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Characterization and computation of robust invariant sets for switching systems under dwell-time consideration

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Abstract— **This paper introduces the concept of Disturbance-Dwell-Time invariance (DDT-invariance) for constrained switching systems with additive disturbance under dwell-time switching. This concept is useful for the characterization of invariant sets for such systems. Necessary and sufficient conditions for DDT-invariance of a set are also provided and algorithms for the numerical computation of the minimal and maximal constraint admissible convex DDT-invariant sets are proposed.**

I. Introduction

This paper considers the following constrained discrete-time switched linear system with additive disturbance:

$$x(t+1) = A_{\sigma(t)}x(t) + w(t) \tag{1a}$$

$$x(t) \in \mathcal{X}, w(t) \in W, \quad \forall t \in \mathbb{Z}^+ \tag{1b}$$

where $x(t) \in \mathbb{R}^n$, $w(t) \in \mathbb{R}^n$ are the state and disturbance variables respectively, $W \subset \mathbb{R}^n$ is the disturbance set, $\mathbb{Z}^+$ is the set of non-negative integers and $\sigma(t) : \mathbb{Z}^+ \to \mathcal{I}_N := \{1, \cdots, N\}$ is a time-dependent switching signal that indicates the current mode of the system among N possible modes in $\mathcal{A} := \{A_1, \cdots, A_N\}$. The set $\mathcal{X} \subset \mathbb{R}^n$ models constraints imposed on physical state of the system, including those arising from the actuator via some appropriate state feedback when (1) is seen as a feedback system.

The study of switching systems is quite active in the past years. Most of the literature [1], [2], [3], [4] is concerned with conditions that ensure stability of the system when $\sigma(\cdot)$ is an arbitrary switching function while others [5], [6], [7], [8] consider stability of switching systems when $\sigma(\cdot)$ respects some dwell-time consideration. A few of them also consider the presence of constraints and/or disturbances [9], [10], [11], [12]. This work is concerned with the characterization and computation of suitably defined disturbance-invariant sets for system (1) when $\sigma(\cdot)$ is an admissible switching function that respects the dwell-time consideration. Since only dwell-time switching is allowed, the invariance condition is termed Disturbance Dwell-Time invariance (DDT-invariance). Other contributions of this work include algorithms for the computations of the maximal and the minimal convex DDT-invariant sets for system (1). In the limiting case where the dwell-time is one sample period, $\sigma(\cdot)$ becomes an arbitrary switching function, and the corresponding invariant sets and their computations have appeared in the literature, see for example, [9] and [11], [12]. Hence, this work can also be seen as a generalization of those obtained for arbitrary switching systems.

The rest of this paper is organized as follows. This section ends with a description of the notations used. Section II reviews some standard terminology and preliminary results for switching systems. Section III shows the main result on the characterization of the DDT-invariant set for system (1) and its properties. Sections IV and V consider, respectively, computations of the minimal and the maximal convex DDT-invariant sets for (1). Section VI contains numerical examples followed by the conclusions in Section VII. All proofs for the results stated in this paper can be found in forthcoming paper [13].

Standard notations are followed. Given $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$, A_j and b_k are the corresponding j-th row and the k-th element respectively while $\rho(A)$ denotes its spectral radius. The floor function $\lfloor a \rfloor$ is the largest integer that is less than a. Standard 2-norm is indicated by $\|\cdot\|$ while other p-norms are $\|\cdot\|_p, p = 1, \infty$. $\mathcal{B}(\eta) := \{x \in \mathbb{R}^n : \|x\| \leq \eta\}$ refers to the 2-norm ball of radius η. Suppose $\alpha > 0$ and $X, Y \subset \mathbb{R}^n$ are compact sets that contain 0 in their interiors. Then, scaling of X is $\alpha X := \{\alpha x : x \in X\}$, image of X is $AX := \{y : y = Ax$ and $x \in X\}$ for some appropriate matrix A, the Minkowski sum is $X \oplus Y := \{z \in \mathbb{R}^n : z = x + y, x \in X, y \in Y\}$, the Pontryagin (or Minkowski) difference is $X \ominus Y := \{z \in \mathbb{R}^n : z + y \in X, \forall y \in Y\}$ and $A(X \oplus Y) = AX \oplus AY$. The boldface $\mathbf{1}$ indicates the vector of all 1s and $co\{\cdot\}$ denotes the convex-hull. The distance between $x \in \mathbb{R}^n$ and a set $Y \subset \mathbb{R}^n$ is $d(x, Y) := \inf_{y\in Y} \|x - y\|$. The distance between tow sets X, Y is measured by Hausdorff metric $\mathcal{H}(X, Y) := \max\big\{\sup_{x\in X} d(x, Y), \sup_{y\in Y} d(y, X)\big\}$. Other notations are introduced as and when needed.

II. Preliminaries

This section reviews definitions of switching time, dwell-time and admissible switching functions or sequences. These definitions have appeared in prior papers [1], [6], [7], [8] but are repeated here for completeness and for setting up the needed notations.

