2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Control of connected Markov chains. Application to congestion avoidance in the Internet.

Alexander Miller and Boris Miller

***Abstract*— The article considers the optimal control for the system of finite number of controlled connected Markov chains (CMC). Such models come from queuing systems with many service lines and/or from the control of resources of multiple connected dams. The state of such CMC is represented as a tensor of the depth d, where d is the number of controlled chains. This tensor form is much more convenient for derivation of the dynamic programming equation. We give a tensor form for the control problems arising in the router control which is aimed to the congestion avoidance with the aid of two telecommunication lines having different properties and cost of services.**

I. Introduction

Behavior of complex systems like broadband networks, power networks, water storage systems and even UAV, operating in abruptly changing environment can be considered as the hybrid systems driven by controlled Markov chains (MC). The controlled MC is used very often as an approximation of the continuous state space system, those state space is separated on a finite number of cells, and the evolution of the systems consists in discrete change of position at random time. The typical examples are: the number of customers currently connected to the network, number of customers of power network, level of storage of some resource in the stock. In all these examples the controls are responsible for the rate of state changes and the random state distribution after the abrupt change of state. In the telecommunication networks such controls are: the intensities of incoming flows of jobs, service rates, access probability and so on, and they can be chosen by controller in order to achieve the best performance of the system. One of the most effective approach to the control of MC is based on the martingale representation and the using of stochastic differential equations describing their evolution [7]. Basing on this approach many problems arising in various areas can be solved rather effectively. Some, but not exhausting examples are: control of the network with the aid of access and the service rate controls [13], congestion avoidance with the aid of RED type protocols [18] based on the using of the optimal stochastic control [10], [12], the network controls with different constraints [15], and control of the large dams [1], [6], [16].

This work was supported in part by Australian Research Council Grant DP0988685 and by Russian Basic Research Foundation Grant 10-01-00710.

Alexander Miller: Institute for Information Transmission Problems, Russian Academy of Sciences, Moscow, Russia. E-mail: amiller@iitp.ru. This work was partially supported by Monash University during his stay in the School of Mathematical Sciences, Monash University, Clayton, 3800, Victoria, Australia

Boris Miller: School of Mathematical Sciences, Monash University, Clayton, 3800, Victoria, Australia and Institute for Information Transmission Problems, Russian Academy of Sciences, Moscow, Russia, E-mail: boris.miller@monash.edu

In most previous works the model of single controlled MC is used, where the current state of the controlled MC with d number of possible states is represented by a unit vectors $X \in \{e_1, ..., e_M\}$ of a type $e_i = (0, ..0, 1, 0, ..0)$ with the unit at i^{th} place, in the space R^M [7].

However, the set of d connected Markov chains (CMC) each having $M_i, i = 1, ..., d$ possible states is represented by the set of unit vectors $X_i \in \mathcal{S}_i = \{e_1, ..., e_{M_i}\}$, with $i = 1, ..., d$ and in general by a tensor of the order d, that is $\mathbf{X} = \{X_1|X_2|...|X_d\}$. The representation of the state of such CMC in the form of unique joint vector, that needs to be of dimension $\prod_{i=1}^{d} M_i$, is cumbersome and rather tedious, particularly from the viewpoint of numerical analysis.

In this article our aim is to give the tensor description of such CMC and to give a generalization of the dynamic programming equation in tensor form. In Section II we describe a general model of controlled CMC, give the optimal control problem statement and introduce the tensor form of the state and dynamic programming equation. In Section III we introduce an active users model which provide a feedback influencing on the users activity (the package sending rate) through the probabilities of accept or reject the current demand. In Section IV we give a detailed description of this model for the case of queuing system with two communication lines: one is the main and receives the demands first, the second one is reserve line, which is usually more fast, but at the same time has more expensive service rate. The aim of the control is the congestion avoidance, taking into account other criteria like the average time in queue and/or the service cost.

II. Model of controlled CMC, optimal control problem statement

A. Controllable CMC model

In order to deal with the system of CMC, we make some assumptions about the behavior of each Markov chain (MC). We assume that each i^{th} controllable MC has M_i possible states and is described by the following stochastic differential equation [7]

$$(X_i)_t = (X_i)_0 + \int_0^t A_i(s, u(s))(X_i)_s ds + (W_i)_t, \qquad (1)$$

where $(X_i)_t \in \mathcal{S}_i = \{e_1, ..., e_{M_i}\}$, $(X_i)_0$ is the initial state of the i^{th} MC. The $(M_i \times M_i)$ matrix $A_i(t, u)$ is the generator of i^{th} MC, matrix valued function A_i is assumed to be continuous on $(t, u) \in [0, T] \times U$, where $T < \infty$ and U is a compact set in R^m.

The process $(W_i)_t$ is a square integrable martingale.

The general state of CMC $\mathbf{X} = \{X_1|X_2|...|X_d\}$ can be described as tensor product of vectors $\mathbf{X} = X_1 \otimes X_2 \otimes ... \otimes X_d$, where $X_i \in \mathcal{S}_i$.

