2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Optimal Topology Design for Dynamic Networks

Ran Dai and Mehran Mesbahi

Abstract— **In this paper, we examine optimization-based methods for designing the network topology when the desired topology should have certain global variational properties, such as minimal power consumption for information exchange links, or supporting fast convergence of consensus-based distributed protocols. We first classify the optimization-based methodology for three distinct categories, namely, for construction of optimal non-geometric networks, time-invariant geometric networks, and time-varying geometric networks. We then proceed to propose optimization-based algorithms for each class of problems that aim to allocate the limited resources, e.g., communication ranges, in the most efficient way while achieving optimal performance. Examples and applications of the developed methodology are also examined.**

I. INTRODUCTION

Consensus based dynamic networks have emerged as a flexible framework for multi-agent information sharing when cooperative task is required, such as spacecraft formation control, mobile robot rendezvous, unmanned aerial vehicle (UAV) flocking, etc [1]–[3]. The communication between the agents which is defined by the topology of the network has prominent effect on the performance of the consensus dynamic system, e.g., the relationship between network connectivity and the convergence speed of the consensus protocol. In this paper, we address the problem of designing the topology of the underlying network supporting distributed operations in order to improve the performance of the protocol. Related works in the area of network design have been pursued predominately for allocating edge weights [4]–[6]. However, in the present work we focus on the more combinatorial problem of determining the topology of the (undirected) network, possibly time varying, that will lead to fast convergence for the protocol. For a non-geometric based network, if the existence and absence of a link between a pair of nodes is represented by binary values, then the off-diagonal entries of the adjacency matrix are all expressed as binary values as well.

Our objective is to determine these binary values in order to optimize the performance index when the number of edges has an upper bound. A related work in [7] has approached this problem using genetic-algorithm to determine the existence of the link between two nodes. Our approach to network topology design explored in this paper is inspired by the works that are based on Mixed-Integer Semidefinite Programming (MISDP) [8]. We then proceed to relax the positive semidefinite constraint in MISDP formulation to a set of quadratic constraints to further improve the algorithmic performance. The relaxation method is discussed in [9] and [10] where the semidefinite programming (SDP) problem is changed to smaller SDPs or second order cone programming problems to get faster algorithmic performance for solving them with similar solution accuracy. Motivated by these works, in this paper we consider formulating the original MISDP problem arising from the network topology deign problem as a Mixed Integer Quadratic Constraint Programming (MIQCP) problem which can be solved by the CPLEX software [11].

When relative positions between agents are considered in the construction of a geometric-based network, the information-exchange link is assumed to exist within specified range; it is further assumed that power strength drops quickly out of this range. This characteristic could be represented by a power function in order to approximately represent the on/off linkage relationship when searching for the maximum second smallest eigenvalues of the Laplacian to increase convergence speed of the protocol [12]. Similar works can also be found in [13] where a distributed method has been applied to solve the connectivity optimization problem. In this paper, we examine the situation where the position of the agents are given and static, while each agent needs to allocate its communication range in order to achieve fast protocol convergence. This problem is distinct from other resource allocation problems [14], [15], as the range selection for each agent is coupled with others when determining the optimal topology.

For time-varying geometry based networks, the Laplacian is assumed to be changing with time. In previous works, more attention has been paid to connectivity control in order to guarantee that the network is connected over the entire maneuver [16], [17]. Among these works, for exmaple, [17] has proposed designing a feedback control law which is dependent on the gradient of the potential function. In this direction, this work has successfully applied this approach for multiple scenarios when connectivity preservation is required for mobile networked systems. In this paper, our contribution concerns network optimization problems that aim to obtain network topologies leading to faster convergence of the underlying protocol while considering the allocation of communication ranges. Since the position of the agents can be obtained by integration of consensus dynamics for a given set of ranges, the topology design problem is shown to be reduced to a parameter optimization problem, which in turn, can be solved by a nonlinear programming (NLP) [18] solver.

The organization of the paper comprises of three sections addressing the problems discussed above. Starting from the

Ran Dai and Mehran Mesbahi are with the Department of Aeronautics and Astronautics, University of Washington, Seattle, WA. Emails: dairan@uw.edu and mesbahi@aa.washington.edu

background introduction for non-geometric network, we will describe the MISDP and MIQCP methods for this type of optimal topology design problem in §II. §III-IV will discuss the time-invariant and time-varying geometric optimal network design problems, respectively. We conclude the paper with a few remarks in §V.

