

Discrete-time switched linear system with constraints: characterization and computation of invariant sets under dwell-time consideration

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Abstract—This paper introduces the concepts of Dwell-Time invariance (DT-invariance) and maximal constraint admissible DT-invariant set for discrete-time switching systems under dwell-time switching. Main contributions of this paper include a characterization for DT-invariance; a numerical computation of the maximal CADT-invariant set; a necessary and sufficient condition for asymptotic stability of the origin of the switching systems under dwell time switching and an algorithm for the computation of the minimal dwell time needed for stability.

I. INTRODUCTION

This paper considers the following constrained discrete-time switched linear system:

$$x(t+1) = A_{\sigma(t)} x(t), \quad (1a)$$

$$x(t) \in \mathcal{X}, \quad \forall t \in \mathbb{Z}^+ \quad (1b)$$

where $x(t) \in \mathbb{R}^n$ is the state variable and $\sigma(t) : \mathbb{Z}^+ \rightarrow \mathcal{I}_N := \{1, \dots, N\}$ is a time-dependent switching signal that indicates the current active mode of the system among N possible modes in $\mathcal{A} := \{A_1, \dots, A_N\}$. The constraint set $\mathcal{X} \subset \mathbb{R}^n$ models physical state constraints imposed on the system, including those arising from the actuator via some appropriate state feedback if (1) is seen as a feedback system.

The study of such a system is quite active in the past decade. Most of the literature [1], [2], [3], [4] is concern with conditions that ensure stability of the system when $\sigma(\cdot)$ is an arbitrary switching function while others [5], [6], [7], [8] consider designing the appropriate switching functions that ensure stability. With a few notable exceptions [9], [10] past literature does not consider the presence of constraints. When constraints are present, one major focus of research is the characterization of invariant sets that are constraint admissible [11]. The existence of such invariant sets for system (1) is predicated on it being stable. Hence, studies of such sets often assume that $A_i, i \in \mathcal{I}_N$ is stable, which is a necessary condition for the stability of the origin of (1) under arbitrary switching. Additional conditions are required. The most common of these are those based on Lyapunov function consideration. For example, the origin of system (1) is stable under arbitrary switching upon the existence of a common quadratic Lyapunov function [2], [12], [13], [4], (pairwise) switched Lyapunov functions [3], multiple Lyapunov functions [14], [15], [16], [7], composite quadratic functions [17], [18] or polyhedral Lyapunov functions [10].

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Another condition for stability is that based on dwell-time consideration. When all A_i is stable, stability of the origin can be ensured if the time duration spent in each subsystem is sufficiently long [2], [8]. Upper bounds of the minimal dwell-time needed have also appeared [8], [19], [20], [21]

This work is concerned with the characterization and computation of invariant sets for system (1) when $\sigma(\cdot)$ is an admissible switching function that respects the dwell-time consideration. In the limiting case where the dwell time is one sample period, $\sigma(\cdot)$ becomes an arbitrary switching function, and the corresponding invariant sets and their computations have appeared in the literature [9]. Hence, this work can also be seen as a generalization of those obtained for arbitrary switching systems. Other contributions of this work include: connection between stability of dwell-time switching systems and the stability of an associated arbitrary switching system, a necessary and sufficient stability condition for dwell-time switching systems, and a procedure that determines the minimal dwell time needed to ensure stability of the origin of system (1).

The rest of this paper is organized as follows. This section ends with a description of the notations used. Section II reviews some standard terminology and results for switching system. Section III shows the main result on the characterization of the invariant set for system (1) and its properties. Section IV describes a procedure for determination of the minimum dwell time needed for a given system. Sections V and VI contain, respectively, numerical examples and conclusions. All proofs for the results stated in this paper can be found in forthcoming paper [22].

The following standard notations are used. \mathbb{Z}^+ is the set of non-negative integers. Given a matrix $A \in \mathbb{R}^{n \times n}$ and a vector $b \in \mathbb{R}^n$, A_j and b_k are the corresponding j -th row and the k -th element respectively. The floor function, $\lfloor a \rfloor$, is the largest integer that is less than scalar a . Positive definite (semi-definite) matrix, $Q \in \mathbb{R}^{n \times n}$, is indicated by $Q \succ 0$ ($\succeq 0$). The p -norm of a vector or a matrix is $\|\cdot\|_p, p = 1, 2, \infty$ with $\|\cdot\|$ refers to the 2-norm. Given a $P \succ 0$, $\|x\|_P^2 = x^T P x$ and $\mathcal{E}(P) := \{x : \|x\|_P^2 \leq 1\}$. Suppose $\alpha > 0$, $\mathcal{X} \subset \mathbb{R}^n$ is a compact set that contains 0 in its interior, then $\alpha\mathcal{X} := \{\alpha x : x \in \mathcal{X}\}$. Boldface **1** indicates the vector of all 1s. Other notations are introduced when needed.

