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Optimal second order consensus protocol with time delay

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***Abstract*— In recent years, stability of consensus protocol with time delay has attracted a lot of interests. On the basis of local information exchange with limited time delay, the states of all the agents reach consensus with a second order consensus protocol under certain conditions, where the convergence speed is determined by eigenvalue spectrum of network Laplacian and system parameters. In this paper, the problem of fast consensus seeking in networked multi-agent systems is considered, which aims to optimize the convergence speed of consensus protocol with time delay. By using the frequency domain method, the optimization problem is decomposed into a series of convex optimization problems that can be solved by SeduMi toolbox. Furthermore, a consensus protocol with multi-hop relay scheme is investigated, where each agent can receive delayed information data transmitted by multi-hop agents. The results show that this scheme can improve the convergence speed without physically changing network topology. Finally, some examples are supplied to demonstrate theoretical results.**

I. INTRODUCTION

The huge advances in Internet and large-scale integration and micro-electromechanical technologies have created the opportunity to interconnect a large number of devices such as wireless sensors which can exchange information and cooperatively accomplish a variety of tasks ranging from military surveillance, unmanned aerial vehicles for intelligence, navigation, formation control and environmental monitoring. Consensus problem as a basis of cooperative control has been employed in many engineering problems, which motivates the fast development of its theoretical research. In past decade, much work [1]- [15], [17] on consensus problem has been done by the researchers from diverse areas including control engineering, computer science, system biology and physics. A lot of work studies the stability of consensus protocol with fixed and switched network topology, and shows the importance of network connectivity on the convergence property, see e.g., Olfati-Saber [3], Ren [5], Moreau [4], Cao [6]. A few work addresses the optimal consensus protocol design for specific goal, e.g., Xiao [7], [9], Bauso [8], Semsar-Kazerooni [10], Carli [11]. In [7], the authors consider the problem of finding the fastest converging linear iteration for distributed averaging consensus problem. The work in [8] designs a consensus protocol to solve individual optimizations performed by the agents. In [10], an optimal designed based consensus protocol minimizing team cost function is proposed. In [11], an optimal synchronization protocol is designed for fastest convergence speed and minimal steady state error in case the protocol is perturbed by an additive noise.

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In real applications, when local information data travel along channels in a large communication network, the effect of communication delay cannot be neglected. Some results have shown that consensus can still be achieved if the network topology is connected with upper bounded time delay [3], [12]. And the larger the second eigenvalue of the Laplacian $\lambda_2(L)$ is (Note that $\lambda_2(L)$ denotes the algebraic connectivity), the faster the convergence speed is. But the larger the largest eigenvalue $\lambda_{max}(L)$ is, the smaller the delay margin is. Thus, a tradeoff exists between the delay margin and the convergence speed. One important challenge is to improve the convergence speed of consensus protocol with communication delay, however, it seems to be less studied in the literature.

In this paper, we study a second order consensus protocol with fixed time delay. A lot of results for the stability of consensus protocol with uniform and non-uniform time delay can be found in [12]- [15]. Here, we focus on optimizing the convergence speed by assuming that the communication delay between one-hop neighbor is the same for commensurate [3] and non-commensurate [12] delay algorithms, respectively. We analyze the convergence property of consensus protocol with time delay by a frequency domain method, which also helps us decompose the problem of finding the fastest convergence speed into solving a series of convex optimization problem. Motivated by the multi-hop relay scheme of wireless sensor network, we study a consensus protocol with multi-hop relay scheme, where each agent can receive the information data travel along multi-hop communication links with time delay besides the information data over one hop links. We derive the delay margin for the stability of the multi-hop consensus protocol by generalized eigenvalue searching method. Finally, we provide some examples to demonstrate the theoretical results.

The rest of the paper is organized as follows. In Section II, we describe the system models, and analyze the convergence property of consensus protocol with commensurate and non-commensurate delay. In Section III, we focus on the problem of fast consensus seeking for consensus protocol with non-commensurate delay. In Section IV, we investigate a consensus protocol with multi-hop relay scheme. In Section V, we demonstrate the theoretical results derived in Section III and IV. Finally, some concluding remarks and future work are given in Section VI.