II. NON-GEOMETRIC OPTIMAL NETWORK DESIGN

We consider a network $\mathcal{G} = (V, E)$ with n mobile agents, denoted as the nodes of the graph, with vertex set $V = \{1, 2, \ldots, n\}$ and edge set E consisting of two element subsets of V. The connection among the nodes in the undirected network $\mathcal{G} = (V, E)$ is expressed by the entries of the adjacency matrix with $[\mathcal{A}(\mathcal{G})]_{ij} = 1$ when $v_i, v_j \in E$ and $[\mathcal{A}(\mathcal{G})]_{ij} = 0$ otherwise. Since the adjacency matrix for a graph on n nodes, $\mathcal{A}(\mathcal{G})$, is symmetric, we use a set of binary variables comprised of $n(n-1)/2$ elements to determine off-diagonal entries of the $\mathcal{A}(\mathcal{G})$. We note that the diagonals are simply zeros. Using such a framework, we assign binary variable a_{ij} to represent the element $[\mathcal{A}(\mathcal{G})]_{ij}$ in $\mathcal{A}(\mathcal{G})$ with $a_{ij} = a_{ji}$, $(i \neq j)$ and a_{ii} is set to be zero. If the cost of constructing a graph with such adjacency is proportional to the cardinality of edge set E, then we can scale the cost as

$$C(E) = \sum_{\{v_i, v_j\} \in V} \frac{a_{ij}}{2}; \quad a_{ij} = a_{ji}, \ (i \neq j). \qquad (1)$$

The degree matrix $\Delta(\mathcal{G})$ of the graph can also be expressed in terms of the binary variables a_{ij} as $\Delta(\mathcal{G})_{ii} = \sum_{j=1}^{n} a_{ij}$ and $\Delta(\mathcal{G})_{ij} = 0$ $(i \neq j)$. Therefore, the Laplacian $\mathcal{L}(\mathcal{G}) = \Delta(\mathcal{G}) - \mathcal{A}(\mathcal{G})$ is completely determined by these binary variables.

Recall that when x_i denotes the state of dynamic agent i in the connected network $\mathcal{G}$, the consensus protocol of the overall system is represented by $\dot{x} = -\mathcal{L}(\mathcal{G})x$, which will drive each agent to the consensus set $\mathcal{C} = \{x \in \mathbb{R}^n \mid x_i = x_j, \forall\, v_i, v_j \in V\}$ by exchanging state information with connected agents in the specified network $\mathcal{G}$. We denote the eigenvalues of $\mathcal{L}(\mathcal{G})$ by $0 = \lambda_1(\mathcal{G}) \leq \lambda_2(\mathcal{G}) \leq \cdots \leq \lambda_n(\mathcal{G})$ and therefore have $\mathcal{L}(\mathcal{G}) \succeq 0$ with $\succeq$ denoting the positive semidefinite ordering. We quote the following well known Lemma [19] which can be used to determine the rate of convergence to the agreement set. It is known that for a connected graph $\mathcal{G}$, the (undirected) consensus protocol converges to the consensus set $\mathcal{C}$ with a rate of convergence that is dictated by $\lambda_2(\mathcal{G})$.

With $\lambda_2(\mathcal{G})$ as our objective function, our goal is to design the topology of the non-geometric based network, such that construction cost, $C(E)$, bounds the number of edges in the graph. More specifically, one can summarize the optimal network topology problem as

$$\max_{A(\mathcal{G})} \lambda_2(\mathcal{G}) \qquad (2)$$

$$\text{s.t.} \quad a_{ij} = a_{ji},\ a_{ij} \in \{0, 1\},\ \forall\, v_i, v_j \in V,\ i \neq j \qquad (3)$$
$$|E| \leq C(E)$$
$$\lambda_2(\mathcal{G}) > 0.$$

A. MISDP Method

Now we will make our first attempt to solve a relaxed form of the problem (2)-(3) as an MISDP [8] by transforming the constraints in (3) into linear matrix equalities or inequalities. We first present a positive semidefinite inequality constraint to relax the graph Laplacian to a diagonal matrix via the following proposition.

An orthogonal matrix $P = [p_1, p_2, \ldots, p_{n-1}, \mathbf{1}/\sqrt{n}]$ is constructed here with unit vectors p_i's chosen as $p_i^T \mathbf{1} = 0$ $(i = 1, 2, \ldots, n-1)$ and $p_i^T p_j = 0$ $(i \neq j)$. By the similarity transformation, we get

$$\mathcal{L} \sim P^T \mathcal{L}(\mathcal{G}) P, \qquad (4)$$

where symbol '$\sim$' indicates the similarity between two matrices.

Proposition 1: For a graph Laplacian $\mathcal{L}(\mathcal{G})$, if $\alpha \geq \lambda_2$, we have $\mathcal{L}(\mathcal{G}) + \alpha \mathbf{1}\mathbf{1}^T/n \succeq \lambda_2 I$.