All processes are defined on the probability space $\{\Omega, \mathcal{F}, \mathbf{P}\}$. Specifically, we define $(X_i)_t$, $i = 1, ...d$, where $(X_i)_t \in \mathcal{S}_i, t \in [0, T]$ for $T < \infty$, as a controlled jump Markov process with piecewise constant right-continuous paths. We also make the following assumptions about the controls $u(t)$.

Assumption 1: Assume that the set of admissible controls, $u(\cdot)$ is the set of $\mathcal{F}_t^{\mathbf{X}}$-predictable controls taking values in U, where $\mathbf{X} = X_1 \otimes X_2 \otimes ... \otimes X_d$.

Remark 1: Assumption 1 ensures that if the number of jumps of i^{th} MC up to the current time $t \in [0, T]$ is N_t, τ_k is the time of the k^{th} jump and

$$(X_i)_0^t = \{(X_i)_0, 0), (X_i)_1, \tau_1), ..., ((X_i)_{N_t}, \tau_{N_t})\}$$

is the set of states and jump times, then for $\tau_{N_t} \leq t < \tau_{N_{t+1}}$ the controls $u(t) = u(t, \mathbf{X}_0^t)$ are measurable with respect to t and $\mathbf{X}_0^t$, where $\mathbf{X}_0^t = (X_1)_0^t \otimes ... \otimes (X_d)_0^t$ [7].

B. General performance criterion

Let $f_0(s, p(s), \mathbf{X}_s)$ be the running cost function when the CMC is in state $\mathbf{X}_s$ at time $s \in [0, T]$. Then a general performance criterion to be minimized has the form

$$J[u(\cdot), \mathbf{X}(\cdot)] = \mathbb{E}\left[\phi_0(\mathbf{X}_T) + \int_0^T f_0(s, p(s), \mathbf{X}_s) ds\right]. \tag{2}$$

Here $\phi_0(X_T) = \langle \phi_0, \mathbf{X}_T \rangle$ and $f_0(s, p(s), \mathbf{X}_s) = \langle f_0(s, u(s)), \mathbf{X}_s \rangle$ with $\langle \cdot, \cdot \rangle$ as a standard inner product, and ϕ_0 and $f_0(s, u(s))$ are the tensors of the order d.

Assumption 2: For each $\mathbf{X} \in \mathcal{S} = \mathcal{S}_1 \otimes \mathcal{S}_2 \otimes ... \otimes \mathcal{S}_d$, the elements of $f_0(s, u))$ are bounded below and continuous on $[0, T] \times U$.

C. Value function and its representation

The value function of CMC is a function which gives minimum total cost for CMC starting at time $t \in [0, T]$ and state $\mathbf{X}_t = \mathbf{X} \in \mathcal{S}$. It has the form

$$\begin{aligned} &V(t, \mathbf{X}) = \inf_{u(\cdot)} J[u(\cdot), \mathbf{X}(\cdot)|\mathbf{X}_t = \mathbf{X}], \\ &J[u(\cdot), \mathbf{X}(\cdot)|\mathbf{X}_t = \mathbf{X}] = \\ &\mathbb{E}\left[\phi_0(\mathbf{X}_T) + \int_t^T f_0(s, u(s), \mathbf{X}_s) ds|\mathbf{X}_t = \mathbf{X}\right]. \end{aligned} \tag{3}$$

Assumption 2 ensures that this infimum exists.

We now represent $V(t, \mathbf{X})$ as $V(t, \mathbf{X}) = \langle \phi(t), \mathbf{X} \rangle$, where $\phi(t)$ is a tensor of the order d with measurable components.

D. Dynamic programming approach and its extension to tensor state

Here we develop and extend to the tensor state the general approach described in [13], Thm. 2.8 Let us consider $\hat{\phi}(t)$ be of the same form as $\phi(t)$, and define the *dynamic programming equation* with respect to $\hat{\phi}(t)$

$$\begin{aligned} &\frac{\langle d\hat{\phi}(t), \mathbf{X} \rangle}{dt} = -\min_{u \in U}\Big[\langle \hat{\phi}(t), A_1(t, u) X_1 \otimes X_2 \otimes ... \otimes X_d + \\ &X_1 \otimes A_2(t, u) X_2 \otimes ... \otimes X_d + ... + \\ &X_1 \otimes X_2 \otimes ... \otimes A_d(t, u) X_d \rangle + \langle f_0(t, u), \mathbf{X} \rangle\Big] = \\ &-\min_{u \in U} H(t, \hat{\phi}(t), u, \mathbf{X}) \end{aligned} \tag{4}$$

with boundary condition $\hat{\phi}(T) = \phi_0$ [5], [7], [13], [15]. Since $H(t, \hat{\phi}, u, \mathbf{X})$ is continuous in (t, u) and affine in $\hat{\phi}$, for any $(t, \mathbf{X}) \in [0, T] \times \mathcal{S}$, $\mathcal{H}(t, \hat{\phi}, \mathbf{X})$ is Lipschitz in $\hat{\phi}$.

Proposition 1: With Assumption 2 held equation (4) has a unique solution on $[0, T]$.