A switching sequence of (1) is denoted by $\mathcal{S}_\tau(t) = \{\sigma(t-1), \cdots, \sigma(1), \sigma(0)\}$ with $\sigma(\cdot) \in \mathcal{I}_N$. Suppose $t_{s_0}, t_{s_1}, \cdots, t_{s_k}, \cdots$ are the switching instants with $t_{s_0} = 0$ and $t_{s_k} < t_{s_{k+1}}$. It follows that $\sigma(t_{s_k}) \neq \sigma(t_{s_{k+1}})$ and $\sigma(t_{s_k}) = \sigma(t_{s_k} + 1) = \cdots = \sigma(t_{s_{k+1}} - 1)$ for all $k \in \mathbb{Z}^+$. Let $\{i\}^\ell := \{i, i, \cdots, i\}$ be a sequence

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of ℓ elements of i with $i \in \mathcal{I}_N$ and $\mathcal{W}_\ell$ be the set of sequences $\{w(\cdot)\}$ of length ℓ with every $w(\cdot) \in W$. Then, a switching sequence can equivalently be represented by $\mathcal{S}_\tau(t) = \{\{i_m\}^{k_m}, \cdots, \{i_1\}^{k_1}, \{i_0\}^{k_0}\}$ for some appropriate $i_j \in \mathcal{I}_N$ for all $j = 0, \cdots, m$ and $\sum_{j=0}^{m} k_j = t$.

Definition 1: An admissible switching sequence of system (1), $\mathcal{S}_\tau(t)$, with switching instants $t_{s_0}, t_{s_1}, \cdots, t_{s_k}, \cdots$ has a dwell-time of τ means that $t_{s_{k+1}} - t_{s_k} \geq \tau$ for all $k \in \mathbb{Z}^+$. In addition, suppose t_{last} is the last switching time for an admissible sequence $\mathcal{S}_\tau(t)$, then $t - t_{last} \geq \tau$.

Remark 1: The last condition in *Definition* 1 requires further qualification. Suppose $\mathcal{I}_N = \{1, 2\}$ and $\tau = 3$ then $\mathcal{S}_3^a(6) = \{1, 1, 1, 2, 2, 2\}$ is an admissible sequence. However, $\mathcal{S}_3^b(6) = \{1, 1, 2, 2, 2, 2\}$ is not an admissible sequence because $t - t_{last} < 3$ and the dwell-time consideration may be violated if $\sigma(6) = 2$. On the other hand, if $\sigma(6) = 1$ means $\mathcal{S}_3^b(6)$ is a truncated subsequence of an admissible sequence. This is a key point that distinguishes systems under dwell-time consideration and under arbitrary switching. Following the same reasoning, $\mathcal{S}_\tau(t)$ for $t < \tau$ is also not an admissible sequence.

System (1) is also assumed to satisfy the following assumptions: **(A1)** The spectral radius of each individual subsystem $A_i, i \in \mathcal{I}_N$ is less than 1; **(A2)** The constraint set $\mathcal{X}$ is a non-empty polytope represented by $\mathcal{X} = \{x : Rx \leq \mathbf{1}\}$ for some appropriate matrix $R \in \mathbb{R}^{q \times n}$; **(A3)** (A_i, R) are observable for all $i \in \mathcal{I}_N$. **(A4)** A value of $\tau \geq 1$ has been identified such that the disturbance-free system (1a) with dwell-time τ is asymptotically stable. (**A5**) W is a polytope and contains 0 in its interior.

Assumption (A1) defines the family of systems considered in this work and is a common requirement in past works of similar nature. The polyhedral assumptions of (A2) and (A5) are made to facilitate numerical computations described in sections IV and V. They are not needed for the theoretical development of section III. If (A3) is not satisfied, then system (1) can be reformulated to consider only the observable subsystem of A_i. Assumption (A4) poses minimal restriction as procedure for obtaining a minimal dwell-time for a disturbance-free system is known [8].

III. Main Results

This section begins with definitions of Disturbance-Dwell-Time-invariant (DDT-invariant) set and Constraint Admissible Disturbance Dwell-Time-invariant (CADDT-invariant) set for system (1a) with admissible input sequences.

Definition 2: A set $\Omega \subset \mathbb{R}^n$ is said to be DDT-invariant w.r.t. (1a) with dwell-time τ, if $x(0) \in \Omega$ implies $x(t) \in \Omega$ for every admissible sequence $S_\tau(t)$ and for every allowable disturbance sequence $\{w(0), \cdots, w(t-1)\} \in \mathcal{W}_t$.

Definition 3: A set $\Omega \subset \mathbb{R}^n$ is said to be CADDT-invariant w.r.t. (1a) with dwell-time τ, if it is DDT-invariant and $x(t) \in \mathcal{X}$ for all $t \in \mathbb{Z}^+$.

The definition of DDT-invariant is closely related to the definition of an admissible sequence. Using the example of *Remark* 1 and assuming that $x(0) \in \Omega$, $\mathcal{S}_3^a(6)$ will result in $x(6) \in \Omega$ but $\mathcal{S}_3^b(6)$ may not result in $x(6) \in \Omega$. Similarly, $x(7) \in \Omega$ if $\mathcal{S}_3^b(7)$ is obtained from $\mathcal{S}_3^b(6)$ with $\sigma(6) = 1$.