II. PRELIMINARIES

This section begins with a review of the definitions of switching time, dwell time and admissible switching sequence/function. Suppose $t_{s_0}, t_{s_1}, \dots, t_{s_k}, \dots$ are the switching instants of (1) with $t_{s_0} = 0$ and $t_{s_k} < t_{s_{k+1}}$. By

definition, this means that $\sigma(t_{s_k}) \neq \sigma(t_{s_{k+1}})$ and $\sigma(t_{s_k}) = \sigma(t_{s_k+1}) = \dots = \sigma(t_{s_{k+1}} - 1)$ for all $k \in \mathbb{Z}^+$.

Definition 1: An admissible switching sequence of system (1), $\mathcal{S}_\tau(t) = \{\sigma(t-1), \dots, \sigma(1), \sigma(0)\}$, with switching instants $t_{s_0}, t_{s_1}, \dots, t_{s_k}, \dots$ has a dwell time of τ means that $t_{s_{k+1}} - t_{s_k} \geq \tau$ for all $k \in \mathbb{Z}^+$. In addition, suppose t_{last} is the last switching time for an admissible sequence $\mathcal{S}_\tau(t)$, then $t - t_{last} \geq \tau$.

Remark 1: As defined, the dwell time condition corresponds to the minimal duration of stay in each mode required of the system. The last condition in *definition 1* requires further qualification. Suppose $\mathcal{A} = \{A_1, A_2\}$ and $\tau = 3$ then $\mathcal{S}_3^a(6) = \{1, 1, 1, 2, 2, 2\}$ is an admissible sequence. However, $\mathcal{S}_3^b(6) = \{1, 1, 2, 2, 2, 2\}$ is not an admissible sequence because $t - t_{last} < 3$ and the dwell time consideration may be violated if $\sigma(6) = 2$. On the other hand, if $\sigma(6) = 1$ means $\mathcal{S}_3^b(6)$ is a truncated subsequence of an admissible sequence. This is a key point that distinguishes systems under dwell time consideration and under arbitrary switching. Following the same reasoning, $\mathcal{S}_\tau(t)$ for $t < \tau$ is also not meaningful.

Throughout this paper, system (1) is assumed to satisfy the following assumptions:

(A1) The spectral radius of each individual subsystem $A_i, i \in \mathcal{I}_N$ is less than 1;

(A2) The constraint set \mathcal{X} is a polytope represented by $\mathcal{X} = \{x : Rx \leq \mathbf{1}\}$ for some appropriate matrix $R \in \mathbb{R}^{q \times n}$;

(A3) (A_i, R) are observable for all $i \in \mathcal{I}_N$.

(A4) A value of $\tau \geq 1$ has been identified such that the origin of the unconstrained switched system (1) with dwell time τ is asymptotically stable.

Assumption (A1) defines the family of systems considered in this work and is a reasonable requirement. The polyhedral assumption of (A2) is made to facilitate numerical computations of the invariant set of (1). If (A3) is not satisfied, then system (1) can be reformulated to consider only the observable subsystem of A_i . Assumption (A4) follows from (A1) and is made out of convenience. For example, a conservative dwell time can be estimated under (A1) [8]. More exactly, it is known that for each $i \in \mathcal{I}_N$ there exist scalars h_i and λ_i with $|\lambda_i| \in (0, 1)$ such that $\|A_i^k\| < h_i |\lambda_i|^k$ for all $k \in \mathbb{Z}^+$. This means that

$$\begin{aligned} \|x(t)\| &= \|A_{\sigma(t-1)} A_{\sigma(t-2)} \cdots A_{\sigma(1)} A_{\sigma(0)} x(0)\| \\ &= \|A_{\sigma(t_{s_k})}^{t-t_{s_k}} A_{\sigma(t_{s_{k-1}})}^{t_{s_k}-t_{s_{k-1}}} \cdots A_{\sigma(t_{s_0})}^{t_{s_1}-t_{s_0}} x(0)\| \end{aligned} \quad (2)$$

