of ones of leaders. Let $\tilde{x}_f = x_f - x_d$, $\tilde{v}_f = v_f - v_d$ then $E_1 = \mathcal{T}x_f + \mathcal{T}_d x_l = \mathcal{T}\tilde{x}_f$, $E_2 = \mathcal{T}v_f + \mathcal{T}_d v_l = \mathcal{T}\tilde{v}_f$. Thus $x_f = x_d$ and $v_f = v_d$ is true when $E_1 = 0$ and $E_2 = 0$ from the nonsingularity of matrix $\mathcal{T}$, that is, the followers remain within the convex hull spanned by the leaders, particularly, they have the desired positions and velocities. ■

IV. Main Result

Suppose that the positions and velocities of leaders are available to only a subset of followers. And there exists an estimator embedded in each follower. So we propose a distributed architecture for containment control is shown in Fig.1. The hierarchical architecture consists of three layers: desired states estimation layer, containment control layer and followers layer. In estimation level, $\hat{x}_{di} = x_{di}$, $\hat{v}_{di} = v_{di}$, $i \in F$ can be achieved in finite time. Based on the accurate estimations of desired position and velocity of followers, the states of the followers converge to the convex hull formed by those of leaders in finite time in control level. So the whole scenario is decoupled into two subtasks, i.e., accurate estimation and containment control. In the following sections, robust finite time containment control based on finite time estimation are discussed under two cases: dynamic leaders with constant velocity and variable velocity.

A. Dynamic leaders with constant velocity

Propose the following distributed homogeneous estimator

$$\begin{aligned}\dot{\hat{x}}_{di} &= \hat{v}_{di} - \beta\sum_{j=1}^{m} T_{ij} sig[\sum_{k=1}^{n} a_{jk}(\hat{x}_{dj} - \hat{x}_{dk})]^{\gamma_1} \\ \dot{\hat{v}}_{di} &= -\beta\sum_{j=1}^{m} T_{ij} sig[\sum_{k=1}^{n} a_{jk}(\hat{x}_{dj} - \hat{x}_{dk})]^{\gamma_2}\end{aligned} \tag{6}$$

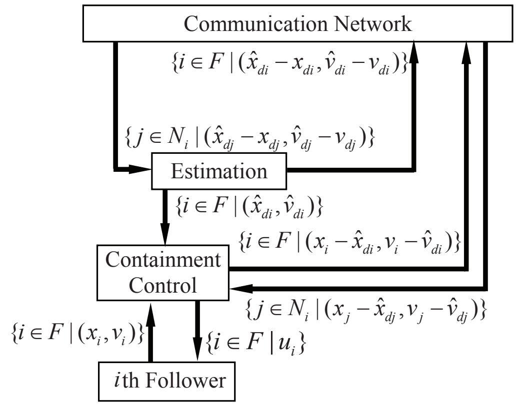

Fig. 1. Distributed architecture for containment control

Where $\beta > 0$, $0 < \gamma_2 < 1$, $\gamma_1 = \dfrac{\gamma_2 + 1}{2}$, $i \in F$. Moreover, $\hat{x}_{di}$ and $\hat{v}_{di}$ represent the estimations of desired position and velocity of ith follower, obviously, $\hat{x}_{di} = x_i$, $\hat{v}_{di} = v_i$, $i \in L$.

Theorem 4.1: Consider the directed network given by the dynamics (3). Assume that Assumption 3.1 holds, then $\hat{x}_{di} = x_{di}$, $\hat{v}_{di} = v_{di}$, $i \in F$ can be achieved in finite time if estimator (6) is used.