II. SECOND ORDER CONSENSUS PROTOCOL WITH TIME DELAY

We firstly introduce some notations and concepts that will be used through this paper. We model a multi-agent system composed by n agents as an undirected graph $G = (V, E)$ with the nodes $V = (1, 2, \cdots, n)$ being the n agents, and the edges $E \subset V \times V$ representing the communication links. Here, we assume the edge is bidirectional, such as in a homogeneous wireless sensor networks, where all sensors have the same sensing/influencing radius. Define the one-hop neighbor agents of agent i by $N_{1i} = \{j : (i,j) \in E\}$. Let $d_{1i} = |N_{1i}|$ be the number of one-hop neighbor agents of agent i. We specify the interconnection topology of a network by an nonnegative symmetric adjacency matrix $A = [a_{ij}]$, where $a_{ii} = 0$ and $a_{ij} = \frac{1}{d_{1i}}$ if $(i,j) \in E$; otherwise, $a_{ij} = 0$. Then the Laplacian of the weighted graph is defined as $L = [l_{ij}]$, where $l_{ii} = \sum_j a_{ij}$, and $l_{ij} = -a_{ij}$, $i \neq j$. Note that $L = L^T$. Define the eigenvalues of L as $\lambda_1 = 0 \leq \lambda_2 \leq \cdots \leq \lambda_n$. In this paper, we assume that G is connected, thus L has a simple zero eigenvalue($\lambda_2 > 0$) [16].

In recent years, a varieties of second order consensus protocol have been proposed, see e.g., [18], [19], [20]. In this paper, we consider a general linear consensus protocol [18]. Suppose that each agent is a second order integrator with dynamics

$$\begin{aligned} \dot{x}_i &= v_i, \\ \dot{v}_i &= u_i, i \in (1, \cdots, n), \end{aligned} \tag{1}$$

where $x_i \in \mathbb{R}^m$ and $v_i \in \mathbb{R}^m$ are the agent positions and velocities, $u_i \in \mathbb{R}^m$ is the control input which is taken as

$$u_i = -\sum_{i=1}^{n} a_{ij}[(x_i(t) - x_j(t)) + \gamma(v_i(t) - v_j(t))]. \tag{2}$$

It has been shown that consensus can be achieved if the undirected communication network is connected with $\gamma > 0$. Here, we consider delays along communication links caused by communication constraints frequently in large-scale communication networks. Assume that all the one hop communication delays are the same as $\tau > 0$. We discuss the consensus protocol with commensurate delay(e.g. [3]) and non-commensurate delay(e.g. [12]) as follows.

A. *consensus protocol with commensurate delay*

The control input u_i is

$$u_i = -\sum_{i=1}^{n} a_{ij}[(x_i(t-\tau) - x_j(t-\tau)) + \gamma(v_i(t-\tau) - v_j(t-\tau))]. \tag{3}$$

Denote $U(t) = [X(t), V(t)]$, where $X(t) = [x_1^T, \cdots, x_n^T]^T$, $V(t) = [v_1^T, \cdots, v_n^T]^T$. Here, we assume $m = 1$ for simplicity. However, all the results hereafter can be derived for $m > 1$, using the Kronecker product. Substituting (3) into (1), the system dynamics is:

$$\dot{U}(t) = \Gamma \cdot U(t) + H \cdot U(t - \tau), \tag{4}$$

where

$$\Gamma = \begin{pmatrix} 0 & I_n \\ 0 & 0 \end{pmatrix}, \ H = \begin{pmatrix} 0 & 0 \\ -L & -\gamma L \end{pmatrix}.$$

Taking the Laplacian transform of (4), we have

$$sU(s) = \Gamma U(s) - HU(s)e^{-\tau s}. \tag{5}$$

Then the characteristic polynomial is

$$p(s, e^{-\tau s}) = det(sI_{2n} - \Gamma - He^{-\tau s}). \tag{6}$$

Recall that L is symmetric. Let $TLT^{-1} = \Lambda$. The equation (6) is equivalent to

$$p(s, e^{-\tau s}) = s^2 \cdot \Pi_{i=2}^{n}(s^2 + \gamma \cdot \lambda_i \cdot e^{-\tau s}s + \lambda_i \cdot e^{-\tau s}), \tag{7}$$

where $\lambda_i, \ i \in (1, \cdots, n)$ are the eigenvalues of L. Let ω denotes the crossing frequency. Substituting $s = j\omega$ into (7), we have at least one equation as following

$$\omega^2 = (\gamma\lambda_i\omega j + \lambda_i) \cdot e^{-\tau\omega j}.$$

According to the magnitude and phase condition, we get

$$\begin{aligned} \omega^2 &= \frac{\lambda_i^2\gamma^2 + \sqrt{\lambda_i^4\gamma^4 + 4\lambda_i^2}}{2}, \\ \tau &= \frac{arctan\gamma\omega}{\omega}. \end{aligned}$$

It is easy to see that τ decreases with λ_i increasing. Thus, the delay margin is $\tau^\star = \frac{arctan\gamma\omega^\star}{\omega^\star}$ with $\omega^{\star 2} = \frac{\lambda_n^2\gamma^2 \pm \sqrt{\lambda_n^4\gamma^4 + 4\lambda_n^2}}{2}$.