While stating the requirements of DDT-invariance and CADDT-invariance, the above definitions are of limited practical usefulness since the reachable set of system (1) for all admissible switching input and disturbance sequences of length t have to be considered. Clearly, such an approach is not computationally tractable. This difficulty can be circumvented using a useful characterization of Dwell-Time invariance from the work of [8]. They consider system (1) without disturbance and show that any admissible sequence of the form

$$\mathcal{S}_\tau(t) = \{\{i_m\}^{k_m}, \cdots, \{i_1\}^{k_1}, \{i_0\}^{k_0}\} \tag{2}$$

with $i_j \in \mathcal{I}_N$, $k_j \geq \tau$ for all $j = 0, \cdots, m$ and $\sum_{j=0}^{m} k_j = t$ can be written as a unique ordering of a finite number of subsequences as

$$\mathcal{S}_\tau(t) = \{\{i_m\}^{q_m \tau}, \{i_m\}^{r_m}, \cdots, \{i_0\}^{q_0 \tau}, \{i_0\}^{r_0}\} \tag{3}$$

where, for all $j = 0, \cdots, m$, $q_j = \lfloor \frac{k_j - \tau}{\tau} \rfloor$ is the remainder of $k_j - \tau$ when divided by τ and $r_j \in \mathbb{T}$ with

$$\mathbb{T} := \{\tau, \tau + 1, \cdots, 2\tau - 1\}. \tag{4}$$

Motivated by this result, a parameterization of all admissible sequences can be obtained using an alternative representation of (2). This takes the form of

$$\mathcal{S}_\tau(t) = \{\ \{j_{p-1}\}^{\ell_{p-1}}, \cdots, \{j_1\}^{\ell_1}, \{j_0\}^{\ell_0}\} \tag{5}$$

for some appropriate integers $\ell_0, \ell_1, \cdots \ell_{p-1}$ with $\sum_{i=0}^{p-1} \ell_i = t$ where each $\ell_i \in \mathbb{T}$, $j_i \in \mathcal{I}_N$ for $i = 0, \cdots, p-1$. This form shows that an admissible sequence is a concatenation of p-stage subsequences (as opposed to a m-mode subsequences of (2)): the first stage is in mode j_0 for ℓ_0 steps, the second in mode j_1 for ℓ_1 steps and so on with the possibility that $j_i = j_{i+1}$. Such a representation facilitates the representation of all admissible sequences up till time t. For this purpose, several operations are introduced. They are slight modifications of well-known one-step forward (backward) operator for standard linear system.

Given a set $\Omega \subset \mathbb{R}^n$, let $\hat{P}(\Omega, A, W) := \{Ax + w : x \in \Omega, w \in W\} = A\Omega \oplus W$ be the set of reachable states in one time step from Ω with respect to system $x(t+1) = Ax(t) + w(t)$ driven by disturbance $w(\cdot) \in W$. Repeating this operation ℓ times lead to the ℓ-step reachable set of Ω given by

$$\begin{aligned} &\hat{P}_\ell(\Omega, A, W) = \hat{P} \cdots \hat{P}(\Omega, A, W) \\ &= \{A^\ell x + A^{\ell-1} w + \cdots + Aw + w : x \in \Omega, w \in W\} \\ &= A^\ell \Omega \oplus A^{\ell-1} W \oplus \cdots \oplus AW \oplus W \end{aligned} \tag{6}$$

In the case of (5), mode j_i can be any index of $\mathcal{I}_N$ and ℓ_i is any element in $\mathbb{T}$. This motivates the definition of

$$\begin{aligned} P(\Omega, W) &:= \bigcup_{\ell \in \mathbb{T}} \Big\{ \bigcup_{i \in \mathcal{I}_N} \hat{P}_\ell(\Omega, A_i, W) \Big\} \\ &= \bigcup_{\ell \in \mathbb{T}, i \in \mathcal{I}_N} \hat{P}_\ell(\Omega, A_i, W) \end{aligned} \tag{7}$$

and it characterizes the reachable set of one stage based on the representation given by (5). This means that $x(\ell_0) \in P(\{0\},W)$ and $x(\ell_1+\ell_0) \in \bigcup_{\ell\in\mathbb{T},i\in\mathcal{I}_N} \hat{P}_\ell(P(\{0\},W),A_i,W)$=$P(P(\{0\},W),W)$=$P_2(\{0\},W)$. This continues till the p-th stage where

$$x(\ell_{p-1}+\cdots+\ell_0) \in \bigcup_{\ell\in\mathbb{T},i\in\mathcal{I}_N} \hat{P}_\ell\Big(P_{p-1}(\{0\},W),A_i,W\Big) = P_p(\{0\},W). \tag{8}$$

Another interpretation of the above is that the family of *all* admissible sequences up to time $p(2\tau-1)$ is

$$\bigcup_{\ell_0\in\mathbb{T},\cdots,\ell_{p-1}\in\mathbb{T}} \Big(\bigcup_{j_0\in\mathcal{I}_N,\cdots,j_{p-1}\in\mathcal{I}_N} \{\,\{j_{p-1}\}^{\ell_{p-1}},\cdots,\{j_0\}^{\ell_0}\}\Big) \tag{9}$$