Proof: It is easy to confirm that the matrix $\alpha \mathbf{1}\mathbf{1}^T/n$ has one eigenvalue equal to α with corresponding eigenvector of $\mathbf{1}$ and the remaining eigenvalues are all equal to zero. Let us assume that the eigenvectors of matrix $\alpha \mathbf{1}\mathbf{1}^T/n$ are denoted by $P = [p_1, p_2, \ldots, p_{n-1}, \mathbf{1}/\sqrt{n}]$ where all elements of P satisfies conditions stated above. From (4), one has

$$P^T \mathcal{L}(\mathcal{G}) P \sim P^T \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{pmatrix} P,$$

where $\lambda_1 = 0$ with eigenvector $\mathbf{1}$. We can now proceed to determine the eigenvalues of matrix $\mathcal{L}(\mathcal{G}) + \alpha \mathbf{1}\mathbf{1}^T/n$,

$$P^T(\mathcal{L}(\mathcal{G}) + \alpha \mathbf{1}\mathbf{1}^T/n) P \sim P^T \begin{pmatrix} \lambda_1 + \alpha & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{pmatrix} P.$$

As we assigned $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, $\mathcal{L}(\mathcal{G}) + \alpha \mathbf{1}\mathbf{1}^T/n \succeq \lambda_2 I$ is satisfied if and only if $\alpha \geq \lambda_2$. ■

By now, the optimization problem proposed in (2)-(3) has been transformed into an MISDP summarized in the following formulation:

$$\min_{A(\mathcal{G})} -\lambda_2(\mathcal{G}) \qquad (5)$$

$$s.t. \quad a_{ij} = a_{ji},\ a_{ij} \in \{0, 1\},\ \forall\, v_i, v_j \in V,\ i \neq j \qquad (6)$$
$$\textstyle\sum_{i,j=1, i \neq j}^{n} a_{ij}/2 \leq C_E$$
$$\lambda_2(\mathcal{G}) > 0$$
$$\mathcal{L}(\mathcal{G}) + \alpha \mathbf{1}\mathbf{1}^T/n \succeq \lambda_2(\mathcal{G}) I.$$

Although the MISDP solver will generate the optimal solution, the computational speed is generally slow, as the branch-and-bound algorithm is invoked in search of the integer variables. In order to improve the computational performance, the traditional method is to relax the binary variables to continuous variables on the unit interval $[0, 1]$ [20], [21]. Here we propose yet another method by relaxing the semidefinite constraint for the last inequality function in (6) to a group of quadratic constraints and formulate the problem as an MIQCP.

B. MIQCP Method

The MIQCP problem is defined as

$$\min_x f(x) = c^T x, \quad x_i \in \mathbb{I} \tag{7}$$

$$\begin{aligned} s.t. \quad & x_L \leq x \leq x_U & (8)\\ & b_L \leq Ax \leq b_U \\ & x^T Q(i)x + a(i)^T x \leq r(i),\ i = 1, \ldots, n_{qc}, \end{aligned}$$

where $x \in \mathcal{R}^n$ is the unknown state variables to be determined, $c \in \mathcal{R}^n$ is the vector of coefficients, $b_U \in \mathcal{R}^m$ and $b_L \in \mathcal{R}^m$ are the upper and lower bounds for the linear constraints, respectively. Matrix $A \in \mathcal{R}^{m \times n}$ specifies the linear (composite) equality or inequality constraints. $Q \in \mathcal{R}^{n \times n}$, $a \in \mathcal{R}^n$, and $r \in \mathcal{R}^1$ define the quadratic constraints and qc is the number of quadratic constraints. We first recall a property of positive-semidefinite matrices.

Lemma 2: Let M be a positive semidefinite matrix. Then every principal submatrix of M is positive semidefinite.

Accordingly, positive semidefiniteness of the matrix M can be partially represented by a group of 2×2 principal submatrices whose determinant are all nonnegative, expressed as

$$m_{ii} m_{jj} \geq m_{ij} m_{ji}, \quad \forall\, i, j \in \{0, \ldots, n\},\ i \neq j. \tag{9}$$

Since $L(\mathcal{G})$ in the semidefinite constraint is symmetric and the diagonal elements of $L(\mathcal{G})$ are included in the quadratic constraints, we assign the upper triangular elements of L as unknown integer variables of x in the MIQCP. Together with continuous variable λ_2, x in (7) is represented by $x = [l_{11}, l_{12}, \ldots, l_{1n}, l_{22}, \ldots, l_{2n}, \ldots, l_{ii}, \ldots, l_{in}, \ldots, l_{nn}, \lambda_2]^T$. The constraint in (6) can now be relaxed by a group of quadratic constraints according to (9) and expressed as

$$\begin{aligned} (l_{ii} + \alpha - \lambda_2)(l_{jj} + \alpha - \lambda_2) - (l_{ij} + \alpha)^2 \geq 0, \quad (10)\\ \forall\, i, j \in \{0, \ldots, n\},\ i \neq j. \end{aligned}$$

Since $\alpha \geq \lambda_2$, we assign $\alpha = \lambda_2$ to simplify (10) as

$$l_{ii} l_{jj} - (l_{ij} + \lambda_2)^2 \geq 0, \quad \forall\, i, j \in \{1, \ldots, n\},\ i \neq j. \tag{11}$$