$$\begin{aligned} &< h_{\sigma(t_{s_k})} |\lambda_{\sigma(t_{s_k})}|^{t-t_{s_k}} \cdots h_{\sigma(t_{s_0})} |\lambda_{\sigma(t_{s_0})}|^{t_{s_1}-t_{s_0}} \|x(0)\| \\ &< (h_m^{1/\tau})^t \lambda_m^t \|x(0)\| = (\bar{h} \lambda_m)^t \|x(0)\| \end{aligned} \quad (3)$$

where $h_m := \max\{h_i : i \in \mathcal{I}_N\}$, $\lambda_m := \max\{|\lambda_i| : i \in \mathcal{I}_N\}$, $\bar{h} := h_m^{1/\tau}$ and τ is the smallest positive integer such that $\bar{h} < \frac{1}{\lambda_m}$. The last inequality above ensures that $x(t) \rightarrow 0$ for all admissible switching functions.

III. MAIN RESULTS

This section begins with several definitions needed to precisely state the invariance condition for system with dwell time consideration. For notational convenience, $A_{\mathcal{S}_\tau(t)}$ refers to the product $\prod_{r=0}^{t-1} A_{\sigma(r)}$ with the admissible sequence

$\mathcal{S}_\tau(t) = \{\sigma(t-1), \dots, \sigma(0)\}$ and may be expressed in the form of (2).

Definition 2: A set $\Omega \subset \mathbb{R}^n$ is t -step invariant with respect to $x(t+1) = Ax(k)$ if $x \in \Omega$ implies $A^t x \in \Omega$.

Definition 3: A set $\Omega \subset \mathbb{R}^n$ is said to be DT-invariant (Dwell-Time invariant) with respect to system (1a) with a dwell time τ if $x \in \Omega$ implies $A_{\mathcal{S}_\tau(t)} x \in \Omega$ for all admissible switching sequences $\mathcal{S}_\tau(t)$ and for all time t .

While stating the requirement of DT-invariance for system (1), the above definition is of limited practical usefulness since $A_{\mathcal{S}_\tau(t)} x \in \Omega$ has to be satisfied by an infinite number of admissible sequences for all time t . The next theorem shows how the infinite sequences can be avoided.

Theorem 1: Suppose (A1) and (A4) are satisfied. A set $\Omega \subset \mathbb{R}^n$ is DT-invariant for system (1a) with dwell time τ if and only if it is t -step invariant for all $t = \tau, \tau+1, \dots, 2\tau-1$ and for all $A_i \in \mathcal{A}$.

Proof: (i) (\Rightarrow): The solution of (1) under an admissible switching function at time t is $x(t) = A_{\mathcal{S}_\tau(t)} x_0$ where

$$A_{\mathcal{S}_\tau(t)} = \cdots A_{i_\ell}^{k_\ell} \cdots A_{i_1}^{k_1} A_{i_0}^{k_0} \quad (4)$$

for some appropriate switching sequence $\mathcal{S}_\tau(t) = \{i_\ell, i_\ell, \dots, i_\ell, i_{\ell-1}, \dots, i_{\ell-1}, i_{\ell-2}, \dots, i_0\}$ where $i_j \in \mathcal{I}_N$ and $k_j := t_{s_{j+1}} - t_{s_j}, j = 0, 1, \dots, \ell$ being the corresponding duration times in each mode. Due to the dwell time requirement, each $k_j \geq \tau$. Without loss of generality, consider any of the A_i^k on the right hand side of (4). This term can be decomposed into a product of matrices involving A_i^τ and one matrix from $\{A_i^\tau, A_i^{\tau+1}, \dots, A_i^{2\tau-1}\}$. To see this, let $q = \lfloor \frac{k-\tau}{\tau} \rfloor$ and

$$A_i^k = (A_i^\tau)^q A_i^{k-q\tau} \quad (5)$$

Here, the superscript $k - q\tau$ of the last term corresponds to the remainder of $k - \tau$ when divided by τ and hence, assumes a value from $\{\tau, \tau+1, \dots, 2\tau-1\}$. Consider the rightmost term of (5). Since Ω is t -step invariant for all $\tau \leq t \leq 2\tau-1$ and for all $A_i \in \mathcal{A}$, $A_{i_0}^{k_0 - q_0 \tau} x_0 \in \Omega$ for any $x_0 \in \Omega$. Similarly, $(A_{i_0}^\tau)^{q_0} A_{i_0}^{k_0 - q_0 \tau} x_0 \in \Omega$ as Ω is t -step invariant when $t = \tau$ under (A1) and (A4). Repeating this process for the rest of the terms in (4) and for all admissible sequences completes the proof.