The above analysis is based on the forward operation of $\hat{P}(\cdot,\cdot,\cdot)$. Another operation needed in the sequel is that given by the one-step backward operator. Formally, this one-step and ℓ-step backward sets of a given non-empty $\Omega\subset\mathbb{R}^n$ w.r.t. system $x(t+1)=Ax(t)+w(t)$ are known respectively to be $\hat{Q}(\Omega,A,W)=\{x: Ax+w\in\Omega, w\in W\}=\{x: Ax\in(\Omega\ominus W)\}$ and

$$\begin{aligned}\hat{Q}_\ell(\Omega,A,W) &= \hat{Q}\cdots\hat{Q}(\Omega,A,W)\\ &= \{x: A^\ell x+\cdots+Aw+w\in\Omega, w\in W\}\\ &= \big\{x: A^\ell x\in\big(\Omega\ominus W\ominus\cdots\ominus A^{\ell-1}W\big)\,\big\}\end{aligned} \tag{10}$$

In the characterization of (5), the first stage consists of ℓ_0 time steps where ℓ_0 can be any element in $\mathbb{T}$ while j_0 can be any element of $\mathcal{I}_N$. Hence, the set of state that can be brought into Ω for one stage of an admissible sequence is

$$\begin{aligned}Q(\Omega,W) &:= \bigcap_{\ell\in\mathbb{T}}\Big\{\bigcap_{i\in\mathcal{I}_N}\hat{Q}_\ell(\Omega,A_i,W)\Big\}\\ &= \bigcap_{\ell\in\mathbb{T},i\in\mathcal{I}_N}\hat{Q}_\ell(\Omega,A_i,W)\end{aligned} \tag{11}$$

and it is the backward set for one stage in an admissible sequence.

Theorem 1: Suppose (A1), (A4) and (A5) are satisfied and a non-empty set Ω is given. Let $P(\cdot,\cdot)$ and $Q(\cdot,\cdot)$ be as defined by (7) and (11) respectively. The following statements are equivalent:
(i) A set $\Omega\subset\mathbb{R}^n$ is DDT-invariant for system (1a);
(ii) $P(\Omega,W)\subseteq\Omega$;
(iii) $\Omega\subseteq Q(\Omega,W)$.

Proof: $(i)\Rightarrow(ii)$: This proof is by contradiction. Suppose Ω is DDT-invariant but (ii) is not satisfied. This means there exists an $\ell\in\{\tau,\tau+1,\cdots,2\tau-1\}$ and $i\in\mathcal{I}_N$ such that $\hat{P}_\ell(\Omega,A_i,W)\nsubseteq\Omega$. However, $\{i\}^\ell=\{i,i,\cdots,i\}$ is an admissible switching sequence and for any $x(0)\in\Omega$ it follows that $x(t)\in\hat{P}_\ell(\Omega,A_i,W)\nsubseteq\Omega$. This implies $x(t)\notin\Omega$ for some admissible switching sequence, which contradicts the DDT-invariance of Ω.

$(ii)\Rightarrow(iii)$: With (7), condition (ii) holds means $\hat{P}_\ell(\Omega,A_i,W)\subseteq\Omega$ for all $\ell\in\mathbb{T}$ and for all $i\in\mathcal{I}_N$. Applying $\hat{Q}_t(\cdot)$ operator on both sides of the above inclusion yields

$$\hat{Q}_\ell\Big(\hat{P}_\ell(\Omega,A_i,W),A_i,W\Big)\subseteq\hat{Q}_\ell(\Omega,A_i,W),\quad \forall\ell\in\mathbb{T},\forall i\in\mathcal{I}_N \tag{12}$$

because $\hat{Q}_t(\Omega_1,A,W)\subseteq\hat{Q}_t(\Omega_2,A,W)$ for any Ω_1 and Ω_2 such that $\Omega_1\subseteq\Omega_2$. The left-hand side of (12) is $\hat{Q}_\ell\big(\hat{P}_\ell(\Omega,A_i,W),A_i,W\big)=\Omega$ and taking the intersection of $\hat{Q}_\ell(\Omega,A_i,W)$ over all $\ell\in\mathbb{T}$ and $i\in\mathcal{I}_N$ leads to $\Omega\subseteq Q(\Omega,W)$.

$(iii)\Rightarrow(i)$: Let $x(0)\in\Omega$, this implies $x(0)\in Q(\Omega,W)$ by (iii). Consider all admissible sequence, $\mathcal{S}_\tau(t)$ of the form (5). It follows that $x(\ell_0)=A_{j_0}^{\ell_0}\Omega+A_{j_0}^{\ell_0-1}w_0+\cdots+A_{j_0}w_{\ell_0-2}+w_{\ell_0-1}\in\Omega$ for any $j_0\in\mathcal{I}_N$ and any $\ell_0\in\mathbb{T}$. Repeating this for all stages until the last stage of ℓ_p-1 shows that $x(t)\in\Omega$. This shows that Ω is DDT-invariant. ■

Theorem 1 requires that $x(t)\in\Omega$ for all $t\in\mathbb{T}$, but no mention is made of the constraints $x(t)\in\mathcal{X}$ stipulated in (1b). Clearly, the constraint admissibility requires more condition than $\Omega\subseteq\mathcal{X}$. Imposing $x(t)\in\mathcal{X}$ for $t=0,1,\cdots,\tau-1$ ensures that Ω is CADDT-invariant. This result is therefore stated without a proof.