Proposition 2.1 : Consider the system (1) with the control input (3). The system reaches consensus if and only if $\tau < \tau^\star$.

B. *consensus protocol with non-commensurate delay*

In real applications, time delay can be best modeled by a non-commensurate algorithm. The control input u_i is

$$u_i = -\sum_{i=1}^{n} a_{ij}[(x_i(t) - x_j(t-\tau)) + \gamma(v_i(t) - v_j(t-\tau))], \tag{8}$$

and further

$$\dot{U}(t) = \widetilde{\Gamma} \cdot U(t) + \widetilde{H} \cdot U(t - \tau), \tag{9}$$

where

$$\widetilde{\Gamma} = \begin{pmatrix} 0 & I_n \\ -I_n & -\gamma I_n \end{pmatrix}, \ \widetilde{H} = \begin{pmatrix} 0 & 0 \\ A & \gamma A \end{pmatrix}.$$

The characteristic polynomial of (9) is

$$p(s, e^{-\tau s}) = \Pi_{i=1}^{n}(s^2 + (s\gamma + 1)(1 + (\lambda_i - 1) \cdot e^{-\tau s})). \tag{10}$$

Let $\mu_i = \lambda_i - 1$. Substituting $s = j\omega$ into (10), we have

$$\begin{aligned} \omega^2 &= \frac{1}{2}\{(\mu_i^2 - 1)\gamma^2 + 2 \pm \\ &\quad \sqrt{((\mu_i^2 - 1)\gamma^2 + 2)^2 + 4(\mu_i^2 - 1)}\}, \\ \omega\tau &= \begin{cases} \pi + arctan\frac{-\gamma\omega^3}{(\gamma^2-1)\omega^2+1}, & \text{if } \mu_i \geq 0 \\ arctan\frac{-\gamma\omega^3}{(\gamma^2-1)\omega^2+1}, & \text{if } -1 \leq \mu_i < 0 \end{cases} \end{aligned}$$

Note that $\omega = 0$ for $\lambda_i = 0$. The delay margin is

$$\tau^\star = \min_{\lambda_i}(\tau).$$

Proposition 2.2 : Consider the system (1) with the control input (8). The system reaches consensus if and only if $\tau < \tau^\star$.

III. Fast consensus seeking in the consensus protocol with time delay

In this section, we focus on the problem of fast consensus seeking in the second order consensus protocol with time delay. For all the applications in cooperative control area, fast consensus convergence speed is necessary and important. Much related work is built on finding optimal weights for a given network topology [9] or changing a network topology into a fast convergence topology by random rewiring. In [11], the authors solve the optimal problem of consensus protocols for double integrators by optimizing the convex function of all the consensus matrix eigenvalues. However, all those work consider the ideal case, namely, no delay. It is unavoidable in practice, especially in large scale communication network. Here, we extend the method proposed in [11] to the fast consensus seeking problem for consensus protocol with time delay.

Suppose that the system (1) with the protocol(3) or the protocol(8) can reach consensus. We solve the problem of fast convergence speed by the frequency domain method, where it is transferred to the problem of pushing all the roots of the characteristic polynomial as far as possible to imaginary axis.

Define

$$r(\lambda_i, \gamma) = \min\{|Re(s)|\},$$

where s are the roots of the characteristic polynomial for λ_i, $Re(\cdot)$ denotes the real part of complex number.

Consider a network with n agents using protocol (3) or protocol (8). Define $\gamma \in \Upsilon$, where Υ is the set for guaranteeing the convergence of the consensus protocol with a given τ. It can be derived from proposition 2.1 or proposition 2.2. We formulate the problem of fast convergence speed as an optimal problem,

$$\begin{aligned} &\text{maximize} \quad r(\lambda_i, \gamma) \\ &\text{subject to} \quad \lambda_2 > 0, \gamma \in \Upsilon. \end{aligned} \tag{11}$$

where $\lambda_i \in \sigma(L)$ and L is defined in Section II. Here, $\sigma(\cdot)$ denotes the eigenvalue spectrum.