(ii) (\Leftarrow) Suppose there exists a $t \in \{\tau, \tau+1, \dots, 2\tau-1\}$ and some $A_i \in \mathcal{A}$ such that Ω is not invariant w.r.t. A_i^t . The sequence $\mathcal{S}_\tau(t) := \{i, i, \dots, i\}$, which is an admissible sequence, violates the DT-invariance of Ω . ■

An example that illustrates the proof is in order. Consider $\mathcal{A} = \{A_1, A_2\}$, $\tau = 3$ and $x(27) = A_{\mathcal{S}_\tau(27)} x_0 = A_1^8 A_2^9 A_1^{10} x_0$. Using the procedure described in the proof above, $x(27) = [A_1^3 A_1^5] [(A_2^3)^2 A_2^3] [(A_1^3)^2 A_1^4] x_0$ and from the t -step invariance of Ω for all $t \in \{3, 4, 5\}$, $A_{\mathcal{S}_\tau(27)} x_0 \in \Omega$ if $x_0 \in \Omega$.

An interesting and important connection can now be established between dwell time stability and stability under arbitrary switching for system (1a). The proof of this result is given in the appendix. While not needed for the rest of this section, this result is needed for the convergence of algorithm described in section IV.

Theorem 2: Consider an associated system of (1a) in the form

$$\begin{aligned} \hat{x}(t+1) &= \hat{A}\hat{x}(t), \\ \hat{A} &\in \{A_i^r : \text{for all } i \in \mathcal{I}_N \text{ and } r = \tau, \dots, 2\tau - 1\} \end{aligned} \quad (6)$$

Then system (1a) is asymptotically stable with dwell time τ if and only if (6) is asymptotically stable under arbitrary switching.

Remark 2: A consequence of Theorem 2 is that properties related to the stability of system (6) is also applicable to the stability of dwell-time switching systems.

Theorem 1 on DT-invariance for a set Ω requires that $x(t) \in \Omega$ for all t with $t \in \{\tau, \tau + 1, \dots, 2\tau - 1\}$ but no mention is made of the $x(t) \in \mathcal{X}$ constraint stipulated in (1b). The next definition imposes this latter requirement for all time instants.

Definition 4: A set Ω is said to be CADT-invariant (Constraint Admissible Dwell Time-invariant) with respect to system (1) with dwell time τ if it is DT-invariant and $x(t) \in \mathcal{X}$ for all $t \in \mathbb{Z}^+$.

Theorem 1 on DT-invariance of Ω requires that $x(t) \in \Omega$ for all $t \in \{\tau, \tau + 1, \dots, 2\tau - 1\}$ but there is no guarantee that $x(t) \in \mathcal{X}$ for all t . Clearly, a necessary condition for constraint admissibility is that $\Omega \subseteq \mathcal{X}$. Imposing $x(t) \in \mathcal{X}$ for $t = 0, 1, \dots, \tau - 1$ ensures that Ω is CADT-invariant. The following result is therefore obvious and is stated without proof.

Theorem 3: A DT-invariant set $\Omega \subset \mathcal{X}$ is CADT-invariant for system (1) with dwell time τ , if and only if for any $x \in \Omega$,

$$A_i^t x \in \mathcal{X}, \text{ for all } i \in \mathcal{I}_N \text{ and for all } t = 0, \dots, \tau - 1. \quad (7)$$