Theorem 2: Suppose (A1), (A4) and (A5) are satisfied and a non-empty set Ω is given. Let $\hat{P}_\ell(\cdot,\cdot,\cdot)$ and $\hat{Q}_\ell(\cdot,\cdot,\cdot)$ be as defined by (6) and (10) respectively. A DDT-invariant set $\Omega\subseteq\mathcal{X}$ is CADDT-invariant for system (1) with dwell-time τ, if and only if

(i) $\hat{P}_\ell(\Omega,A_i,W)\subseteq\mathcal{X},\ \forall\,\ell\in\{0,1,\cdots,\tau-1\}\,,\forall\,i\in\mathcal{I}_N$
or
(ii) $\Omega\subseteq\hat{Q}_\ell(\mathcal{X},A_i,W),\ \forall\,\ell\in\{0,1,\cdots,\tau-1\},\forall\,i\in\mathcal{I}_N$

IV. Minimal DDT-invariant set and its computation

The solution of (1a) is

$$\begin{aligned}x(t) = &A_{\sigma(t-1)}A_{\sigma(t-2)}\cdots A_{\sigma(1)}A_{\sigma(0)}x(0)+\\ &A_{\sigma(t-1)}\cdots A_{\sigma(1)}w(0)+A_{\sigma(t-1)}\cdots A_{\sigma(2)}w(1)\\ &+\cdots+A_{\sigma(t-1)}w(t-2)+w(t-1).\end{aligned} \tag{13}$$

The first term on the righthand side of (13) approaches zero as t approaches infinity for any admissible switching sequence under (A4). The sum of the rest of the terms on the righthand side of (13) characterizes the asymptotic behavior of switching system (1a) in the presence of disturbance sequences. Let $F_t(\mathcal{A},W,\tau)$ be the set of states that can be reached in t steps from the origin for all admissible sequences with dwell-time τ and all disturbance sequences of length t. Using (13) with $x(0)=0$, it follows that

$$F_t(\mathcal{A},W,\tau) := \bigcup_{\sigma(\cdot)\in\mathcal{S}_\tau(t)}\Big(A_{\sigma(t-1)}\cdots A_{\sigma(1)}W\oplus\cdots\oplus A_{\sigma(t-1)}W\oplus W\Big) \tag{14}$$

with $F_0(\mathcal{A},W,\tau):=\{0\}$. For notational simplicity, the dependence of F_t and other derived sets on $(\mathcal{A},W,\tau)$ will be dropped unless warranted by context. The limiting condition of (14), existence of which is shown in Theorem 3, becomes

$$F_\infty = \lim_{t\to\infty} F_t \tag{15}$$

Hence, F_∞ characterizes the asymptotical behavior of (1) and, typically, a small F_∞ set is desirable. The computation of F_t based on the elapsed time t for system (1) is difficult because it has no clear structure. A more useful representation is that given by (5) which characterizes the reachable states by stages instead of time. Let $\mathcal{F}_k$ be the set of reachable states at the k-th stage. Using the same reasoning in Section III leading to equation (9), define

$$\mathcal{F}_k := P(\mathcal{F}_{k-1}, W) = \bigcup_{\ell\in\mathbb{T}, i\in\mathcal{I}_N} \hat{P}_\ell(\mathcal{F}_{k-1}, A_i, W) \tag{16}$$

with $\mathcal{F}_0 = \{0\}$. Since the above union operation is taken over all $\ell \in \mathbb{T}$ and all $i \in \mathcal{I}_N$, (16) captures all admissible sequences of length $k(2\tau-1)$ and hence, $\mathcal{F}_k = F_{k(2\tau-1)}$ for all $k \in \mathbb{Z}^+$. Taking the limit as $k \to \infty$, $F_\infty = \lim_{t\to\infty} F_t = \lim_{k\to\infty} F_{k(2\tau-1)} = \lim_{k\to\infty} \mathcal{F}_k = \mathcal{F}_\infty$.

The union operation of (16) remains problematic computationally as it does not preserve convexity. This problem can be circumvented by computing a convex outer-bound of $\mathcal{F}_k$, denoted by $\mathbb{F}_k$ in the form of $\mathbb{F}_k := co\{\mathcal{F}_k\}$. Similarly, $\mathbb{F}_\infty := \lim_{k\to\infty} \mathbb{F}_k$. Conceptually, the procedure of computing $\mathbb{F}_\infty$ is to first compute $\mathcal{F}_k$ based on (16) at every stage k and then compute its convex hull, starting from $k = 0$. The algorithmic computation of $\mathbb{F}_\infty$ is given below.

Algorithm 1 Computation of $\mathbb{F}_\infty$

`Input:` $\mathcal{A}$, τ and W.