Next, we firstly study the optimal problem of the consensus protocol with non-commensurate delay (8). As well known, it is very hard to get the solution of Equation (10) due to time delays, whose roots number is infinity. Here, we analyze the distribution of roots with λ_i varying by the root locus method. The root locus of (10) is equivalent to one of the closed system with open transfer function

$$G_0(s) = \frac{K(s+\frac{1}{\gamma})e^{-\tau s}}{s^2 + s\gamma + 1}, \tag{12}$$

where $K = |(\lambda_i - 1)|\gamma$ is the gain of the root locus. Note that $K = (\lambda_i - 1)\gamma$ for $\lambda_i \geq 1$ where the closed system has passive feedback control, namely $1 + G_0(s) = 0$; Otherwise, $K = (1 - \lambda_i)\gamma$ for $0 \leq \lambda_i < 1$ where the closed system has positive feedback control, namey, $1 - G_0(s) = 0$. Besides one root locus with $p_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4}}{2}$ be start points and $z_1 = -\frac{1}{\gamma}$, $z_2 = +\infty$ be end points, there are still countless root locus due to time delay. According to the L'Hospital rule, as $s \to -\infty$,

$$\lim_{s \to -\infty} \frac{(s+\frac{1}{\gamma})e^{-\tau s}}{s^2 + s\gamma + 1} = -\infty,$$

furthermore, as $K \to 0$,

$$K + \frac{1}{\frac{(s+\frac{1}{\gamma})e^{-\tau s}}{s^2+s\gamma+1}} = 0.$$

Therefore, $p \to -\infty$ is the start point of the root locus. Let $s = \sigma + j\omega$. According to the phase condition,

$$\begin{aligned} \omega\tau \quad &- \quad arctan\frac{\omega}{\sigma + \frac{1}{\gamma}} + arctan\frac{(2\sigma+\gamma)\omega}{1 + \sigma\gamma + \sigma^2 - \omega^2} \\ &= \begin{cases} 2k\pi, & \text{if } 0 \leq \lambda_i < 1 \\ (2k+1)\pi, & \text{if } \lambda_i \geq 1 \end{cases} \end{aligned}$$

As $s \to -\infty$, $\sigma \to -\infty$, ω is a limited value,

$$\omega = \begin{cases} \pm\frac{2k\pi}{\tau}, & \text{if } 0 \leq \lambda_i < 1 \\ \pm\frac{(2k+1)\pi}{\tau}, & \text{if } \lambda_i \geq 1 \end{cases}$$

Similarly, as $s \to +\infty$,

$$\lim_{s \to +\infty} \frac{(s+\frac{1}{\gamma})e^{-\tau s}}{s^2 + s\gamma + 1} = 0.$$

Thus, $z \to +\infty$ is the end point of the root locus, and the value of ω is the same with the case $s \to -\infty$. Recall that $\sigma \to -\infty$ as $K \to 0$ for all $k > 0$. Thus, the K at the crossing frequency for $k > 0$ must be larger than the one for $k = 0$ since $p_{1,2}$ is limited. Therefore, we only need to consider the problem (11) for $k = 0$. The root locus for $k = 0$ depends on the distribution of the poles $p_{1,2}$ and the zeros $z_{1,2}$. Let λ_c denotes the λ_i of the separation point of the root locus on negative real axis.

Lemma 3.1:$r(\lambda_i, \gamma)$ increases in λ_i for $0 \leq \lambda < \lambda_c$; $r(\lambda_i, \gamma)$ decreases in λ_i for $1 \leq \lambda < \lambda_n$; The varying trend of $r(\lambda_i, \gamma)$ in λ_i depends on the value of γ.

Proof: By directly computing the positions of the poles and zeros, it is easy to get above facts. Moreover, when $0 < \gamma \leq \sqrt{2}$ and $\gamma \geq 2$, $r(\lambda_i, \gamma)$ decreases in λ_i; when $\sqrt{2} < \gamma < 2$, $r(\lambda_i, \gamma)$ increases in λ_i. ■

Let $\mathcal{L}$ denotes the set of rank $n-1$, symmetric Laplacian L, where $L \in \mathcal{L}$ is satisfied with $L1 = 0$ and $\lambda_2(L) > 0$. Define $\bar{\mathcal{L}}$ as a subset of $\mathcal{L}$ composed by the matrices such that $\lambda_n(L) = 1$. Notice that $\mathcal{L} = \bigcup_{\beta > 0} \beta\bar{\mathcal{L}}$. Define the normalization of Laplacian matrix $\mathbf{N}(L)$. Further let $\tilde{\mathcal{L}}$ denotes the set of normalized matrix of $L \in \mathcal{L}$.