We then summarize the MIQCP formulation of the non-geometric network design problem as

$$\min_{L(\mathcal{G})} -\lambda_2(\mathcal{G}) \tag{12}$$

$$\begin{aligned} s.t. \quad & l_{ij} \in \{0, -1\},\ v_i, v_j \in V,\ i \neq j & (13)\\ & l_{ii} = \textstyle\sum_{j=1, j\neq i}^{n} -l_{ij},\ i = \{1, \ldots, n\} \\ & \textstyle\sum_i^n l_{ii}/2 \leq C_E \\ & l_{ii} l_{jj} - (l_{ij} + \lambda_2)^2 \geq 0,\ \forall\, i, j \in \{1, \ldots, n\},\ i \neq j \\ & \lambda_2(\mathcal{G}) > 0. \end{aligned}$$

The quadratic constraints expressed in (11) are not equivalent to the semidefinite constraint in (6). However, it accelerates the computational speed by relaxing the original constraint. Thus the solution is not equivalent to the MISDP solution and becomes a suboptimal solution in general. We will address these aspects of the proposed relaxation procedure next.

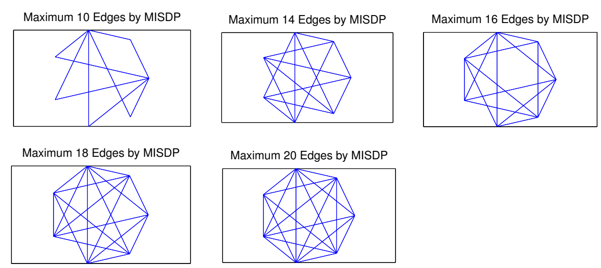
C. An Example

In this part, we will design the topology of a non-geometric network with a given number of agents and upper bound on the total number of edges using the approaches based on MISDP and MIQCP formulation discussed above. In Table I, we provide the results for the corresponding values of λ_2 from the designed networks. Additionally, the computation time T_c from both methods are depicted in order to compare their computational performance for the simulation run on a Lenovo X201 laptop with intel i5 CPU and 4GB RAM. Furthermore, the network topology designs from both methods are shown in Fig. 1. After comparison of the results from MISDP and MIQCP methods in Table I, we can make the statement that the computational performance is improved in the MIQCP method at the expense of optimality properties of the solution in certain cases. This is particularly the case when the maximum number of edges is small compared with the total number of possible edges in the network. Both methods are feasible in small scale network topology design; the preference of accuracy vs. computation time determines which method might be more desirable in practice.

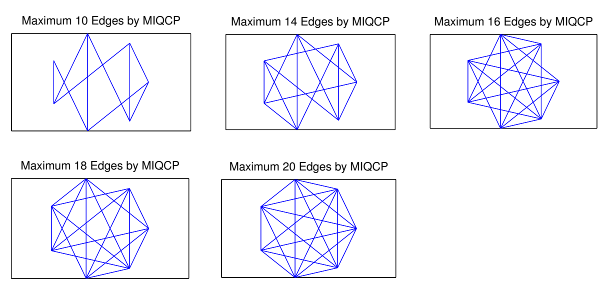
TABLE I

EXAMPLE FOR NON-GEOMETRIC NETWORK DESIGN WITH 7 AGENTS

Maximum Edge No.	10	14	16	18	20
λ_2 from MISDP	2.00	3.20	4.00	5.00	5.00
T_c (sec) from MISDP	880.46	893.04	213.35	4.48	5.60
λ_2 from MIQCP	1.38	2.39	4.00	5.00	5.00
T_c (sec) from MIQCP	0.97	0.33	0.53	0.14	0.31



(a) Network topology from MISDP method.



(b) Network topology from MIQCP method.

Fig. 1. Network topology of 7 agents with maximum edge number from table I

III. TIME-INVARIANT GEOMETRIC OPTIMAL NETWORK DESIGN

A. Problem Formulation and Method

The above discussion assumes the geometric location of each agent has no effect on the communication links. However, in some situation, the strength of the communication link is determined by the distance between the agents and will fade with the increase of the distance. There are a few functions [12] to present this relationship between the distance and the strength of the communication link. Here we use

$$f(d) = 1/(1 + e^{\alpha(d-\rho)}), \tag{14}$$

where $\alpha = \frac{2}{\rho_2-\rho_1}\log(\frac{1-\epsilon}{\epsilon})$ and $\rho = \frac{\rho_1+\rho_2}{2}$. A plot of (14) is shown in Fig. (2).

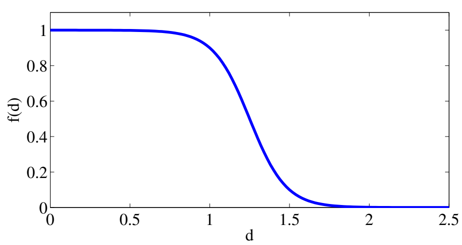


Fig. 2. Illustration figure of function $f(d)$ with $\rho_1 = 1$, $\rho_2 = 1.5$ and $\epsilon = 0.1$.