Proof: Define the estimation errors as $\tilde{x}_{di} = \hat{x}_{di} - x_{di}$, $\tilde{v}_{di} = \hat{v}_{di} - v_{di}$, $i \in F$ and the estimation error vectors as $\tilde{x}_d = [\tilde{x}_{d1}, \cdots, \tilde{x}_{dm}]^T$, $\tilde{v}_d = [\tilde{v}_{d1}, \cdots, \tilde{v}_{dm}]^T$. So (6) can be rewritten as

$$\begin{aligned}\dot{\tilde{x}}_d &= \tilde{v}_d - \beta\mathcal{T}^{-1} sig(\mathcal{T}\tilde{x}_d)^{\gamma_1} \\ \dot{\tilde{v}}_d &= -\beta\mathcal{T}^{-1} sig(\mathcal{T}\tilde{x}_d)^{\gamma_2}\end{aligned} \tag{7}$$

Let $\mathcal{E}_1 = [\varepsilon_1^1, \cdots, \varepsilon_m^1]^T = \mathcal{T}\tilde{x}_d$ and $\mathcal{E}_2 = [\varepsilon_1^2, \cdots, \varepsilon_m^2]^T = \mathcal{T}\tilde{v}_d$. Consider Lyapunov function $V_1 = \sum_{i=1}^{m}[\beta|\varepsilon_i^1|^{1+\gamma_2} + \dfrac{1+\gamma_2}{2}(\varepsilon_i^2)^2]$ for estimation error dynamics (7). Note that V_1 is positive definite with respect to ε_i^1 and ε_i^2, and its derivative is negative semi-definite. Therefore we can obtain that $\mathcal{E}_1 \to 0$ and $\mathcal{E}_2 \to 0$ by LaSalle's invariant principle, that is, system (7) is globally and asymptotically stable. By following the analysis in [15], we know that system (7) is a homogeneous system of degree $k = \dfrac{\gamma_2}{\gamma_1} - 1 < 0$ with respect to the dilation coefficient $(r_1 1_m^T, r_2 1_m^T)$ where $r_1 = \dfrac{1}{\gamma_1}$,$r_2 = 1$. Thus the finite time stability of system (7) is proved by using Lemma 2.1. Let the settling time is T, then x_{di} and v_{di} can be replaced by $\hat{x}_{di}$ and $\hat{v}_{di}$, respectively, when $t \geq T$. ■

To ensure the bounded states, apply the proportional plus derivative control for followers when $t \leq T$. When $t > T$, based on accurate estimation, design the following distributed control protocols (8), where $i \in F$, u_{ideal}^i is the ideal control protocol for ith agent without perturbation and u_{disc}^i is discontinuous control protocol for ith agent to suppress perturbation. Auxiliary variable e_{aux}^i is introduced in the design of the sliding mode variable associated with u_{disc}^i.

Theorem 4.2: Consider the directed network given by the dynamics (3). Assume Assumption 3.1 holds, containment control can be achieved in finite time if estimator (6) and control protocol (8) are used, where coefficients in (8) are satisfied with $\rho \geq \sigma + \delta$, $\delta > 0$, $0 < \alpha_1 < 1$, $\alpha_2 = \frac{2\alpha_1}{1+\alpha_1}$.

$$
\begin{aligned}
u_i &= u^i_{ideal} + u^i_{disc} \\
u^i_{ideal} &= -\sum_{j=1}^{m} \mathcal{T}_{ij}\{sig[\sum_{k=1}^{m}\mathcal{T}_{jk}(x_k - \hat{x}_{dk})]^{\alpha_1} \\
&\quad + sig[\sum_{k=1}^{m}\mathcal{T}_{jk}(v_k - \hat{v}_{dk})]^{\alpha_2}\} \\
u^i_{disc} &= -\rho sign(s_i) \\
s_i &= \sum_{j=1}^{m}\mathcal{T}_{ij}(v_j - \hat{v}_{dj}) + e^i_{aux} \\
\dot{e}^i_{aux} &= -\sum_{j=1}^{m}\mathcal{T}_{ij}u^j_{ideal}
\end{aligned} \tag{8}
$$

Proof: Based on the accurate estimations, we consider the finite time stability of ideal network regardless of the perturbation at first, then introduce a discontinuous control law with auxiliary variable to guarantee system to reach the sliding mode in finite time in spite of perturbation, at last prove that the finite time containment control is achieved on the sliding mode surface.