Remark 2: If we now let $\mathbf{X} = \bigotimes_{k=1}^{d} e^{(i_k)}$, $k = 1, ..., d$, then we get a system of ODE's

$$\frac{d\hat{\phi}^{i_1, i_2, ..., i_d}(t)}{dt} = -\mathcal{H}\left(t, \hat{\phi}(t), \bigotimes_{k=1}^{d} e^{(i_k)}\right), \tag{5}$$

$$i_1 = 1, ..., M_1, i_2 = 1, ..., M_2, ..., i_d = 1, ..., M_d.$$

The simple generalization of the Thm. 2.8 [13] gives the following characterization of the optimal control.

Theorem 1: Let $\hat{\phi}(t)$ be the solution of the system of equations (5), then for each $(t, \mathbf{X}) \in [0, T] \times \mathcal{S}$ there exists $u_0(t, \mathbf{X}) \in U$ such that $H(t, \hat{\phi}(t), u, \mathbf{X})$ achieves a minimum at $u_0(t, \mathbf{X})$. Then

1) There exists an $\mathcal{F}_t^{\mathbf{X}}$-predictable optimal control, $\hat{u}(t, \mathbf{X}_0^t)$ such that $V(t, \mathbf{X}) = J[\hat{u}(\cdot)|\mathbf{X}_t = \mathbf{X}] = \langle \hat{\phi}(t) \mathbf{X} \rangle$.
2) The optimal control can be chosen as Markovian, that is

$$\hat{u}(t, \mathbf{X}_0^t) = u_0(t, \mathbf{X}_{t-}) = \underset{u \in U}{argmin}\, H(t, \hat{\phi}(t), u, \mathbf{X}_{t-}).$$

III. Model of controlled queuing system with active users

A. Model of queuing system with two connected service lines

Consider the queueing system having the possibility to use two different lines for sending the packages and controlled by restriction of access and by changing the service rate. When the income flow enters the router the last one accepts or rejects it. In first case the package queues in the buffer of the volume M_1 and waits for the sending by the first line, if it is rejected by the first line it goes to the second line and is either accepted or rejected completely. If it is accepted by the second line it queues in the buffer of the volume M_2 and waits for the sending by the second line.

The reason of the using two lines is that they could have different characteristics and different service costs. So in the situation of congestion one can use the second line (much more faster, but probably more expensive). The incoming flow of demands is a counting process with random intensity $\lambda(t) \geq 0, t \in [0,T]$ [13], which depends on time-preferences of users and the current probability of rejection [11]. Both service lines use the following controls: the service rates

$$\mu = (\mu_1, \mu_2) \in [\underline{\mu_1}, \overline{\mu_1}] \times [\underline{\mu_2}, \overline{\mu_2}]$$

where $\underline{\mu_1} > 0, \underline{\mu_2} > 0$ and the probabilities of access

$$\mathbf{u}(t) = (u_1(t), u_2(t)) \in [0,1] \times [0,1]$$

for the first and the second lines, respectively. Assuming that the lines work independently, the probability to accept a demand by the system is $U(t) = U_1(t) + U_2(t)$, where

$$U_1(t) = u_1(t), \quad \text{and} \quad U_2(t) = (1 - u_1(t))u_2(t) \tag{6}$$

are the probabilities to accept by the first and second lines, respectively, and the probability to reject a demand by such system is

$$P(t) = 1 - U(t) = (1 - u_1(t))(1 - u_2(t)). \tag{7}$$

Let $M(t) = (M_1(t), M_2(t))$ be the number of demands in queues at time t. Then the general number of possible states is $(M_1 + 1) * (M_2 + 1)$, and the corresponding state of outcomes *S* can be represented by a the tensor product of unit vectors $\mathbf{X} = X_1 \otimes X_2$, where $X_1 \in \mathcal{S}_1, X_2 \in \mathcal{S}_2$. This is the straightforward generalization of general approach to the couple of connected Markov chains [7].

Proposition 2: Assume that the intensity of incoming flow, access and the service rate control, that is the triple $\Lambda = (\lambda, \mathbf{u}, \mu)$, are $\mathcal{F}_t^{\mathbf{X}}$-predictable process. Then the controlled process is described by controlled Markov chain with $(M_1+1)*(M_2+1)$ states with two CMC , described by two generators $A_1(\lambda, \mathbf{u}, \mu), A_2(\lambda, \mathbf{u}, \mu)$, which are represented by the following $(M_1+1)\times(M_1+1)$ and $(M_2+1)\times(M_2+1)$ matrices, respectively,

$$A_1(\Lambda) = \begin{pmatrix} -\lambda U_1 & \mu_1 & \dots & 0 & 0 \\ \lambda U_1 & -\mu_1 - \lambda U_1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -\mu_1 - \lambda U_1 & \mu_1 \\ 0 & 0 & \dots & \lambda U_1 & -\mu_1 \end{pmatrix}, \tag{8}$$

$$A_2(\Lambda) = \begin{pmatrix} -\lambda U_2 & \mu_2 & \dots & 0 & 0 \\ \lambda U_2 & -\mu_2 - \lambda U_2 & \dots & 0 & 0 \\ 0 & \lambda U_2 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \mu_2 & 0 \\ 0 & 0 & \dots & -\mu_2 - \lambda U_2 & \mu_2 \\ 0 & 0 & \dots & \lambda U_2 & -\mu_2 \end{pmatrix}, \tag{9}$$

where the probabilities U_1, U_2 are defined by relations (6).