For every agent i, we assume the power of communication is denoted by the radio range ρ_{1i} and the total power of the network is constrained by $\sum_{i=1}^{n}\rho_{1i} \le p_{max}$. The communication signal is strong within the distance of ρ_1 and decreases quickly to zero when the distance reaches ρ_2. For a set of randomly scattered agents, assigning the power strength of each agent to compose a favorable topology according to their relative location is supposed to possess better performance than a network without elaborate design. Now the problem is how to allocate the power of communication for a network with given location of all the agents so that λ_2 of $L(\mathcal{G})$ is maximized. We formulate this problem as

$$\min_{\rho_1} -\lambda_2(\mathcal{G}),\ \rho_1 = \{\rho_{11}, \dots, \rho_{1n}\} \tag{15}$$
$$s.t.\, a_{ij} = f(\max(\rho_{1i}, \rho_{1j})), \forall i,j \in \{1,\dots,n\}, i \neq j \tag{16}$$
$$\rho_{min} \le \rho_{1i} \le \rho_{max}, \quad \forall i \in \{1,\dots,n\}$$
$$\textstyle\sum_{i=1}^{n}\rho_{1i} \le p_{max}$$
$$\mathcal{L}(\mathcal{G}) + \alpha\mathbf{1}\mathbf{1}^T/n \succeq \lambda_2(\mathcal{G})I$$
$$\lambda_2(\mathcal{G}) > 0.$$

Since $\rho_1 = \{\rho_{11}, \dots, \rho_{1n}\}$ are continuous unknown variables, the above problem can be solved by the iterative algorithm [12] using the SDP solver [8] to maximize the second smallest eigenvalue of $\mathcal{L}(\mathcal{G})$ at each step. One issue of concern is the behavior of the power strength function, determined by the maximum range between agent i and j. We write out the maximum function of (16) as

$$\rho_{mij} = \max(\rho_{1i}, \rho_{1j}) = \frac{\rho_{1i}+\rho_{1j}}{2} + |\frac{\rho_{1i}-\rho_{1j}}{2}| \tag{17}$$

and then for each iterative step k, the elements in the adjacent matrix a_{ij} are updated by linearization of (14) as

$$a_{ij}(k+1) = a_{ij}(k) + \alpha e^{\alpha(d-\rho(k))} f^2(\rho_{mij}(k))$$
$$(\tfrac{\rho_{1i}(k+1)+\rho_{1j}(k+1)}{2} + s(k)\tfrac{\rho_{1i}(k+1)-\rho_{1j}(k+1)}{2} - \rho_{mij}(k)), \tag{18}$$

where $s(k) = \text{sign}(\rho_{1i}(k) - \rho_{1j}(k))$. The approach starts with an initial guess of ρ_1 and the solution process for problem (15)-(16) is repeated and updated by (18) at each step until it convergence to a local optimal solution.

B. An Example

Here we simulate an example to test the proposed algorithm for a time-invariant geometric network design. We have six agents with fixed locations in a two-dimensional space and initial communication ranges specified in Table II. The total power range is constrained by $p_{max} = 16.5$. We also assume $\rho_2 = \rho_1 + 0.5$, which means that the communication signal strength drops close to zero when distance increases more than 0.5 of the designed range ρ_1. The other parameters are set as $\alpha = \frac{1}{\rho_2-\rho_1}\log(\frac{1-\epsilon}{\epsilon})$ and $\epsilon = 0.1$. Based on these assumptions, λ_2 of the initial network with average communication range is 2.69. However, from the simulation result, λ_2 of the designed network increases to the value of 3.65.

TABLE II
EXAMPLE FOR TIME-INVARIANT GEOMETRIC NETWORK DESIGN

Agent No.	1	2	3	4	5	6
x	1.60	3.20	2.72	4.80	4.48	5.60
y	1.60	0.80	4.48	4.00	1.60	3.20
Initial ρ_1	2.75	2.75	2.75	2.75	2.75	2.75
Maximum ρ_1	4.00	4.00	4.00	4.00	4.00	4.00
Minimum ρ_1	2.00	2.00	2.00	2.00	2.00	2.00
Optimal ρ_1	3.81	3.53	2.96	2.20	2.00	2.00

IV. TIME-VARYING GEOMETRIC OPTIMAL NETWORK DESIGN

A. Problem Formulation and Method

In the time-invariant geometric network design, it was assumed the locations (states) of the agents are fixed. In certain scenarios, e.g., rendezvous, formation flight and flocking, the locations of agents are included in the dynamic states and are changing with time. For these cases, the graph Laplacian matrix is also changing with time since the distances between the agents that determine the entries of the Laplacian are updated at each sampling time. In such a setup, the elements of the adjacency matrix $\mathcal{A}(\mathcal{G})$ are determined by

$$a_{ij} = f(\max(\rho_{1i}, \rho_{1j}), x_i, x_j), \quad \forall i,j \in \{1,\dots,n\}, i \neq j. \tag{19}$$