Let $u_{ideal} = [u^1_{ideal}, \cdots, u^m_{ideal}]^T$,$s = [s_1, \cdots, s_m]^T$, $u_{disc} = [u^1_{disc}, \cdots, u^m_{disc}]^T$, $E_{aux} = [e^1_{aux}, \cdots, e^m_{aux}]^T$. Then we obtain

$$
\begin{aligned}
u_f &= u_{ideal} + u_{disc} \\
u_{ideal} &= -\mathcal{T}^{-1}[sig(E_1)^{\alpha_1} + sig(E_2)^{\alpha_2}] \\
u_{disc} &= -\rho sgn(s) \\
s &= E_2 + E_{aux} \\
\dot{E}_{aux} &= -\mathcal{T}u_{ideal}
\end{aligned} \tag{9}
$$

Regardless of perturbation, we have $u_f = u_{ideal}$, then the network error dynamics is

$$
\begin{aligned}
\dot{E}_1 &= E_2 \\
\dot{E}_2 &= -sig(E_1)^{\alpha_1} - sig(E_2)^{\alpha_2}
\end{aligned} \tag{10}
$$

We take the similar procedure as Theorem 4.1, choose Lyapunov function $V_2 = \sum_{i=1}^{m}[(\dfrac{1}{1+\alpha_1}|e^1_i|^{1+\alpha_1}) + \frac{1}{2}(e^2_i)^2]$ for error dynamics (10). Note that V_2 is positive definite with respect to e^1_i and e^2_i. Then take the derivative of V_2 gives $\dot{V}_2 = -\sum\limits_{i=1}^{m}|e^2_i|^{\alpha_2+1} \leq 0$. Therefore we obtain that $E_1 \to 0$ and $E_2 \to 0$ by LaSalle's invariant principle, that is, error dynamics (10) is globally and asymptotically stable. As well, we know that error dynamics (9) is homogeneous of degree $k = \alpha_2 - 1 < 0$ with respect to the dilation coefficient $(r_1 1^T_m, r_2 1^T_m)$, where $r_1 = \dfrac{2}{1+\alpha_1}$, $r_2 = 1$ by following the analysis in [18]. So the finite time stability of error dynamics (10) is proved using Lemma 2.1. Thus the followers converge and remain within the convex hull spanned by the leaders by using lemma 3.1.

In order to suppress perturbation and achieve finite time control, the discontinuous control law with auxiliary variable is introduced. Define the sliding mode variable $s = E_2 + E_{aux}$, thus $\dot{s} = \dot{E}_2 + \dot{E}_{aux}$. Choose Lyapunov function $V_3 = \frac{1}{2}s^T s$, then take derivative of V_3 gives

$$
\dot{V}_3 = s^T\mathcal{T}(u + d - u_{ideal}) \leq -\delta\|\mathcal{T}\|\|s\| = -\sqrt{2}\delta\|\mathcal{T}\| V_3^{\frac{1}{2}}
$$

Thus we obtain that system trajectories evolve on the sliding mode $s \to 0$ in finite time and remain there in spite of perturbation according to Lemma 2.2. In sliding mode surface, the equivalent control of u_{disc}, denoted as u^{eq}_{disc}, obtained by $\dot{s} = 0$, is given by $u^{eq}_{disc} = -d$. Substituting $u_f = u_{ideal} + u^{eq}_{disc}$ into (5), we obtain the equivalent closed-loop error dynamics in sliding mode similar as the error dynamics (10). According to the first step of the proof, we can conclude that finite time containment control can be achieved if estimator (6) and control protocol (8) are used. ■

Remark 4.1: Control protocol (8) is motivated by [9], where studied finite time trajectory tracking of a single system with bounded uncertainty. In contrast, the paper deals with distributed containment control for directed networks with arbitrary perturbation and the states of leaders known to only a portion of followers.

B. Dynamic leaders with variable velocity

Assume that the time-varying control inputs of leaders are unknown to a portion of followers, and $\sup|u_i| \leq \mu, i \in L$.