Lemma 3.2: Define $R(L, \gamma) = \min_{\lambda_i \in \sigma(L) \setminus 0, L \in \tilde{\mathcal{L}}} r(\lambda_i, \gamma)$. Its value depends on only λ_2 and λ_n, namely, $R(L, \gamma) = \min(r(\lambda_2, \gamma), r(\lambda_n, \gamma))$.

Proof: The above facts can be easily deduced from Lemma 3.1. For the easy of notation, we define $R(L, \gamma) := r(\lambda_2, \lambda_n, \gamma)$. ■

According to Lemma 3.2, we solve the optimal problem (11) as finding the optimal values of L, γ and β,

$$(L_{opt}, \gamma_{opt}, \beta_{opt}) \in \arg\max_{L\in\bar{\mathcal{L}},\gamma\in\Upsilon} r(\beta\lambda_2(\mathbf{N}(L)), \beta, \gamma) \quad (13)$$

Firstly, we find the optimal L. If γ and β are fixed, then $r(\beta\lambda_2(\mathbf{N}(L)), \beta, \gamma)$ is non-decreasing in $\lambda_2(\mathbf{N}(L))$. Thus, we have

$$L_{opt} \in \arg \max_{L\in\bar{\mathcal{L}}} \lambda_2(\mathbf{N}(L)). \quad (14)$$

Note that $\bar{\mathcal{L}}$ is not convex. But the above problem is equivalent to the following optimization problem,

$$L_{opt} \in \arg\max_{L\in\mathcal{L},\lambda_n(L)\leq 1} \lambda_2(\mathbf{N}(L)). \quad (15)$$

$\mathcal{L}$ and any nonnegative numbers $\alpha_1, \alpha_2, \ldots, \alpha_r$ such that $\alpha_1 + \alpha_2 + \ldots + \alpha_r = 1$, the Laplacian $\sum_{k=1}^{r} \alpha_k L_k$ is in $\mathcal{L}$. Moreover, $\lambda_n(\alpha_1 L_1 + \alpha_2 L_2 + \ldots + \alpha_r L_r) \leq 1$ is guaranteed according to the Weyl theorem for all $L_i \in \mathcal{L}, \lambda_n(L) \leq 1$. Thereby, $\{L \in \mathcal{L}, \lambda_n(L) \leq 1\}$ is convex set. Furthermore, consider the $(n-1)$-dimensional subspace $\mathbf{P} \subseteq \mathbf{R}^n$ spanned by the unit vectors $p_i \in \mathbf{R}^n$, $i = 1, \ldots, n-1$. Denote $P := [p_1, p_2, \ldots, p_{n-1}] \in \mathbf{R}^{n\times(n-1)}$ with $p_i^T\mathbf{1} = 0$, $p_i^T p_i = 1$ and $p_i^T p_j = 0$, $(i \neq j)$.

Corollary 3.1: The optimization problem (15) is equivalent to

$$\max_{L\in\mathcal{L}} \lambda$$
$$\text{s.t. } \lambda_n(L) \leq 1, P^T\mathbf{N}(L)P \geq \lambda I_{n-1}. \quad (16)$$

Proof: Let $\lambda_2(\mathbf{N}(L)) = \lambda$. Define $x = \alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_{n-1} p_{n-1} \in \mathbf{P}$ for some not all zeros $\alpha_1, \alpha_2, \cdots, \alpha_{n-1} \in \mathbf{R}$. For a connected graph G that is undirected, the following well-known property holds:

$$x^T(\mathbf{N}(L) - \lambda I_{n-1})x \geq (\lambda_2(\mathbf{N}(L)) - \lambda)x^T x \geq 0.$$

The proof follows from the facts that $\mathbf{1}^T x = 0$. Let $x = Py$, where $y := [\alpha_1, \alpha_2, \cdots, \alpha_{n-1}]^T$. Thereby,

$$(Py)^T(\mathbf{N}(L) - \lambda I_{n-1})(Py) \geq 0$$

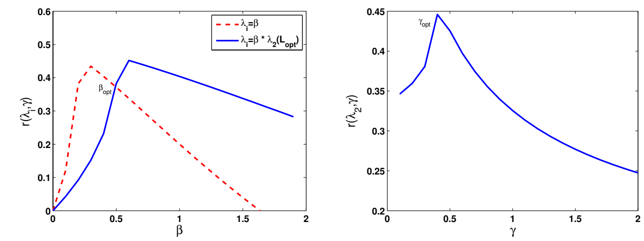
and further,

$$P^T\mathbf{N}(L)P \succeq \lambda I_{n-1}.$$

The above semi-definite program can be solved by SeDuMi tools or other efficient softwares. ■