For the consensus-based dynamics, each agent is treated as a single integrator whose states will converge to the constant agreement offset τ. If the initial and final relative states of the agents are given as vectors x_0 and x_f, respectively, all agents will converge to $x = x_f + \tau$ according to the consensus dynamics. The relative states describe the configuration of the agents in space, i.e., the rendezvous task requires $x_f(i) =$

$x_f(j),\ \forall\, i,j \in \{1,\ldots,n\},\ i \neq j$, while in formation flight, x_f represents the desired shape of the final formation. If we assign $x' = x - x_f$, then the consensus dynamics of the network is represented as

$$\dot{x}' = -\mathcal{L}(\rho_1, (x' + x_f))x'. \quad (20)$$

When x' reaches the consensus offset τ, the required rotationally and translationally invariant configuration, is in the desired relative displacement t_f. Since the topology of the network is dynamic, the objective function is not only to optimize the network topology at a specific time, but also for the entire time interval of interest, from initial state to the final desired target state. For this approach, the system dynamics are required to be satisfied as well as other constraints over the concerned time interval. Correspondingly, the problem formulation assumes the form

$$\min_{\rho_1} \int_0^{t_f} dt,\ \rho_1 = \{\rho_{1_1}, \ldots, \rho_{1_n}\} \quad (21)$$

$$\begin{aligned} s.t. \quad & x(0) = x_0, \quad x(t_f) = x_f + \tau \quad (22) \\ & \rho_{min} \le \rho_{1_i} \le \rho_{max}, \quad \forall\, i \in \{1,\ldots,n\} \\ & \textstyle\sum_{i=1}^{n} \rho_{1_i} \le p_{max} \\ & \dot{x}' = -\mathcal{L}(\rho_1, (x' + x_f))x',\ x' = x - x_f, \end{aligned}$$

where $\rho_{1_i}\ i \in \{1,\ldots,n\}$ is the radio range to be determined with upper bound ρ_{max} and lower bound ρ_{min}.

We may consider the above described problem as path planning problem which requires optimal resource allocation so that the agents in the system will reach the desired targets in minimum time. The algebraic connectivity, λ_2, is evaluated as a objective function in the non-geometric and time-invariant geometric cases. Although the other eigenvalues also contribute to the rate of convergence, λ_2 plays the prominent role. Minimizing the time from initial states to final states with consensus dynamics includes the contribution of all eigenvalues of the changing Laplacian during the entire time period. To solve this two point boundary optimization problem with highly nonlinear dynamics, we will apply the NLP method to search for the optimization solution.

Combined with the direct collocation and NLP method [18], the network design procedure is transformed into a parameter optimization problem. Assume that the overall time period from initial states to the target is t_f and that this time interval has been divided into h segments as $0 = t_1 < t_2 < \ldots < t_{h+1} = t_f$. Then, the system dynamics are discretized into a series of separate nodes represented by state variables. All of these variables, together with the unknown parameters ρ_1 and t_f, compose the NLP variables as

$$x = [x_1, x_2, \ldots, x_i, \ldots, x_{h+1}, \rho_1, t_f], \quad (23)$$

where x at each node includes the state variables of every agent. The trapezoidal integration rule is now applied to enforce the state variables and unknown parameters at these discrete points, approximately satisfying the system dynamics

$$d_i = x_{i+1} - x_i - \frac{\Delta t_i}{2}(\dot{x}_i + \dot{x}_{i+1}),\ i \in \{1,\ldots,h\}, \quad (24)$$

where d_i are called the defect vector, which are expected to be zero in order to reproduce the system dynamics. $\Delta t_i = t_{i+1} - t_i$ is the time interval between two adjacent nodes, $\dot{x}_i$ and $\dot{x}_{i+1}$ are the system first-order derivative functions at node i and $i+1$, which are expressed in last equation of (22) with respect to x_i and x_{i+1}, individually.

The NLP solver used for the problem (21)-(22) is SNOPT [22]. SNOPT solves optimization problems with nonlinear objective and constraints. According to the above discussion, all NLP variables are listed in (23) and the performance index has been evaluated as (21). The nonlinear constraints, including the "defect" vector constraints, are set as $d_i = 0$, $i \in \{1,\ldots,h\}$. In the ideal consensus condition one has $\lim_{t\to\infty} x = x_f + \tau$. However, the exact value of τ is not known ahead of time. In order to choose a judicious stopping criteria, we assume that the agents will reach the target when $\text{std}(x(h+1)) \le \epsilon$, where 'std' is the standard deviation function of the states at the final boundary points and ϵ is the termination threshold. Finally, the problem described in (21)-(22) is transformed into an NLP summarized as follows:

$$\min_x t_f,\ x = [x_1, x_2, \ldots, x_i, \ldots, x_{h+1}, \rho_1, t_f] \quad (25)$$

$$\begin{aligned} s.t. \quad & x_1 = x_0 \quad (26) \\ & \text{std}(x(h+1)) \le \epsilon \\ & \rho_{min} \le \rho_{1_i} \le \rho_{max}, \quad i \in \{1,\ldots,n\} \\ & \textstyle\sum_{i=1}^{n} \rho_{1_i} \le p_{max} \\ & d_i = c_U = c_L = 0, \quad i \in \{1,\ldots,h\}. \end{aligned}$$