Propose the following distributed sequential estimator

$$
\dot{\hat{x}}_{di} = \hat{v}_{di} - \beta\sum_{j=1}^{m}T_{ij}sig[\sum_{k=1}^{n}a_{jk}(\hat{x}_{dj} - \hat{x}_{dk})]^{\gamma} \tag{11a}
$$

$$
\dot{\hat{v}}_{di} = -\beta\sum_{j=1}^{m}T_{ij}sign[\sum_{k=1}^{n}a_{jk}(\hat{v}_{dj} - \hat{v}_{dk})] \tag{11b}
$$

Where $\beta > 0$, $0 < \gamma < 1$, $i \in F$. Moreover, $\hat{x}_{di}$ and $\hat{v}_{di}$ represent the estimations of desired position and desired velocity of ith follower, obviously, $\hat{x}_{di} = x_i$,$\hat{v}_{di} = v_i$,$i \in L$.

Theorem 4.3: Consider the directed network given by the dynamics (3). Assume Assumption 3.1 holds and $\beta > \mu\|\Gamma_d\|_\infty$, then $\hat{x}_{di} = x_{di}$, $\hat{v}_{di} = v_{di}$, $i \in F$ can be achieved in finite time if estimator (11) is used.

Proof: At first, we prove $\hat{v}_{di} \to v_{di}$, $i \in F$, then prove $\hat{x}_{di} \to x_{di}$, $i \in F$ based on the accurate estimation of the desired velocity of followers.

According to (11b), Choose Lyapunov function $V_4 = \frac{1}{2}\tilde{v}^T_d\mathcal{T}^T\mathcal{T}\tilde{v}_d$, then take the derivative of V_4 gives

$$
\begin{aligned}
\dot{V}_4 &= \tilde{v}^T_d\mathcal{T}^T\mathcal{T}(-\beta\mathcal{T}^{-1}sign(\mathcal{T}\tilde{v}_d) - \dot{v}_d) \\
&= (\mathcal{T}\tilde{v}_d)^T(-\beta sign(\mathcal{T}\tilde{v}_d) - \mathcal{T}\dot{v}_d) \\
&\leq -\sqrt{2}(\beta - \mu\|\mathcal{T}_d\|_\infty)V_4^{\frac{1}{2}}
\end{aligned}
$$

Here we have used Hölder inequality$|x^T y| \leq \|x\|_1\|y\|_\infty$, matrix norm compatibility $\|AB\|_\infty \leq \|A\|_\infty\|B\|_\infty$ and $\|A\|_1 \geq \|A\|_2$ for the inequality step. Then we obtain that $\hat{v}_{di} \to v_{di}$, $i \in F$ in settling time $T_1 = \dfrac{\sqrt{2V_4(0)}}{(\beta - \mu\|\mathcal{T}_d\|_\infty)}$

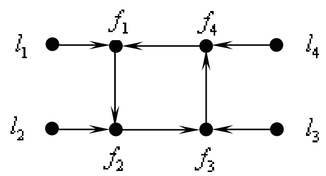

Fig. 2. Directed network topology for a group of eight agents

according to Lemma 2.2. And the accurate estimation of desired velocity of followers is achieved when $t > T_1$, and we can use $\hat{v}_{di}$ to replace v_{di}, $i \in F$.

According to (11a), choose Lyapunov function $V_5 = \frac{1}{2}\tilde{x}_d^T \tilde{x}_d$, where $\hat{x}_d = [\hat{x}_{d1}^T, \cdots, \hat{x}_{dm}^T]^T$, so we have

$$\begin{aligned}\dot{V}_5 &= \tilde{x}_d^T(-\beta \mathcal{T}^{-1} sig(\mathcal{T}\tilde{x}_d)^\gamma) \\ &\le -\beta\lambda_{\min}[(\mathcal{T}\mathcal{T}^T)^{-1}] \sum_{i=1}^{m} |(\mathcal{T}\tilde{x}_d)_i|^{\gamma+1} \\ &\le -2^{\frac{\gamma+1}{2}}\beta\lambda_{\min}[(\mathcal{T}\mathcal{T}^T)^{-1}][\lambda_{\min}(\mathcal{T}^T\mathcal{T})]^{\frac{\gamma+1}{2}} V_5^{\frac{\gamma+1}{2}}\end{aligned}$$