Proof: Detailed proof can be accomplished in the same way as in [13]. The difference is that the original income flow of demands, which is a counting process with the random intensity $\lambda(t)$, is separated into three processes, namely: the flow of demands accepted by the first line having the intensity

$$\lambda(t)U_1(t) = \lambda(t)u_1(t);$$

the flow of demands rejected by the first line, but accepted by the second one, having the intensity

$$\lambda(t)U_2(t) = \lambda(t)(1 - u_1(t))u_2(t);$$

the flow of demands rejected by the second line, and therefore, rejected at all with intensity

$$\lambda(t)(1 - U(t)) = \lambda(t)(1 - u_1(t))(1 - u_2(t)).$$

The remaining proof is the same as in [13]. ■

Remark 3: It has to be stressed that the class of $\mathcal{F}_t^{\mathbf{X}}$ predictable controls includes all standard existing protocols as the possible option for optimization, for example, the RED type protocol.

B. Model of the CMC with active users

Here we use the model of active users such that each of them tries to maximize the own utility function, taking into account the price of traffic [4], [18], which depends on the probability of rejection and the traffic intensity itself. As in [11], [12] we assume the following form of the utility function

$$f_i(v_i, t) = -\frac{a_i(t)}{v_i}, \quad i = 1, .., N, \tag{10}$$

where N is a number of users (customers), $v_i \geq 0$ is the intensity of the package sending by i^{th} users and a_i is a seasonal coefficient, which corresponds to the seasonally preferable sending rate of the i^{th} user. This utility function corresponds to the case of *minimum potential delay fairness* [18]. As we underlined before [12] the real aim for i^{th} active user is different from the maximization of the utility function (10) itself since its maximization leads to $v_i \to \infty$. Therefore his aim is to maximize *utility function - expenses*, where the expenses depend on the probability of the package rejection established by router. Therefore, his function to be maximized is equal to

$$f_i(v_i, t, P) = -\frac{a_i(t)}{v_i} - \lambda_0 v_i - \lambda_0 v_i P, \quad i = 1, .., N, \tag{11}$$

where λ_0 is the price of traffic, and P is the probability of the package reject, which is established by router (7) as a predictable function of current time t and the history of service, that is $P = P(t, \mathbf{X}_0^t)$. So we assume that each rejected package is sent again and the customer pays twice for traffic.

Remark 4: One can assume that each rejected package may be sent as many times as necessary, since even if it is rejected and have been sent second time it can be rejected again. In this case the utility function has a form [12]

$$f_i(v_i,t,P) = -\frac{a_i(t)}{v_i} - \frac{\lambda_0 v_i}{1-P}. \qquad (12)$$

We assume that the router has the information about the utility functions of users, particularly about the seasonal intensity of the general preferable traffic. By maximizing the utility functions of the users we get the following dependence of the intensity of the incoming flow upon the probability P : for the first type of the utility function (11)

$$\lambda(t,P) = \frac{C(t)}{\sqrt{\lambda_0(1+P)}}, \qquad (13)$$

and for the utility function (12)

$$\lambda(t,P) = C(t)\sqrt{\frac{1-P}{\lambda_0}}, \qquad (14)$$

where

$$C(t) = \sum_{i=1}^{N} \sqrt{a_i(t)}$$

is a function which characterizes the seasonal preferences of the whole set of users.

By substitution of values

$$U = 1-P,$$

$$\lambda(t,P)U_1 = \frac{C(t)u_1}{\sqrt{\lambda_0(1+(1-u_1)(1-u_2))}}, \qquad (15)$$

$$\lambda(t,P)U_2 = \frac{C(t)(1-u_1)u_2}{\sqrt{\lambda_0(1+(1-u_1)(1-u_2))}}$$

into expression (8), (9) we get the representation of the generator matrices $A_1(t,\mathbf{u},\mu), A_2(t,\mathbf{u},\mu)$ for the case of active users applying the utility function (11).

For the set of users applying the utility function (12) we get

$$\lambda(t,P)U_1 = C(t)u_1\sqrt{\frac{1-(1-u_1)(1-u_2)}{\lambda_0}},$$

$$\lambda(t,P)U_2 = C(t)(1-u_1)u_2\sqrt{\frac{1-(1-u_1)(1-u_2)}{\lambda_0}}. \qquad (16)$$

IV. Dynamic programming and the optimal control for CMC with two lines

A. Dynamic programming equation for two CMC

Consider the general optimal control problem with criterion to be minimized

$$J[u(\cdot),\mu(\cdot)] =$$
$$\mathbf{E}\left\{\sum_{i=1}^{M_1}\sum_{j=1}^{M_2}\phi_0^{i,j} I\{(X_1)_T = e_i\} I\{(X_2)_T = e_j\}\right\} +$$
$$\mathbf{E}\{\sum_{i=1}^{M_1}\sum_{j=1}^{M_2}\int_0^T f_0^{i,j}(s,\mathbf{u}(s),\mu(s))\times$$
$$I\{(X_1)_s = e_i\} I\{(X_2)_s = e_j\} ds\} \to \min_{\mathbf{u}(\cdot),\mu(\cdot)}, \qquad (17)$$