(a) $k = 0$, and initialize $\mathbb{F}_0 = \{0\}$.

(b) Compute $\hat{P}_\ell(\mathbb{F}_k, A_i, W)$ for $\ell = \tau, \tau+1, \cdots, 2\tau-1$ and for all $A_i \in \mathcal{A}$ and let

$$\mathbb{F}_{k+1} = \mathrm{co}_{t\in\mathbb{T}, A_i\in\mathcal{A}}\left\{\hat{P}_\ell(\mathbb{F}_k, A_i, W)\right\}$$

(c) If $\mathbb{F}_{k+1} \equiv \mathbb{F}_k$ set $\mathbb{F}_\infty = \mathbb{F}_k$ and stop, else set $k = k+1$ and goto step (b).

Step **(b)** of Algorithm 1 computes

$$\begin{aligned}\mathbb{F}_{k+1} &= co\{P(\mathbb{F}_k, W)\}\\ &= co\{\hat{P}_\ell(\mathbb{F}_k, A_i, W) : \ell \in \mathbb{T}\,\text{and}\, i \in \mathcal{I}_N\}.\end{aligned} \tag{17}$$

This step can be computed when W is a polytope under (A5). Properties of the $\mathbb{F}_\infty$ set obtained from Algorithm 1 are stated next.

Theorem 3: Suppose system (1) satisfies assumptions (A1), (A4), (A5) and $\mathbb{F}_k$ is generated based on Algorithm 1. Then:

(i) $\mathbb{F}_k \equiv co\{\mathcal{F}_k\}$ for all k.

(ii) $0 \in \mathbb{F}_k$ and $\mathbb{F}_k \subseteq \mathbb{F}_{k+1}$ for all k.

(iii) $\mathbb{F}_k \supseteq \mathcal{F}_k = F_{k(2\tau-1)}$ for all k.

(iv) $\mathcal{F}_\infty := \lim_{k\to\infty} \mathcal{F}_k$ exists and it is bounded.

(v) $\mathbb{F}_\infty := \lim_{k\to\infty} \mathbb{F}_k$ exists and the set sequence $\{\mathbb{F}_k : k \in \mathbb{Z}^+\}$ of Algorithm 1 converges to $\mathbb{F}_\infty$, in the sense that $\mathcal{H}(\mathbb{F}_\infty, \mathbb{F}_k) \to 0$ as $k \to \infty$.

(vi) $\mathbb{F}_\infty = co\{F_\infty\}$.

(vii) $\mathbb{F}_\infty$ is DDT-invariant.

(viii) $\mathbb{F}_\infty$ is the minimal convex DDT-invariant set.

(ix) The state of system (1a) starting from any $x(0)$ converges to F_∞ for every admissible sequence in the sense that $d(x(t), F_\infty) \to 0$ as $t \to \infty$.

V. Maximal Constraint Admissible DDT-invariant set

This section deals with the characterization and computation of the maximal constraint admissible DDT-invariant set, $\mathbb{O}_\infty(\mathcal{A}, \mathcal{X}, W, \tau)$, for system (1). This set defines the largest region starting from which system (1) remains constraint admissible for all admissible switching sequences. A necessary assumption for the existence of such a set is that **(A6)** F_∞ should be CADDT-invariant, which implies that $\mathcal{X} \supset F_\infty$. Clearly, this means that the effect of disturbance should not lead to the state exceeding the constraint set.

Let $\mathbb{O}_1(\mathcal{A}, \mathcal{X}, W, \tau)$ be the set of states that can be brought into constraint set $\mathcal{X}$ in one stage for system (1) under an appropriate $\mathcal{S}_\tau(t)$. This means that $x(t) \in \mathcal{X}$ if $x(0) \in \mathbb{O}_1(\mathcal{A}, \mathcal{X}, W, \tau)$ for all appropriate t for one stage. Using (5), $x(0)$ belongs to the set $\mathbb{O}_1(\mathcal{A}, \mathcal{X}, W, \tau) := Q(\mathcal{X}, W) = \bigcap_{\ell\in\mathbb{T}, i\in\mathcal{I}_N} \hat{Q}_\ell(\mathcal{X}, A_i, W)$ since (j_0, ℓ_0) of (5) can be any element of $\mathbb{T} \times \mathcal{I}_N$ and $x(t) \in \mathcal{X}$ has to be satisfied for all such sequences. Using the above recursively leads to

$$\begin{aligned}\mathbb{O}_k(\mathcal{A}, \mathcal{X}, W, \tau) &:= Q(\mathbb{O}_{k-1}(\mathcal{A}, \mathcal{X}, W, \tau), W)\\ &= \bigcap_{\ell\in\mathbb{T}, i\in\mathcal{I}_N} \hat{Q}_\ell(\mathbb{O}_{k-1}(\mathcal{A}, \mathcal{X}, W, \tau), A_i, W)\end{aligned} \tag{18}$$

with $\mathbb{O}_0 := \mathcal{X} \bigcap_{i\in\mathcal{I}_N, \ell=1,2,\cdots\tau-1} \hat{Q}_\ell(\mathcal{X}, A_i, W)$. The detailed algorithmic computation of $\mathbb{O}_\infty$ is given below. Hereafter, the dependence of $\mathbb{O}_k$ on $(\mathcal{A}, \mathcal{X}, W, \tau)$ is dropped for notational convenience unless needed.