A. Computation of polyhedral CADT-invariant sets

The results of Theorems 1 and 3 can be used to compute the maximal CADT-invariant set for (1). This set, denoted by \mathcal{O}_∞ , is the largest CADT-invariant set inside \mathcal{X} in the sense that $x(t) = A_{\mathcal{S}_\tau(t)}x(0) \in \mathcal{O}_\infty$ if $x(0) \in \mathcal{O}_\infty$ for any admissible switching sequence $\mathcal{S}_\tau(t)$. For this purpose, let

$$\hat{Q}(\Omega, A) = \{x : Ax \in \Omega\}$$

denote the one time backward set of Ω under system A . It corresponds to the set of x that can be brought into Ω by system A in one time step. Similarly, repeating the above ℓ times lead to

$$\hat{Q}_\ell(\Omega, A) = \hat{Q} \cdots \hat{Q}(\Omega, A) = \{x : A^\ell x \in \Omega\} \quad (8)$$

and is referred to as the ℓ -step backward set of Ω of system A . Define

$$Q_\ell(\Omega) := \bigcap_{i \in \mathcal{I}_N} \hat{Q}_\ell(\Omega, A_i) \quad (9)$$

as the intersection of $\hat{Q}_\ell(\Omega, A_i)$ over all $A_i \in \mathcal{A}$. With this definition, the algorithm for computing the \mathcal{O}_∞ set using Theorems 1 and 3 is now given.

Algorithm 1 Computation of maximal CADT-invariant set

Input: \mathcal{A} , \mathcal{X} and τ .

- 1) Set $k = 0$ and let $\mathcal{O}_0 := \mathcal{X} \cap_{1 \leq t \leq \tau-1} Q_t(\mathcal{X})$.
 - 2) Compute $Q_t(\mathcal{O}_k)$ for $t = \tau, \tau + 1, \dots, 2\tau - 1$ and let $\mathcal{O}_{k+1} := \mathcal{O}_k \cap_{\tau \leq t \leq 2\tau-1} Q_t(\mathcal{O}_k)$.
 - 3) If $\mathcal{O}_{k+1} \equiv \mathcal{O}_k$ set $\mathcal{O}_\infty = \mathcal{O}_k$ then stop, else set $k = k + 1$ and goto step (2).
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Step (1) of Algorithm 1 imposes the constraint according to Theorem 3. Similarly, step (2) imposes the condition of Theorem 1. More exactly, each $Q_t(\mathcal{O}_k)$ of step (2) is $\bigcap_{i \in \mathcal{I}_N} \hat{Q}_t(\mathcal{O}_k, A_i)$ of (8) and is the intersection for t -step backward set for each mode A_i . By letting $t = \tau, \dots, 2\tau - 1$, step (2) captures all possible admissible sequences defined in Theorem 1. Obviously, the \mathcal{O}_∞ obtained using the above algorithm depends on the choices of \mathcal{A} , \mathcal{X} and τ . For notational convenience, such dependencies are not shown unless warranted.

Remark 3: When $\mathcal{X} = \{x : Rx \leq \mathbf{1}\}$ is a non-empty polytope as given under (A2), the associated computations of step (2) can be obtained noting that $\hat{Q}(\mathcal{X}, A) = \{x : RAx \leq \mathbf{1}\}$, $Q(\mathcal{X}) = \bigcap_{i \in \mathcal{I}_N} \{x : RA_i x \leq \mathbf{1}\}$ and $Q_t(\mathcal{X}) = \bigcap_{i \in \mathcal{I}_N} \{x : RA_i^t x \leq \mathbf{1}\}$.

While not stated in Algorithm 1, fewer computations result if redundant inequalities are removed from \mathcal{O}_{k+1} at the end of step (2). Properties of the \mathcal{O}_∞ set obtained from the algorithm are stated next.

Theorem 4: Suppose system (1) satisfies assumptions (A1)-(A4) and \mathcal{O}_k is generated based on Algorithm 1. Then, (i) $\mathcal{O}_k \subset \mathcal{X}$ and $\mathcal{O}_k \subseteq \mathcal{O}_{k-1}$ for all k . (ii) $\mathcal{O}_\infty := \lim_{k \rightarrow \infty} \mathcal{O}_k \subset \mathcal{X}$ exists, contains the origin and is finitely determined. (iii) \mathcal{O}_∞ is the largest CADT-invariant set in the sense of Definition 4 and is the largest constraint-admissible domain of attraction under admissible switching sequences. (iv) When \mathcal{O}_∞ is the largest CADT-invariant set for system (1) with constraint set \mathcal{X} , $\beta\mathcal{O}_\infty$ is the corresponding set for system (1a) with constraint $\beta\mathcal{X}$ for any $\beta > 0$.