or in tensor form

$$J[u(\cdot),\mu(\cdot)] = \mathbf{E}\{\langle\phi_0,(X_1)_T\otimes(X_2)_T\rangle\} +$$
$$\left\{\int_0^T \langle f_0(s,\mathbf{u}(s),\mu(s)),(X_1)_s\otimes(X_2)_s\rangle ds\right\} \to \min_{\mathbf{u}(\cdot),\mu(\cdot)}, \qquad (18)$$

where ϕ_0 and f_0 are tensors of the order 2. For the tensor ϕ of the order 2, which is represented as a $(M_1+1)\times(M_2+1)$ matrix one can determine the inner product as a following bilinear form

$$\langle\phi,\mathbf{X}\rangle = X_1^T\phi X_2$$

if $X_1 \in R^{M_1+1}, X_2 \in R^{M_2+1}$.

To derive the optimality condition let define the Hamiltonian

$$H(t,\phi,\mathbf{u},\mu,\mathbf{X}) = \langle f_0(t,\mathbf{u},\mu),\mathbf{X}\rangle +$$
$$(X_1)^T A_1^T(t,\mathbf{u},\mu)\phi X_2 + (X_1)^T\phi A_2(t,\mathbf{u},\mu)X_2 \qquad (19)$$

where

$$\mathbf{X} = X_1 \otimes X_2,$$

and ϕ and f_0 are the tensors of the order 2.

Then the dynamic programming equation (4) can be written as follows

$$\frac{d\langle\phi(t),\mathbf{X}\rangle}{dt} = -\min_{\mathbf{u},\mu} H(t,\phi(t),\mathbf{u},\mu,\mathbf{X}), \qquad (20)$$

with terminal condition

$$\phi(T) = \phi_0.$$

Let $\phi(t)$ is the solution of equation (20), then according to the Theorem 1

$$\min_{\mathbf{u}(\cdot),\mu(\cdot)} J[\mathbf{u}(\cdot),\mu(\cdot)|\mathbf{X}_t = \mathbf{X}] = \langle\phi(t),\mathbf{X}\rangle.$$

B. Definition of the running cost function

As in our previous works [11], [12] we determine the running cost, corresponding to the minimization of the following criterions:

1) Average time in queue: The average time in queue in the buffer of the volume M can be evaluated as follows

$$J_1 = \mathbf{E}\left\{\int_0^T \frac{M(\tau)}{\mu(\tau)}d\tau\right\} = \mathbf{E}\left\{\int_0^T \frac{\langle \mathbf{l}, X_\tau\rangle}{\mu(\tau)}d\tau\right\},$$

where $M(\tau)$ is the number of jobs in queue, and $\mu(t)$ is the service rate, vector $X(\tau)$ corresponds to the current state of the MC, where vector $X(\tau)$ has the unit at $M(\tau)$ place and all other entries are zero,

$$\mathbf{l}^* = (0, 1, 2, ..., M) \in R^{M+1}.$$

This is not exactly the expected value in queue, however, if the service rate is still constant and equals μ, then if the service time is exponentially distributed, the average time in queue is equal to $1/\mu$, therefore, if the demand comes when there are $M(t)$ demands in queue, then the average waiting time equals $M(t)/\mu$. If $\mu(t)$ is controlled variable this criterion gives some characterization of waiting time. Moreover, in case of two different lines, where the first line accepts the demand with the current probability $u_1(t)$ and the second line with the probability $(1 - u_1(t))u_2(t)$, more relevant criterion could be

$$J_1 = \mathbf{E}\int_0^T \left\{u_1(\tau)\frac{\langle \mathbf{l_1}, (X_1)_\tau\rangle}{\mu_1(\tau)} + (1 - u_1(\tau))u_2(\tau)\frac{\langle \mathbf{l_2}, (X_2)_\tau\rangle}{\mu_2(\tau)}\right\} d\tau, \tag{21}$$

where

$$\mathbf{l_i}^* = (0, 1, 2, ..., M_i) \in R^{M_i+1}, \quad i = 1, 2.$$

2) Average number of rejected demands: With two lines the probability to reject a demand at all is the probability to reject it by the second line, under condition that the demand is rejected already by the first line, therefore the average number of rejected demands is given by the following formula [13]

$$J_2 = \mathbf{E}\left\{\int_0^T (1 - U_2(\tau)\langle \mathbf{1_2}, (X_2)_\tau\rangle)\lambda(\tau)d\tau\right\},$$

where

$$\mathbf{1_2}^* = (1, 1, ..., 1, 0) \in R^{M_2+1}.$$

The substitution of expressions (13) gives

$$J_2^{'} = \mathbf{E}\left\{\int_0^T \frac{C(\tau)(1 - U_2(\tau)\langle \mathbf{1_2}, (X_2)_\tau\rangle)}{\sqrt{\lambda_0(1 + (1 - u_1(\tau))(1 - u_2(\tau))}}d\tau\right\},$$

and the substitution of expressions (14)

$$J_2^{''} = \mathbf{E}\left\{\int_0^T \frac{C(\tau)(1 - U_2(\tau)\langle \mathbf{1_2}, (X_2)_\tau\rangle)U_2^{1/2}(\tau)}{\sqrt{\lambda_0}}d\tau\right\},$$