Furthermore, we start to find the optimal value of β and γ for the optimization problem (13). Figure 1(a) shows the root of the system (12) varying with β for $\gamma = 0.4$, where $\beta_{opt} \approx 0.512$. Conversely, Figure 1(b) shows the root varying with γ for $\beta = 0.512$, where $\gamma_{opt} = 0.4$. Thus, there exists a unique set $\{\beta_{opt}, \gamma_{opt}\}$ which can be solved by linear searching in the set Υ.

Note that the above results can be easily extended to the optimization problem (13) of consensus protocol with commensurate delay. Here, we omitted the analysis for brevity.



(a) (b)

Fig. 1. The root of the system (12) (a) varies with β for $\tau = 1$ and $\gamma = 0.4$; (b) varies with γ for $\tau = 1$ and $\beta = 0.512$, $\lambda_2 = 0.4$.

IV. Consensus protocol with multi-hop relay scheme

In this section, we investigate the problem of fast consensus seeking when L is given. In some engineering applications, such as the surveillance of the coal mine and the circumstance by WSNs, the location of the sensors are fixed, namely, the network topology can not be changed.

Much work in the literature assumes that each agent can only receive the information of its one-hop neighbors. In [17], the authors propose a multi-hop relay protocol for fast consensus seeking, where each agent can also receive the information transmitted along a multi-hop path with a certain time delay. Each agent with this scheme extends its information despite of time delay, which improves the second smallest eigenvalue of L without changing the physical network topology. Therefore, the convergence speed of consensus protocol increases. In this paper, we extend this scheme to the non-commensurate delay case .

Suppose that the transportation time delay between one-hop neighbors is τ, and then the time delay between r-hop neighbors is $r\tau$. In real applications such as WSNs, the value of r is usually $2 \sim 3$. At each time t, agent i has the following information,

$$I_t^i \triangleq \{x_i(t), x_j(t-\tau), \ldots, x_k(t-r\tau), j \in N_{1i}, k \in N_{ri}\}.$$

We rewrite the control input (8) as the following,

$$\begin{aligned} u_i \quad = \quad & -\sum_{j\in N_i} a_{ij}[(x_i(t) - x_j(t-\tau)) + \\ & \gamma(v_i(t) - v_j(t-\tau)) + \\ & \sum_{k\in N_j} a_{jk}((x_i(t) - x_k(t-2\tau)) \\ & +\gamma(v_i(t) - v_k(t-2\tau)) + \cdots)], \end{aligned} \quad (17)$$

Denote $\tilde{A} = [a_{i\tilde{j}}]$ as

$$a_{i\tilde{j}} = \begin{cases} a_{ij} \cdots a_{\tilde{i}\tilde{j}}, & \text{when } i \neq \tilde{j}, \tilde{j} \in N_{1\tilde{i}}, j \in N_{1i} \\ 0, & \text{when } i = \tilde{j}, \\ 0, & \text{otherwise,} \end{cases}$$

and $\tilde{D} = diag\{d_{ii}\}$, where $d_{ii} = \sum_{\tilde{j}} d_{i\tilde{j}}$.

The system dynamics (9) is,

$$\dot{U}(t) = \tilde{\Gamma}U(t) + \sum_{q=1}^{r} \tilde{H}_q U(t - q\tau), \quad (18)$$

where

$$\tilde{\Gamma} = \left[\begin{array}{cc} 0_{n\times n} & I_n \\ -\sum_{q=1}^{r} \tilde{D}_q & -\gamma(\sum_{q=1}^{r} \tilde{D}_q) \end{array} \right],$$

$$\tilde{H}_q = \left[\begin{array}{cc} 0_{n\times n} & 0_{n\times n} \\ \tilde{A}_q & \gamma\tilde{A}_q \end{array} \right].$$

Taking the Laplace transform of (13), we have

$$sU(s) = \tilde{\Gamma}U(s) + \sum_{q=1}^{r} e^{-q\tau s}\tilde{H}_q U(s),$$

and further,

$$p(s, e^{-\tau s}) = det(sI - \tilde{\Gamma} - \sum_{q=1}^{r} \tilde{H}_q e^{-q\tau s}). \qquad (19)$$