Remark 4: It is important to highlight the precise meaning of result (iii) of the preceding Theorem. As mentioned in Remark 1 and Definition 1, a sequence that violates the $t - t_{last} \geq \tau$ condition is not admissible, yet it may be a truncated subsequence of an admissible sequence. As Algorithm 1 is for system (1) under all admissible sequences, the presence of such inadmissible sequences results in \mathcal{O}_∞ being CADT-invariant and not positive invariant in the conventional sense. This means that $x(0) \in \mathcal{O}_\infty$ implies $x(\tau) \in \mathcal{O}_\infty$ and $x(t) \in \mathcal{X}$ for all t . There is no requirement that $x(t) \in \mathcal{O}_\infty$ when $t = 1, \dots, \tau - 1$. A set with such property is also known as a returnable set. Figure 1 shows the \mathcal{O}_∞ set based on an example with $\mathcal{X} = \{x \in \mathbb{R}^2 : \|x\|_\infty \leq 1\}$, $N = 2$, $A_1 = \begin{bmatrix} 0.7 & 1 \\ 0 & 0.2 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0.8 & 0 \\ 0.4 & 0.6 \end{bmatrix}$ and $\tau = 2$. Trajectories under admissible sequences of two initial states $(\pm(0.846, 0.408))$ within \mathcal{O}_∞ are shown. Clearly, $x(1) \notin \mathcal{O}_\infty$ but $x(2)$ is.

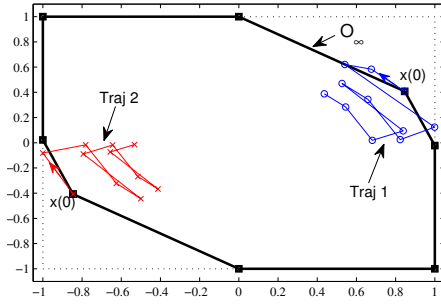


Fig. 1. State trajectories for $x(0) = \pm(0.846, 0.408)^T$

B. Computation of piece-wise quadratic CADT-invariant sets

The results of Theorems 1 and 3 can also be extended to obtain a CADT-invariant set defined by ellipsoidal sets. The next theorem shows the basic results needed.

Theorem 5: Suppose system (1) satisfies assumptions (A1)-(A4) with dwell time τ . If there exist $P_i \succ 0$ for $i = 1, \dots, N$ such that

$$(A_i^k)^T P_j (A_i^k) - P_i \prec 0 \quad \text{for all } (i, j) \in \mathcal{I}_N \times \mathcal{I}_N, \\ \text{and for all } k = \tau, \tau + 1, \dots, 2\tau - 1. \quad (10)$$

Then (i) $\Psi := \bigcap_{i \in \mathcal{I}_N} \mathcal{E}(P_i)$ where $\mathcal{E}(P_i) = \{x : x^T P_i x \leq 1\}$ is a DT-invariant set for system (1). (ii) Let $\mathcal{O}_0 := \mathcal{X} \cap_{1 \leq t \leq \tau-1} Q_t(\mathcal{X})$. There exists an $\bar{\alpha} > 0$, such that $\alpha\Psi$ is CADT-invariant for all $\alpha \leq \bar{\alpha}$ where $\bar{\alpha} := \max\{\alpha : \alpha\mathcal{E}(P_i) \subseteq \mathcal{O}_0, \forall i \in \mathcal{I}_N\}$.

Part (i) of the above can be seen as the equivalence of Theorem 1 but with Ω replaced by Ψ . Like Theorem 1, part (i) does not impose the $x(t) \in \mathcal{X}$ condition. Instead, constraint satisfaction is imposed via the \mathcal{O}_0 set in a similar fashion as Theorem 3 and step (1) of Algorithm 1. Closed-form expression of $\bar{\alpha}$ also exists under (A2). More exactly, when \mathcal{O}_0 is expressed as $\mathcal{O}_0 := \{x : a_j^T x \leq 1 \text{ for all } j \in \mathcal{J}\}$ for some appropriate row vectors $a_j, j \in \mathcal{J}$, the value of $\bar{\alpha}$ of $\bar{\alpha}\Psi$ is obtained by finding the largest α such that $\alpha\mathcal{E}(P_i) \subseteq \mathcal{O}_0$ for all $i \in \mathcal{I}_N$. This is done by considering the largest $\alpha\mathcal{E}(P_i)$ contained in each half space $\{x : a_j^T x \leq 1\}$. It is easy to show that $\bar{\alpha} = \min\{(a_j^T P_i^{-1} a_j)^{-\frac{1}{2}} : i \in \mathcal{I}_N, j \in \mathcal{J}\}$. Figure 2 shows the corresponding $\bar{\alpha}\Psi$ set for the same problem given in Remark 4. Clearly, $\bar{\alpha}\Psi \subset \mathcal{O}_\infty$ since \mathcal{O}_∞ is the largest DT-invariant set.