B. An Example

The example used here for the time-varying geometric network design is the formation control problem. We consider six agents with initial and final positions given in Table III, as well as the initial communication ranges, maximum and minimum bounds on the individual communication ranges. We assume the total power range is constrained by $p_{max} = 16.5$ and $\rho_2 = \rho_1 + 0.5$ and set $\epsilon = 0.01$. In the corresponding simulations, 20 discrete nodes are selected with equal time interval between adjacent nodes to discretize the trajectory from initial states to the final target. More nodes states could be used to increase the solution accuracy at the expense of more computation time. We then constructed the problem as formulated in (25)-(26). By running the NLP solver, we obtain the optimal communication range allocations as listed in Table III. The resulting agents' trajectories composed by the discretized points are illustrated in Fig. (3) with the trajectory

TABLE III

EXAMPLE FOR TIME-VARIANT GEOMETRIC NETWORK DESIGN

Agent No.	1	2	3	4	5	6
Initial x	0.00	1.00	2.00	3.00	4.00	5.00
Initial y	0.00	0.00	0.00	0.00	0.00	0.00
Final x	1.50	1.50	2.50	2.50	3.50	3.50
Final y	1.00	-1.00	2.00	-2.00	1.00	-1.00
Initial ρ_1	2.75	2.75	2.75	2.75	2.75	2.75
Maximum ρ_1	4.00	4.00	4.00	4.00	4.00	4.00
Minimum ρ_1	2.00	2.00	2.00	2.00	2.00	2.00
Optimal ρ_1	3.85	2.73	2.62	2.00	2.00	3.29

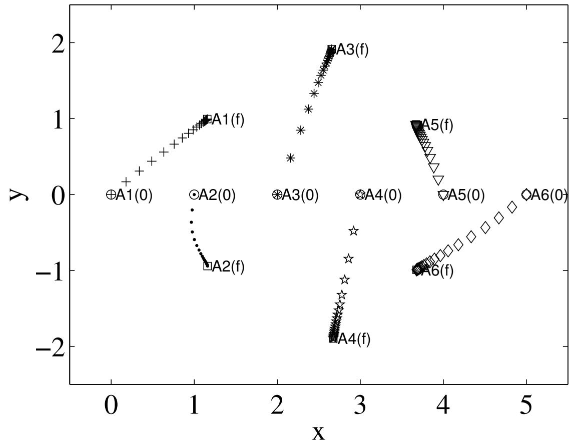
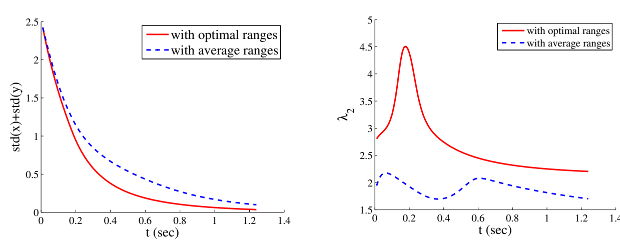


Fig. 3. Simulation result of agents trajectories from initial points to targets

of every agent shown by different markers. The agents' initial positions are labeled by circles and target points as squares with index number $Ai(0)$ and $Ai(f)$, respectively. In order to demonstrate the performance improvement by the proposed algorithms, we compare the time history of the standard deviation for both x and y coordinates using optimal range allocation and average range allocation in Fig. (4-a). The simulation results indicates that the values of $std(x)+std(y)$ reaches below than 0.01 in 0.867 seconds which benchmarks the time required by the average range allocation to reach the same consensus points. Finally, we also compare the time history of λ_2 for the dynamic Laplacian matrix using optimal range allocations and average range allocations in Fig. (4-b). We note that the optimal values of λ_2 for the graph Laplacian using the optimal range allocation method is much higher than the average range allocation and that the graph remains connected over the entire time interval.



(a) Time history of standard deviation of coordinates

(b) Time history of λ_2

Fig. 4. Time history of standard deviation of coordinates and λ_2 with optimal ranges and average ranges

V. CONCLUSIONS

This paper presented the optimal network topology design for consensus based systems with a variety of optimal based methods. We categorized the topology design problem into three categories and formulated the corresponding optimization formulation for each case. The time-invariant non-geometry and geometry based networks were then analyzed in order to design the optimal topology for time-invariant networks. We also considered the situations when the state of the agents is dynamic and considered the optimal allocation of resources for this class of time-varying geometric-based networks. The proposed algorithms are of importance in many situations of practical interest, such as constructing a network for high performance UAV flocking and distributed estimation for sensor networks.