Let

$$G(s) = \left[\begin{array}{cccc} 0_{2n\times 2n} & I_{2n\times 2n} & \cdots & 0_{2n\times 2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{2n\times 2n} & 0_{2n\times 2n} & \cdots & I_{2n\times 2n} \\ -sI + \tilde{\Gamma} & \tilde{H}_1 & \cdots & \tilde{H}_{r-1} \end{array} \right],$$

$$H = \left[\begin{array}{cccc} I_{2n\times 2n} & \cdots & 0_{2n\times 2n} & 0_{2n\times 2n} \\ \vdots & \ddots & \vdots & \vdots \\ \vdots & \cdots & I_{2n\times 2n} & 0_{2n\times 2n} \\ 0_{2n\times 2n} & \cdots & 0_{2n\times 2n} & -\tilde{H}_r \end{array} \right].$$

By the determinant properties of block matrix, we have

$$p(s, e^{-\tau s}) = det(G(s) - e^{-\tau s}H). \qquad (20)$$

When $s = j\omega(\omega \neq 0)$ moves along the imaginary axis from 0 to $j\infty$, there exists at most $2n$ frequency ω_k^i so that $\|\lambda_i(G(j\omega_k^i), H)\|_2 = \|e^{-j\tau_k^i\omega_k^i}\|_2 = 1$.

Lemma 4.1: Suppose that $rank(\tilde{H}_r) = p$. Define

$$\bar{\tau}_i = \min_{1\leq k\leq 2n} \theta_k^i/\omega_k^i$$

when the generalized eigenvalues $r_i(G(s), H)$ satisfy the following equation:

$$r_i(G(j\omega_k^i), H) = e^{-j\theta_k^i}$$

for some $\omega_k^i \in (0, \infty)$ and $\theta_k^i \in [0, 2\pi)$. Then the consensus delay margin of (19) is

$$\tau^\star(r) = \min_{1\leq i\leq 2n(r-1)+p} \bar{\tau}_i.$$

Proof: The proof is similar to one in [17]. Here we omit it for brevity due to the limitations of space. ■

When $s = 0$, we can get

$$(\tilde{\Gamma} + \sum_{q=1}^{r} \tilde{H}_q)\bar{U} = 0,$$

where $\tilde{\Gamma} + \sum_{q=1}^{r} \tilde{H}_q$ is a Laplacian matrix, and $\bar{U}$ is the DC-offset, thus $\bar{U}$ is an eigenvector of $\tilde{\Gamma} + \sum_{q=1}^{r} \tilde{H}_q$ associated with eigenvalue zero. It is easy to see that $\bar{x}_1 = \ldots = \bar{x}_n = c$, where $c \in \mathbb{R}$ is an unknown constant, and $\bar{v}_1 = \ldots = \bar{v}_n = 0$.

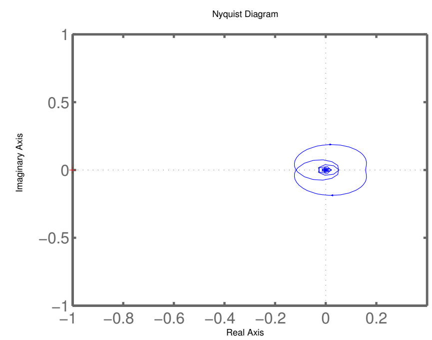


Fig. 2. The Nyquist plot of the system (12) with $\tau = 1$ and the optimal set $(L_{opt}, \gamma_{opt}, \beta_{opt})$.

In mathematical area, many algorithms have been proposed to search generalized eigenvalue. It is much easier to solve the above problem than to solve the roots of the equation (19) directly. However, this method is only efficient for the small-scale network due to the sizes of G and H.

V. Numerical Examples

In this section, we illustrate the results derived in the above sections by numerical simulations using the SeDuMi and Simulink toolbox in Matlab. Firstly, we simulate the algorithm described in section III in Example 5.1, then we verify the multi-hop consensus protocol with time delay in section IV in Example 5.2.