Where $\mathcal{T}^T\mathcal{T}$ and $\mathcal{T}\mathcal{T}^T$ are positive definite symmetric matrices, $\lambda_{\min}[\cdot]$ denotes its smallest eigenvalue of the matrix, $(\mathcal{T}\tilde{x}_d)_i$ denotes the ith entry of $\mathcal{T}\tilde{x}_d$. Let $k_1 = 2^{\frac{\gamma+1}{2}}\beta\lambda_{\min}[(\mathcal{T}\mathcal{T}^T)^{-1}][\lambda_{\min}(\mathcal{T}^T\mathcal{T})]^{\frac{\gamma-1}{2}}$, so we obtain that $\hat{x}_{di} \to x_{di}$, $i \in F$ in settling time $T_2 = T_1 + \dfrac{2V_5(T_1)^{\frac{1-\gamma}{2}}}{k_1(1-\gamma)}$ according to Lemma 2.2. And the accurate estimation of desired velocity of followers is achieved when $t > T_2$, we can use $\hat{x}_{di}$ to replace x_{di}, $i \in F$. ■

Theorem 4.4: Consider the directed network given by the dynamics (3). Assume that $\sup|u_i| \le \mu, i \in L$ and Assumption 3.1 holds, containment control can be achieved in finite time if estimator (11) and control protocol (8) are used, where coefficients in (8) are satisfied with $\rho \ge \sigma\|\mathcal{T}\| + \delta + \mu\|\mathcal{T}_d\|$, $\delta > 0$, $0 < \alpha_1 < 1$, $\alpha_2 = \frac{2\alpha_1}{1+\alpha_1}$.

Proof: We can prove by taking the same procedure as one in the proof of Theorem 4.2, omitted here. ■

V. Simulation

In this section, we present several simulation results to validate the theoretical results. We consider a group of eight agents. The directed topology is shown in Fig.2, where $l_i, i = 1, 2, 3, 4$ are denoted as leaders and $f_i, i = 1, 2, 3, 4$ are denoted as followers. We consider the containment control in the two-dimensional space. Here $x_i \in R^2$, $v_i \in R^2$, $u_i \in R^2$, $i = 1, 2, \cdots, n$. And the perturbation is chosen as $d_i = [0, 0.01\sin t]^T$, $i \in F$.

When the leaders have constant velocity, we choose parameters of estimator (6) as $\gamma_1 = 0.6$, $\gamma_2 = 0.2$, and ones of control law (8) as $\alpha_1 = 0.5$, $\alpha_2 = 0.667$, $\rho = 0.02$. Fig.3 and Fig.4 show the 2-norm of estimation errors of desired positions and velocities of followers using estimator (6), respectively. It can be seen that we can obtain accurate estimations of the desired positions and velocities of followers after $4s$. Simulation result about trajectories of agents using control law (8) in Case A is shown in Fig.5. Squares and circles denote the positions of the dynamic leaders and follows, respectively. We can see that all followers converge and remain within the dynamic convex hull formed by the leaders after about 15s with aids of two snapshots at $t = 15s$ and $t = 25s$.

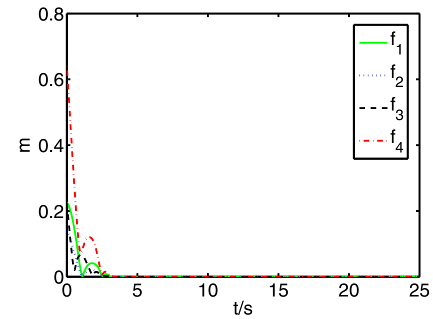

Fig. 3. The norm of desired position estimation errors using estimator (6)