3) Service cost: If we assume the linear dependence of running cost upon the service rate then it gives

$$J_3 = \mathbf{E}\int_0^T [u_1(\tau)\mu_1(\tau)\langle \mathbf{I_1}, (X_1)_\tau\rangle + (1 - u_1(\tau))u_2(\tau)\mu_2(\tau)\langle \mathbf{I_2}, (X_2)_\tau\rangle]\, d\tau, \tag{22}$$

where

$$\mathbf{I_i}^* = (0, 1, ..., 1) \in R^{M_i+1}, \quad i = 1, 2.$$

However, for the sake of simplicity one can use the simplified formula

$$J_3 = \mathbf{E}\left\{\int_0^T [\mu_1(\tau)\langle \mathbf{I_1}, (X_1)_\tau\rangle + \mu_2(\tau)\langle \mathbf{I_2}, (X_2)_\tau\rangle]\, d\tau\right\}. \tag{23}$$

Later the using of (23) permits to separate the numerical minimization over us and μs.

4) Penalty for the router workload: The criterion $J_2^{''}$ is equal zero if $U_2 = 0$, it means that such control will be optimal since there are no jobs and no service at all in this case. So in order to make the problem more reasonable we add the penalty for the lines workload. This penalty is minimal if the line is in intermediate workload and maximal is the line is either completely congested or idle.

So we consider the following criterion which characterizes the load of lines

$$J_4 = \mathbf{E}\left\{\int_0^T \langle \mathbf{g}, \mathbf{X}_\tau\rangle \mathbf{d}\tau\right\} = \mathbf{E}\left\{\int_0^T (X_1)_\tau^T \mathbf{g}(X_2)_\tau d\tau\right\}$$

where $\mathbf{g}^{i,j}$ is a matrix of the order $(M_1 + 1, M_2 + 1)$.

5) Terminal states: Sometimes the problem is to resolve the congestion. In this case the terminal state is rather important and the part of criterion which is responsible for it is given as

$$\mathbf{E}\left\{\langle \phi_0, \mathbf{X}_T\rangle\right\},$$

where the tensor ϕ_0 represents the set of relative weights of the terminal states.

6) Weighted criterion: In the real problem we have to achieve the balanced value of criteria. Moreover, as shown in [14] the problem with constrained criteria can be solved as the problem with weighted criterion, where the coefficients of weighting have to be found as a solution of some maxmin problem. So here we consider the following weighted criterion

$$J = k_1 J_1 + k_2 J_2 + k_3 J_3 + k_4 J_4 \to \min, \quad k_1, k_2, k_3, k_4 > 0.$$

In our numerical example we chose different coefficients for service cost privided by the first and the second lines, namely $k_{3,1}$ and $k_{3,2}$. Therefore, the running cost in criterion for the case of $J_2^{'}$ has a form

$$f_0(t,\mathbf{u},\mu,\mathbf{X}) = \\ k_1\left\{u_1\frac{\langle \mathbf{l}_1,X_1\rangle}{\mu_1}+(1-u_1)u_2\frac{\langle \mathbf{l}_2,X_2\rangle}{\mu_2}\right\}+ \\ k_2\frac{C(t)(1-U_2\langle \mathbf{1_2},X_2\rangle)}{\sqrt{\lambda_0(1+(1-u_1)(1-u_2)}}+ \\ k_{3,1}\mu_1\langle \mathbf{I_1},X_1\rangle + k_{3,2}\mu_2\langle \mathbf{I_2},X_2\rangle + k_4 X_1^T\mathbf{g}X_2, \tag{24}$$

or for the case of $J_2^{''}$

$$f_0(t,\mathbf{u},\mu,\mathbf{X}) = \\ k_1\left\{u_1\frac{\langle \mathbf{l}_1,X_1\rangle}{\mu_1}+(1-u_1)u_2\frac{\langle \mathbf{l}_2,X_2\rangle}{\mu_2}\right\}+ \\ k_2 C(t)(1-U_2\langle \mathbf{1_2},X_2\rangle)\sqrt{\frac{U_2}{\lambda_0}}+ \\ k_{3,1}\mu_1\langle \mathbf{I_1},X_1\rangle + k_{3,2}\mu_2\langle \mathbf{I_2},X_2\rangle + k_4 X_1^T\mathbf{g}X_2. \tag{25}$$

C. Dynamic programming equation for the case of two service lines

The system of differential equations of the dynamic programming (20) is given below for different $X_1 \in \mathcal{S}_1$ and $X_2 \in \mathcal{S}_2$

$$\min_{\mathbf{u},\mu}(X_1)^T((A_1(t,\mathbf{u},\mu))^T\phi(t)+\dot{\phi}(t) \\ +\phi(t)A_2(t,\mathbf{u},\mu)+f_0(t,\mathbf{u},\mu))X_2 = 0,$$

here

$$X_1^i = e_i \in R^{M_1}, \quad X_2^j = e_j \in R^{M_2}, \tag{26}$$

where $i = 0,...,M_1$ and $j = 0,...,M_2$.