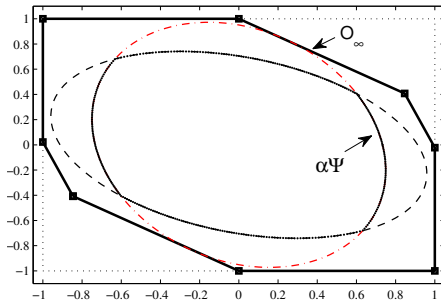


Fig. 2. Illustration of $\bar{\alpha}\Psi$ and \mathcal{O}_∞ sets.

Remark 5: If $P_i = P_j = P$ for all $(i, j) \in \mathcal{I}_N \times \mathcal{I}_N$ in (10), then the resulting DT-invariant set is quadratic and characterized by $\mathcal{E}(P) = \{x : x^T P x \leq 1\}$.

IV. COMPUTATION OF THE MINIMAL DWELL TIME

An algorithm that finds the minimal dwell time which ensures stability of the origin of system (1) can be obtained based on Algorithm 1. This is motivated by the observation that an empty \mathcal{O}_∞ set results if the τ used in Algorithm 1 does not satisfy (A4). Since τ is a scalar, it is easy to use, for example, a bisection search to find the minimal τ using Algorithm 1 as a sub-algorithm. Such an approach, however, suffers from two drawbacks.

- (I1) The implication of violation of assumption (A4): when a given τ is not known *a priori* to satisfy (A4), there is no guarantee that the origin is asymptotically stable even when Algorithm 1 terminates successfully. Only Lyapunov stability can be ascertained.
- (I2) The implication of the characterization of the \mathcal{O}_∞ set in Algorithm 1. If Algorithm 1 fails for a given τ when \mathcal{O}_∞ is polyhedral, does there exist a different characterization of \mathcal{O}_∞ (quadratic or otherwise) for which the origin is asymptotically stable?

These issues are now addressed. As (A4) is no longer assumed in this section, a new definition of a DT-invariant set for asymptotically stable origin is needed, motivated from the definition of standard contractive set [23].

Definition 5: A set $\Omega \subset \mathbb{R}^n$ containing the origin is said to be DT-contractive (with contraction λ) w.r.t. (1), if there exist a $\lambda \in (0, 1)$ such that $x \in \Omega$ implies $A_{\mathcal{S}_\tau(t)} x \in \lambda\Omega$ for all admissible switching sequences $\mathcal{S}_\tau(t)$ and for all time t .

Again, the above definition is of limited applicability since all admissible sequences are needed. The adaption of DT-contractive set to a result similar to Theorem 1 is therefore desirable and can be easily achieved. More exactly, a set $\Omega \subset \mathbb{R}^n$ is DT-contractive, with contraction $\lambda \in (0, 1)$, if and only if $A_i^t \Omega \subseteq \lambda\Omega$ for all $i \in \mathcal{I}_N$ and for all $t \in \{\tau, \dots, 2\tau - 1\}$. With this, a necessary and sufficient condition for stability of (1) with dwell time τ , is now given.

Theorem 6: Suppose (A1)-(A3) are satisfied. The origin of system (1) is asymptotically stable under admissible switching with dwell time τ if and only if system (1) admits a polyhedral DT-contractive set, that contains the origin, for some $\lambda \in (0, 1)$.

A polyhedral DT-contractive set can be computed by a slight modification to Algorithm 1 by incorporating a choice of $\lambda \in (0, 1)$ as follows:

Algorithm 1a Computation of polyhedral CADT-contractive set

Input: \mathcal{A} , \mathcal{X} , λ and τ .