When the leaders have time-varying velocity,we choose parameters of estimator (11) as $\gamma = 0.6$, $\beta = 0.2$ and ones of control law (8) as $\alpha_1 = 0.6$, $\alpha_2 = 0.75$, $\rho = 0.04$. Fig.6 and Fig.7 show the 2-norm of estimation errors of desired positions and velocities of followers using estimator (11), respectively. It can be seen that we can obtain accurate estimations of the desired velocities and positions of followers in $4s$ and $12s$, respectively. Simulation result about trajectories of agents using control law (8) in Case B is shown in Fig.8. Squares and circles denote the positions of the dynamic leaders and followers, respectively. We can see that all followers converge and remain within the dynamic convex hull formed by the leaders after about $25s$ with aids of two snapshots at $t = 25s$ and $t = 50s$.

VI. Conclusion

We have proposed an efficient distributed architecture to achieve finite time containment control of second-order directed networks with nonlinear perturbation. It includes estimation layer, containment control layer and followers layer. Thus the whole scenario can be decoupled into two subtasks, that is, distributed estimation and containment control. Based on finite time stability theory, two different estimators were designed to achieve the accurate estimation of desired position and velocity of followers in finite time. Then the accurate estimations obtained were employed by containment control level. In two cases of dynamic leaders with constant velocity and variable velocity, we designed a distributed control protocol composed of homogeneous part and switching part with an auxiliary variable, which not only made followers converge to the dynamic convex hull spanned by the leaders in finite time but also suppressed perturbation effectively. Finally, several simulation results are used to validate the correctness of theoretical analysis.

References

[1] S. P. Bhat and D. S. Bernstein, "Finite-time stability of continuous autonomous systems", *SIAM Journal of Control and Optimization*, vol. 38, no. 3, 2000, pp. 751-766.

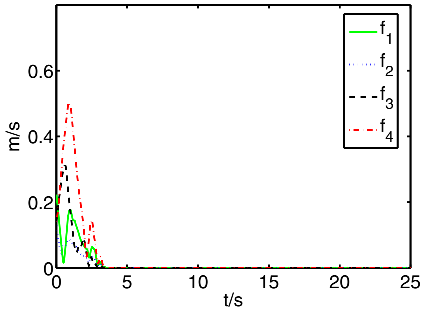

Fig. 4. The norm of desired velocity estimation errors using estimator (6)

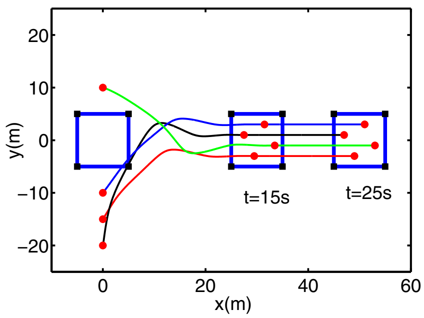

Fig. 5. Trajectories of eight agents using control protocol (8) in Case A

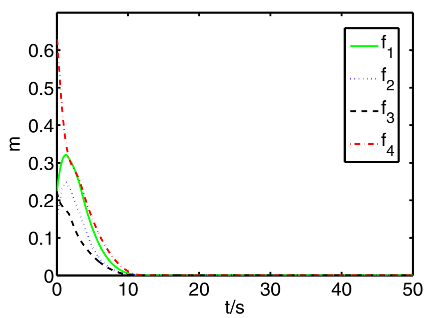

Fig. 6. The norm of desired position estimation errors using estimator (11)

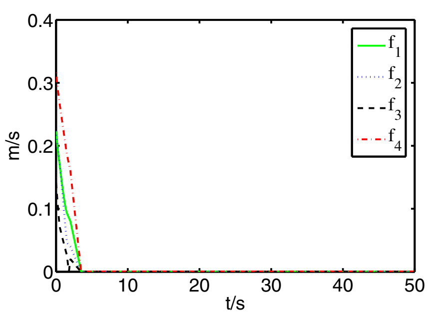

Fig. 7. The norm of desired velocity estimation errors using estimator (11)