For different values of (i,j) we get the following equations where for the sake of brevity the dependences of $\phi^{i,j}$ and c on t are ommited. Below the intensity of the income flow is equal

$$\lambda = \frac{c}{\sqrt{\lambda_0}}\frac{1}{\sqrt{1+(1-u_1)(1-u_2)}} \tag{27}$$

for criterion (24). Moreover, we use the simplified running cost function (23) instead of exact formula (22), where the running cost for service jointly depends on (u_1,u_2) and (μ_1,μ_2) and whereby simplify the realization of numerical procedure. Below we gave some of equations of the system (20) just to show the typical cases.

Case: $(i=0, j=0)$,

since both lines are in the state $i=0$ and $j=0$, then both controls $\mu_1 = 0, \mu_2 = 0$. Substitution of λ gives

$$\dot{\phi}^{00} = \min_{\substack{u_1\in[0,1]\\u_2\in[0,1]}}\left[\phi^{00}(\lambda u_1+\lambda(1-u_1)u_2)-\phi^{10}\lambda u_1\right. \\ \left.-\phi^{01}\lambda(1-u_1)u_2 - k_2(1-u_2)\lambda(1-u_1)-k_4 g^{00}\right] = \\ = \min_{\substack{u_1\in[0,1]\\u_2\in[0,1]}}\left[\frac{c}{\sqrt{\lambda_0}}\frac{u_1}{\sqrt{1+(1-u_1)(1-u_2)}}(\phi^{00}-\phi^{10})\right. \\ +\frac{c}{\sqrt{\lambda_0}}\frac{(1-u_1)u_2}{\sqrt{1+(1-u_1)(1-u_2)}}(\phi^{00}-\phi^{01})- \\ \left.k_2\frac{c}{\sqrt{\lambda_0}}\frac{(1-u_1)(1-u_2)}{\sqrt{1+(1-u_1)(1-u_2)}}-k_4 g^{00}\right].$$

Case: $(0 < i < M_1, 0 < j < M_2)$,

$$\dot{\phi}^{i,j} = \min_{\substack{\mu_1\in[\underline{\mu_1},\overline{\mu_1}],\\ \mu_2\in[\underline{\mu_2},\overline{\mu_2}]}}\left\{\mu_1(\phi^{ij}-\phi^{i-1,j}-k_{3,1})+\right. \\ \left.-k_1\left(\frac{i}{\mu_1}+\frac{j}{\mu_2}\right)+\mu_2(\phi^{ij}-\phi^{i,j-1}-k_{3,2})\right\}+ \\ \min_{\substack{u_1\in[0,1],\\u_2\in[0,1]}}\left[\frac{c}{\sqrt{\lambda_0}}\frac{u_1}{\sqrt{1+(1-u_1)(1-u_2)}}(\phi^{ij}-\phi^{i+1,j})+\right. \\ \frac{c}{\sqrt{\lambda_0}}\frac{(1-u_1)u_2}{\sqrt{1+(1-u_1)(1-u_2)}}(\phi^{ij}-\phi^{ij+1})- \\ \left.k_2\frac{c}{\sqrt{\lambda_0}}\frac{(1-u_1)(1-u_2)}{\sqrt{1+(1-u_1)(1-u_2)}}-k_4 g^{ij}\right]. \tag{28}$$

Case: $(i = M_1, j = M_2)$,

since both lines are in states $i = M_1, j = M_2$, then $u_1 = u_2 = 0$. We have

$$\dot{\phi}^{M_1,M_2} = \min_{\substack{\mu_1\in[\underline{\mu_1},\overline{\mu_1}]\\ \mu_2\in[\underline{\mu_2},\overline{\mu_2}]}}\left[\mu_1(\phi^{M_1,M_2}-\phi^{M_1-1,M_2}-k_{3,1})\right. \\ -k_1\left(\frac{M_1}{\mu_1}+\frac{M_2}{\mu_2}\right)+\mu_2(\phi^{M_1,M_2}-\phi^{M_1,M_2-1}-k_{3,2})- \\ \left.k_2\frac{c}{\sqrt{2\lambda_0}}\right].$$

Equations for not listed cases may be derived in a same way.

D. Numerical analysis

Explicit solution of the minimization procedure in the dynamic programming equation (20) is impossible for any cases, however this minimization procedure admits the numerical solution with the aid of existing packages like Maple, MatLab or Matematika. Meanwhile, the minimization procedure over μ_1, μ_2 admits the explicite solution like in [12], and only the numerical minimization over u_1, u_2 is necessary. Since (20) is a system of odinary differential equations, it is rather easy for qualitative analysis and for determining of the optimal solutions. For real problems this model needs rather large dimension of state variables, corresponding to buffers' capacities M_1, M_2, however, with the aid of the parallel computing approach one can demonstrate the possibility to speed up the calculation [16], [17], particularly for the large number of states. Moreover, for practical reasons it is not necessary to consider all possible states of buffers and for qualitative analysis it is possible to separate their loads just into three different states such as: "low", "middle" and "congestion". Below we give the result of numerical modelling for the system with $M_1 = M_2 = 3$, and give the value of controls u_1, u_2 and $\mu_1 \in [1, 2], \mu_2 \in [3, 6]$. The time of the process is equal to 1, function $C(t) = 5 + 4.5 \sin 10t$. Coefficients are chosen as follows

$$k_1 = 0.5, k_2 = 5, k_{3,1} = 1.0, k_{3,2} = 10, k_4 = 1.5, \lambda_0 = 0.01,$$

$$\begin{array}{llll} g_{00} = 5 & g_{01} = 5 & g_{02} = 10 & g_{03} = 20 \\ g_{10} = 5 & g_{11} = 5 & g_{12} = 15 & g_{13} = 20 \\ g_{20} = 5 & g_{21} = 5 & g_{22} = 15 & g_{23} = 20 \\ g_{30} = 10 & g_{31} = 10 & g_{32} = 15 & g_{33} = 20 \end{array}$$