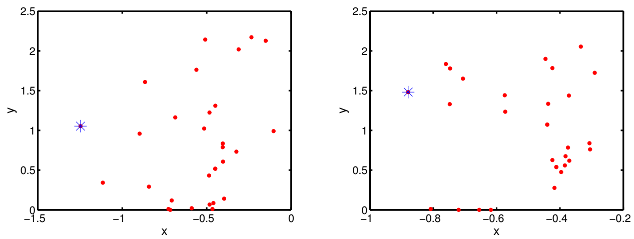
Example5.1 : Consider a multi-agent system with n nodes. Each node is a second order dynamic integrator (1) with the control input (8). The convergence properties have been investigated in Section II. Assuming that the proposition 2.2 is satisfied, we consider the optimization problem (11). This example is to show the effectiveness of transferring the optimization problem into finding an optimal set (L, γ, β). The constants n, τ are chosen to be 20 and 1, respectively.

Here we use SeDuMi toolbox to solve the semi-definite problem (16), and get a complete graph and the value $\beta_{opt} = 1.1$ and $\gamma_{opt} = 1.6$ by linear searching. From Figure 2, we find the system (12) is stable since Nyquist curve does not encircle $(-1, 0)$. To verify the optimal results, we also construct 30 connected random graphs for comparison, and set the optimal value γ for each graph. As Figure 3(a) shows that, the root with the optimal set $(L_{opt}, \gamma_{opt}, \beta_{opt})$ is farthest from the imaginary axis.

In most cases, it is hard to construct a complete graph since the constrained communication or bandwidth. Next, we simulate the optimization problem for a network topology with a certain constrained communication among some nodes. Assume that

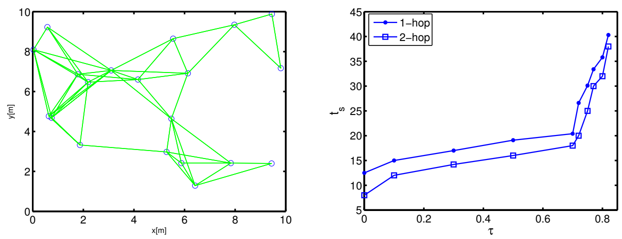
$$\{l_{14}, l_{15}, l_{23}, l_{32}, l_{34}, l_{35}, l_{41}, l_{43}, l_{45}, l_{51}, l_{53}, l_{54}\} = 0,$$

where l_{ij} is defined in Section II. We again use the SeDuMi toolbox to solve the optimization problem and get a non-complete graph with $\beta'_{opt} = 1.2$, $\gamma'_{opt} = 1.5$. As observed in Figure 3(b), the optimal set push the root of the system farthest away the imaginary axis.



(a) no constrained communication (b) with constrained communication

Fig. 3. Plot of the points $R(\mathbf{N}(L),\gamma)$, where $*$ denotes $R(\mathbf{N}(L_{opt}),\gamma_{opt})$.



(a) (b)

Fig. 4. (a)A connected graph with $n = 20$ nodes; (b)The convergence time t_s varies with time delay τ.

Example5.2 : In this example we analyze consensus protocol with multi-hop relay scheme. We randomly construct a connected graph with $n = 20$ nodes as Figure 4(a) shows. Each node is a second order dynamic integrator (1) with the control input (18). We compute the delay margin for multi-hop relay scheme according to Lemma 4.1. Let t_s denotes the convergence time at which the states of agents reach consensus.

Note that the magnitude of the generalized eigenvalue exceeds 1 inevitably as ω increases. Actually we find the delay margin $\tau^*(r)$ over a finite frequency interval for different r. They are 1.52, 1.03, 0.45, 0.13 for $r = 2, 3, 4, 5$, respectively. The results are consistent with the fact that the delay margin decreases with network connectivity increasing. Figure 4(b) shows that the convergence time t_s of 2-hop relay scheme is less than one of 1-hop relay scheme, and the gap between both relay schemes decreases with τ increasing.

VI. Conclusion

In this paper, we investigate the problem of fast consensus seeking in networked multi-agent systems with homogeneous communication delay. We analyze the convergence properties of consensus protocol with commensurate delay and non-commensurate delay, respectively. On the basis of these results, we transfer the problem of optimizing convergence speed into finding an optimal set whose elements are network topology L and system parameter γ by root locus method. We also solve the optimal set using Sedumi toolbox and linear searching over finite interval. An example demonstrates the optimality of the method. We further analyze the multi-hop relay scheme for consensus protocol with non-commensurate delay. We derive the delay margin by searching the frequency at which the magnitude of the generalized eigenvalue of a pair of extension matrix is equal to 1. The simulation results show that the multi-hop relay scheme can improve the convergence speed for the case that the physical topology of the network is fixed, while decreases the delay margin.

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