Algorithm 2 Computation of maximal CADDT-invariant set

`Input:` $\mathcal{A}$, $\mathcal{X}$, W and τ.

(a) Set $k = 0$ and let

$$\mathbb{O}_0 := \mathcal{X} \bigcap_{1\le t\le\tau-1,\, A_i\in\mathcal{A}} \hat{Q}_t(\mathcal{X}, A_i, W)$$

(b) Compute $\hat{Q}_t(\mathbb{O}_k, A_i, W)$ for $t = \tau, \tau+1, \cdots, 2\tau-1$ and for all $A_i \in \mathcal{A}$. Then, let

$$\mathbb{O}_{k+1} = \mathbb{O}_k \bigcap_{\ell\in\mathbb{T},\, A_i\in\mathcal{A}} \hat{Q}_\ell(\mathbb{O}_k, A_i, W)$$

(c) If $\mathbb{O}_{k+1} \equiv \mathbb{O}_k$ set $\mathbb{O}_\infty = \mathbb{O}_k$ then stop, else set $k = k+1$ and goto step (b).

Step **(a)** of Algorithm 2 imposes the constraints for the first $\tau-1$ steps to ensure the constraint admissibility of $\mathbb{O}_\infty$ according to *Theorem* 2. Similarly, step **(b)** imposes (18) and captures all possible admissible switching sequences.

When $\mathcal{X}$ and W are polytopes under assumptions (A2) and (A5), so is $\mathbb{O}_k$. The corresponding numerical operations for each step of Algorithm 2 are also straight forward, including the computation of $\hat{Q}(\mathcal{X}, A, W)$ (see [14]). More exactly, $\hat{Q}(\mathcal{X}, A, W) = \{x : R(Ax + w) \leq \mathbf{1}, \forall w \in W\} = \{x : RAx \leq \mathbf{1} - \max_{w \in W} Rw\}$. Hence,

$$\hat{Q}_\ell(\mathcal{X}, A_i, W) = \{x : RA_i^\ell x \leq \mathbf{1} - \max_{w \in W} Rw - \max_{w \in W} RA_i w - \cdots - \max_{w \in W} RA_i^{\ell-1} w\}. \quad (19)$$

If $\left(1 - \max_{w \in W} R_j A_i w - \cdots - \max_{w \in W} R_j A_i^{r-1} w\right)$ of (19) is negative for any of its rows, Algorithm 2 terminates with $\mathbb{O}_\infty = \emptyset$. While not stated in Algorithm 2, fewer computations results if redundant inequalities are removed from $\mathbb{O}_{k+1}$ at the end of step (2). Properties of the $\mathbb{O}_\infty$ obtained from Algorithm 2 are stated next.

Theorem 4: Suppose system (1) satisfies assumptions (A1)-(A5) and $\mathbb{O}_k$ is generated based on Algorithm 2, such that $\mathbb{O}_k \neq \emptyset$ for all k. Then:
(i) $\mathbb{O}_k \subset \mathcal{X}$ and $\mathbb{O}_{k+1} \subseteq \mathbb{O}_k$ for all k.
(ii) $\mathbb{O}_\infty := \lim_{k \to \infty} \mathbb{O}_k$ exists, contains the origin in its interior and is finitely determined.
(iii) $\mathbb{O}_\infty$ is the largest CADDT-invariant set contained in $\mathcal{X}$.
(iv) For every $x(0) \in \mathbb{O}_\infty$, $x(t)$ converges to F_∞ for every admissible switching sequence in the sense that $d(x(t), F_\infty) \to 0$ as $t \to \infty$.

Remark 2: Suppose system (1) satisfies assumptions (A1)-(A5) and $\mathbb{O}_\infty \neq \emptyset$. Then, minimality of F_∞ implies that $F_\infty \subset \mathbb{O}_\infty \subset \mathcal{X}$. Conversely, (A6) implies the existence of at least one CADDT-invariant set in $\mathcal{X}$. Thus, $\mathbb{O}_\infty \neq \emptyset$ if and only if (A6) is satisfied.

Remark 3: It is important to highlight the precise meaning of result (iii) of the preceding theorem. As mentioned in *Remark* 1 and *Definition* 1, a sequence that violates the $t - t_{last} \geq \tau$ condition is not admissible, yet it may be a truncated subsequence of an admissible sequence. As Algorithm 2 is for system (1) under all admissible sequences, the presence of such inadmissible sequences results in $\mathbb{O}_\infty$ being CADDT-invariant and not *robustly positive invariant* in the conventional sense. This means that $x(0) \in \mathbb{O}_\infty$ implies $x(\tau) \in \mathbb{O}_\infty$ and $x(t) \in \mathcal{X}$ for all t. There is no requirement that $x(t) \in \mathbb{O}_\infty$ when $t = 1, \cdots, \tau - 1$. A set with such property is also known as a constraint admissible *returnable set*.