One can observe that if $u_1 = 0$ and the principal line is closed the second line is open. It shows that model captures the necessity to serve the input flow even if the principal line is closed and it is necessary to use the second even more expensive line.

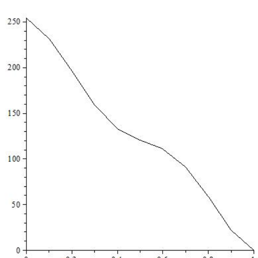

Fig. 1. Solution of the system (20) for the state $(i = 1, j = 2)$.

REFERENCES

[1] V. Abramov. *Optimal Control of Large Dam.* Journal of Applied Probability, v. 44, 2007, pp. 249–258.

[2] E. Altman. *Constrained Markov Decision Processes*. Boca Raton, FL: Chapman & Hall/CRC, 1999.

[3] P. Bremaud. *Optimal thinning of a point processes.* SIAM J. Control and Optimization, v. 17, n 2, 1979, pp. 222–230.

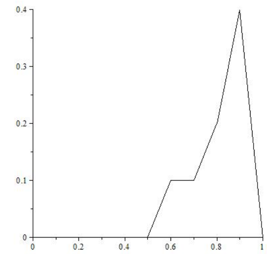

Fig. 2. Control u_1 for the state $(i = 1, j = 2)$.

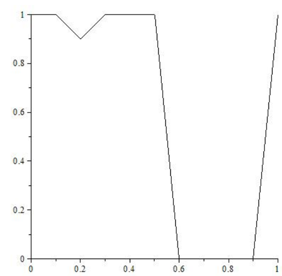

Fig. 3. Control u_2 for the state $(i = 1, j = 2)$.

[4] V. N. Burkov *Foundations of Mathematical Theory of Active Systems.* Moscow: Nauka, 1977 [in russian].

[5] M.H.A. Davis *Markov Models and Optimization.* London: Chapman and Hall, 1993.

[6] F. Delebecque and J.-P. Quadrat. *Optimal control of Markov Chains Admitting Strong and Weak Interactions.* Automatica, v. 17, n 2, 1981, pp. 281–296.

[7] R. J. Elliott, L. Aggoun, and J. B. Moore *Hidden Markov Models. Estimation and Control.* New York: Springer Verlag, 1995.

[8] S. Floyd and V. Jacobson *Random Early Detection Gateways for Congestion Avoidance.* IEEE/ACM Transactions on networking. v. 1, n 4, 1993, pp. 393–413.

[9] S. H. Low, F. Paganini, and J.C. Doyle *Internet Congestion Control.* IEEE Control Systems Magazine, 2002, pp. 28-43.

[10] A. Miller. Dynamic control of access with active users. Information processes, v. 9, n 1, 2009, pp. 1-17.

[11] A. B. Miller, *Using Methods of Stochastic Control to Prevent Overloads in Data Transmission Networks.* Automation and Remote Control, Vol. 71, No. 9, 2010, pp. 1804-1815.

[12] A. Miller and B. Miller *Congestion avoidance with the aid of stochastic control.* Proceedings of the 49th IEEE CDC 2010, Atlanta, Geogia, USA, December, pp. 552–558, 2010.

[13] B. M. Miller *Optimization of queuing systems via stochastic control.* Automatica, v. 45, n 6, 2009, pp. 1423–1430.

[14] B. Miller, G. Miller and K. Siemenikhin *Optimal control of Markov chains with constraints.* Proceedings of Joint 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference Shanghai, P.R. China, December 16-18, 2009, pp. 512–518, 2009.

[15] B. Miller, G. Miller, and K. Siemenikhin *Torwards the optimal control of markov chains with constraints.* Automatica, vol. 46, pp. 1495–1502, 2010.

[16] B. Miller and D. McInnes *Management of dam systems via optimal price control.* Procedia Computer Science, v. 4, 2011, pp. 1373–1382.

[17] B. Miller and D. McInnes *Optimal Management of a Two Dam System via Stochastic Control: Parallel Computing Approach.* Acepted to IEEE CDC 2011.

[18] R. Srikant *The mathematics of Internet congestion control.* Boston : Birkhuser, 2004.

[19] V. A. Vasenin and G. I. Simonova *Mathematical Models of Traffic Control in Internet: New Approaches Based of TCP/AQM Schemes.* Automation and Remote Control, v. 66, n 8, 2005, pp. 1274–1286.

[20] M. Welzl *Network Congestion Control : Managing Internet Traffic* Chichester : John Wiley & Sons, Ltd., 2005.