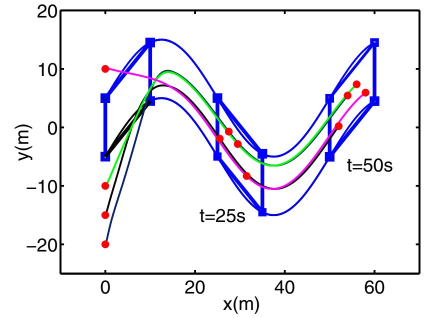

Fig. 8. Trajectories of eight agents using control protocol (8) in Case B

[2] S. T. Venkataraman and S. Gulati, "Terminal slider control of nonlinear systems," *in 1990 International Conference of Advanced Robotics*, Pisa, Italy, 1990, pp. 2513-2514.

[3] J. Cortes, "Finite-Time Convergent Gradient Flows with Applications to Network Consensus", *Automatica*, vol.42, no. 11, 2006, pp. 1993-2000.

[4] Q. Hui, "Finite-Time Rendezvous Algorithms for Mobile Autonomous Agents", *IEEE Transactions on Automatic Control*, vol. 56, no. 1, 2011, pp. 201-211.

[5] S. P. Bhat and D. S. Bernstein, "Finite-Time Stability of Homogeneous Systems", *in 1997 American Control Conference*, Albuquerque, New Mexico, 1997, pp. 2513-2514.

[6] X. Wang and Y. Hong, "Finite-time Consensus for Multi-Agent Networks with Second Order Agent Dynamics", *in 17th World Congress of the International Federation of Automatic Control*, 2008, pp. 15185-15190.

[7] A. Levant, "Homogeneity Approach to High-order Sliding Mode Design", *Automatica*, vol.41, no. 5, 2005, pp. 823 - 830.

[8] S. Khoo and L. Xie and Z. Man, "Robust Finite-Time Consensus Tracking Algorithm for Multirobot Systems", *IEEE/ASME transactions on mechatronics*, vol. 14, no. 2, 2009, pp. 219-228.

[9] M. Defoort and T.Floquet and A.Kokosy, "A Novel Higher Order Sliding Mode Control Scheme", *Systems and Control Letters*, vol. 58, no. 2, 2009, pp. 102-108.

[10] M. Ji and G. Ferrari-Trecate and M. Egerstedt and A. Buffa, "Containment Control in Mobile Networks", *IEEE Transactions on Automatic Control*, vol. 53, no. 8, 2008, pp. 1972-1975.

[11] Y. Cao and W. Ren, "Containment Control with Multiple Stationary or Dynamic Leaders under a Directed Interaction Graph", *in 2009 IEEE Conference on Decision and Control*, Shanghai, China, 2009, pp. 3014-3019.

[12] Y. Cao and D. Stuart and W. Ren, "Distributed Containment Control for Multiple Autonomous Vehicles With Double-Integrator Dynamics:Algorithms and Experiments", *IEEE Transaction on control system technology*, to be published.

[13] Z. Meng and W. Ren and Z. You, "Distributed Finite-Time Containment Control for Multiple Lagrangian Systems", *in 2010 American Control Conference*, 2010, pp. 2885-2890.

[14] Z. Meng and W. Ren and Z. You, "Distributed Finite-Time Attitude Containment Control for Multiple Rigid Bodies", *Automatica*, vol.46, no. 12, 2010, pp. 2092-2099.

[15] Y. Hong and Y. Xu and J. Huang, "Finite-Time Control for Robot Manipulators", *Systems and Control Letters*, vol. 46, no. 4, 2002, pp. 243-253.

[16] S. Yu and X. Yu and B. Shirinzadeh and Z. Man, "Continuous Finite Time Control for Robotic Manipulators with Terminal Sliding Mode", *Automatica*, vol. 41, no. 11, 2005, pp. 1957-1964.

[17] Z. Qu, Cooperative Control of Dynamical Systems: Applications to Autonomous Vehicles, Springer-verlag, New York; 2009.

[18] S. P. Bhat and D. S. Bernstein, "Geometric Homogeneity with Applications to Finite Time Stability", *Math. Control, Signals and Systems*, vol. 17, no. 2, 2005, pp. 101-127.