REFERENCES

[1] T. Balch, R. C. Arkin, and S. Member, "Behavior-based formation control for multi-robot teams," *IEEE Transactions on Robotics and Automation*, vol. 14, pp. 926–939, 1999.

[2] A. W. Beard, J. Lawton, and F. Y. Hadaegh, "A coordination architecture for spacecraft formation control," *IEEE Transactions on Control Systems Technology*, vol. 9, pp. 777–790, 2001.

[3] R. Olfati-saber, "Flocking for multi-agent dynamic systems: Algorithms and theory," *IEEE Transactions on Automatic Control*, vol. 51, pp. 401–420, 2006.

[4] L. Xiao and S. Boyd, "Fast linear iterations for distributed averaging," *Systems and Control Letters*, vol. 53, pp. 65–78, 2003.

[5] Y. Shafi, M. Arcak, and L. El Ghaoui, "Designing node and edge weights of a graph to meet laplacian eigenvalue constraints," in *Prof. Allerton Conference*, 2010.

[6] A. Ghosh, S. Boyd, and A. Saberi, "Minimizing Effective Resistance of a Graph," *SIAM Review*, vol. 50, no. 1, pp. 37–66, 2008.

[7] Z. Wu and R. Wang, "The Consensus in Multi-Agent System with Speed-Optimized Network," *International Journal of Modern Physics B*, vol. 23, no. 10, pp. 2339–2348, 2009.

[8] J. Lfberg, "Yalmip : A toolbox for modeling and optimization in MATLAB," in *Proceedings of the CACSD Conference*, Taipei, Taiwan, 2004. [Online]. Available: http://users.isy.liu.se/johanl/yalmip

[9] Y. Y. Z. Wang, S. Zheng and S. Boyd, "Further Relaxations of the Semidefinite Programming Approach to Sensor Network Localization," *SIAM Journal on Optimization*, vol. 19, no. 2, p. 655673, 2008.

[10] S. Kim and M. Kojima, "Exact Solutions of Some Nonconvex Quadratic Optimization Problems via SDP and SOCP Relaxations," *Computational Optimization and Applications*, vol. 26, no. 2, pp. 143–154, 2003.

[11] (2010) Ibm ilog cplex optimization studio. [Online]. Available: http://www-01.ibm.com/software/integration/optimization/cplex-optimization-studio/

[12] Y. Kim and M. Mesbahi, "On Maximizing the Second Smallest Eigenvalue of a State-Dependent Graph Laplacian," *IEEE Transactions on Automatic Control*, vol. 51, no. 1, pp. 116–120, Jan. 2006.

[13] M. De Gennaro and A. Jadbabaie, "Decentralized control of connectivity for multi-agent systems," in *Decision and Control, 45th IEEE Conference on*, San Diego, CA, 2006, pp. 3628 – 3633.

[14] L. Xiao, M. Johansson, and S. Boyd, "Simultaneous routing and resource allocation via dual decomposition," *IEEE Transactions on Communications*, vol. 52, pp. 1136–1144, 2002.

[15] A. Goldsmith and S. Wicker, "Design challenges for energy-constrained ad hoc wireless networks," *IEEE Wireless Communications Magazine*, vol. 9, pp. 8–27, 2002.

[16] D. P. Spanos and R. M. Murray, "Robust connectivity of networked vehicles," in *Decision and Control, 43th IEEE Conference on*, Bahamas, 2004, pp. 2893 – 2898.

[17] M. M. Zavlanos and G. J. Pappas, "Potential fields for maintaining connectivity of mobile networks," *IEEE Transactions on Robotics*, vol. 23, no. 4, pp. 812–816, 2007.

[18] C. R. Hargraves and S. W. Paris, "Direct trajectory optimization using nonlinear programming and collocation," *J. of Guidance, Control and Dynamics*, vol. 10, pp. 338–342, July-August 1987.

[19] M. Mesbahi and M. Egerstedt, *Graph Theoretic Methods in Multiagent Networks*. Princeton University Press, 2010.

[20] S. Joshi and S. Boyd, "Sensor selection via convex optimization," *Trans. Sig. Proc.*, vol. 57, pp. 451–462, February 2009.

[21] D. Zelazo and M. Mesbahi, "$\mathcal{H}_\infty$ performance and robust topology design of relative sensing networks," in *Proc. American Control Conf. (ACC)*, 2010, pp. 4474–4479.

[22] A. G. K. Holmstrom and M. M. Edvall. (2005) User's guide for tomlab/snopt. [Online]. Available: http://tomlab.biz