Remark 4: Setting $\tau = 1$ in algorithms 1 and 2 retrieves, respectively, the algorithms presented in [9] and [12] for computation of minimal and maximal disturbance invariant sets for arbitrary switching systems. Hence, algorithms 1 and 2 can be seen as a generalization of those obtained for arbitrary switching systems.

VI. Numerical Example

The numerical example is on a switching system with $\mathcal{A} = \{A_1, A_2\}$, $A_1 = \begin{bmatrix} 0.7 & 1 \\ 0 & 0.2 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0.8 & 0 \\ 0.4 & 0.6 \end{bmatrix}$, $\tau = 2$, The constraint and disturbance sets are $\mathcal{X} = \{x \in \mathbb{R}^2 : ||x||_\infty \leq 1\}$ and $W = \{w \in \mathbb{R}^2 : ||w||_\infty \leq 0.01\}$ respectively. It can be verified that the disturbance-free system is asymptotically stable with any dwell-time $\tau \geq 2$. Equivalently, this means that the system is unstable under arbitrary switching and existing computational techniques [9], [11], [12] for arbitrary switched systems cannot be used. With $\tau = 2$, both the minimal and maximal CADDT-invariant sets are computed for this system and are shown in Figure 1. Two state trajectories of this system starting from $x(0) = \pm(0.794, 0.434)$ are also shown. The input sequence used is periodic with $t_{s_{k+1}} - t_{s_k} = 2$ for all $k \geq 0$ while the disturbance sequence is generated from a random uniform distribution over W. That the state moves out of $\mathbb{O}_\infty$ is clear but it comes back in 2 steps, however $x(t) \in \mathcal{X}$ for all times as described in *Remark* 3. It is also evident that the state trajectories converge to $\mathbb{F}_\infty$, as claimed in property (iv) of *Theorem* 4.

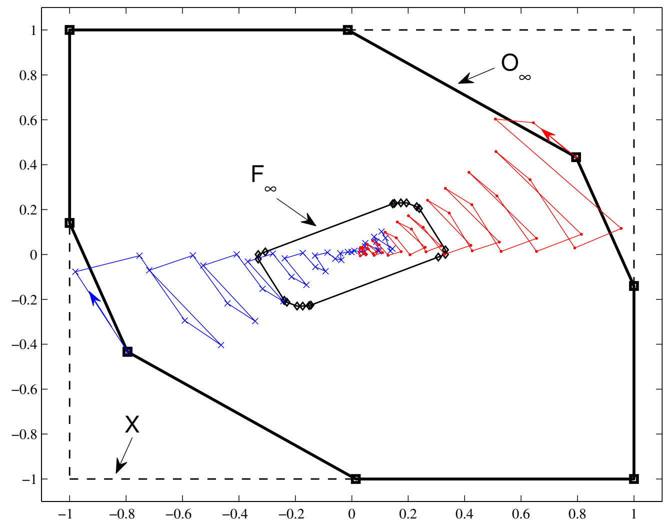

Fig. 1. Illustration of maximal and minimal CADDT-invariant sets

Computational results for this example are presented in Table 1. These results include the computational (wall-clock) time[1] in seconds, a measure of spectral radius of the system $\rho_{\max} := \max\{\rho(A_i^\tau) : i \in \mathcal{I}_N\}$, number of inequalities (#) that represents $\mathbb{O}_\infty / \mathbb{F}_\infty$ and the iteration (it) at which the algorithms converge.

		$\mathbb{O}_\infty$			$\mathbb{F}_\infty$		
τ	$\rho_{\max}$	time	#	it	time	#	it
2	0.2621	0.60	8	3	8.980	18	128

TABLE I
COMPUTATIONAL RESULTS

Characteristics of the complexity of Algorithm 2 are similar to that used for computing the maximal invariant set for standard linear system [14]. For example, much of the computational load is on the verification of $\mathbb{O}_{k+1} \equiv \mathbb{O}_k$; computational load increases when the dimension of the problem increases. Of course, the complexity also increases when $\rho_{\max}$ approaches 1, N increases, or τ increases.

[1]All the algorithms of this paper are implemented in Matlab 7 using multi-parametric programming toolbox solvers [15] and the computations are performed on a dual-core CPU with 3.2 GHz processor.

The computational effort for $\mathbb{F}_\infty$ is much higher than that for $\mathbb{O}_\infty$, due to the Minkowski sum and the convex-hull operations of (6) and (17) needed by Algorithm 1. Of course, the complexity also increases rapidly with the dimension of the system. This complexity issue is likely to remain unless significant improvement is made to the Minkowski sum and convex hull operations. For the time being, it may be desirable to develop efficient outer-approximation algorithms for $\mathbb{F}_\infty$ like those given by [16] and [11].

VII. Conclusions

Definitions of a DDT-invariant set and a CADDT-invariant set are given for discrete-time switching systems under dwell-time switching. The characterization of a DDT-invariant set with dwell-time τ corresponds to the satisfaction of t-step reachable/backward sets for $t = \tau, \cdots, 2\tau - 1$. This characterization allows for numerical algorithms for the computation of the minimal/maximal convex CADDT-invariant set.

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