- 1) Set $k = 0$ and let $\mathcal{O}_0^\lambda := \mathcal{X} \cap_{1 \leq t \leq \tau-1} Q_t(\mathcal{X})$.
 - 2) Compute $Q_t(\lambda\mathcal{O}_k^\lambda)$ for $t = \tau, \tau + 1, \dots, 2\tau - 1$ and let $\mathcal{O}_{k+1}^\lambda := \mathcal{O}_k^\lambda \cap_{\tau \leq t \leq 2\tau-1} Q_t(\lambda\mathcal{O}_k^\lambda)$.
 - 3) If $\mathcal{O}_{k+1}^\lambda \equiv \mathcal{O}_k^\lambda$ set $\mathcal{O}_\infty^\lambda = \mathcal{O}_k^\lambda$ then stop, else set $k = k + 1$ and goto step (2).
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It is worthy to note that step (1) above ensures constraint satisfaction according to Theorem 3 and, hence, does not require the consideration of λ .

Theorem 7: Suppose the origin of system (1) is asymptotically stable under dwell time switching with dwell time τ . Algorithm 1a with dwell time τ yields a non-empty $\mathcal{O}_\infty^\lambda$ for some $\bar{\lambda} \in (0, 1)$. In addition, Algorithm 1a with dwell time τ will yield a non-empty $\mathcal{O}_\infty^\lambda$ for any $\lambda \in [\bar{\lambda}, 1)$.

Together, Theorems 6 and 7 address issues (I1) and (I2). Successful termination of Algorithm 1a means that $x(t) \rightarrow 0$ for any $x(0) \in \mathcal{O}_\infty^\lambda$ and hence issue (I1) is resolved. While the use of a polyhedral set is both necessary and sufficient for determining the asymptotic stability by Theorem 6. Theorem 7 also shows that there is a range of λ , $[\bar{\lambda}, 1)$, that is admissible for Algorithm 1a. In practice, it is prudent to chose λ close to 1, say $\lambda = 0.999$.

With the above observations, the next algorithm outlines the steps for finding the minimal dwell-time needed for stability. It is based on a bisection search on τ starting with an initial τ_0 that satisfies (A4).

Algorithm 2 Computation of minimum dwell time

Input: $\mathcal{A}, \mathcal{X}, \tau_0$

Initialization: Let $\tau_u = \tau_0$ and $\tau_\ell = 1$.

while $\tau_u > \tau_\ell + 1$

- 1) Let $\tau = \lfloor (\tau_u + \tau_\ell)/2 \rfloor$ and invoke Algorithm 1 using \mathcal{A}, \mathcal{X} and τ .
- 2) If $\mathcal{O}_\infty = \emptyset$, then $\tau_\ell = \tau$, else $\tau_u = \tau$.

end while

Let $\bar{\tau} := \tau$.

- 3) Invoke Algorithm 1a using $\mathcal{A}, \mathcal{X}, \bar{\tau}$ and $\lambda = 0.999$.
 - 4) If $\mathcal{O}_\infty^\lambda \neq \emptyset$, then $\tau_{\min} = \bar{\tau}$ and terminate, else $\bar{\tau} = \bar{\tau} + 1$. Goto step (3)
-

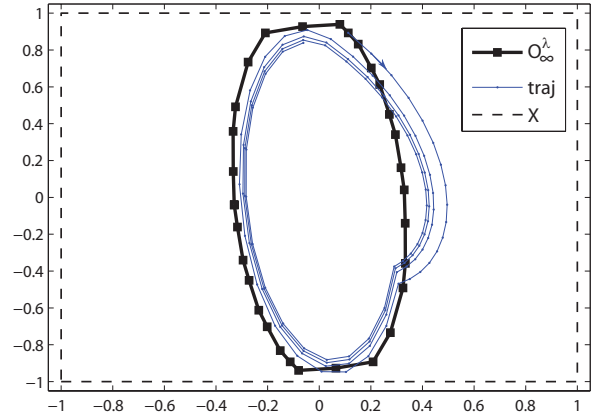
The “while” loop in Algorithm 2 compute \mathcal{O}_∞ based on Algorithm 1. Following the discussion of (II) above, the second part of algorithm 2 is needed to ensure that all $x \in \mathcal{O}_\infty$ converges to the origin. Clearly, if only DT-invariance is needed but not asymptotic stability of the origin, this second part can be omitted.

Remark 6: The $\mathcal{O}_\infty^\lambda$ obtained from Algorithm 1a can be interpreted as a “generalized” Lyapunov function for switching system (1). Since $\mathcal{O}_\infty^\lambda$ is a polytope and contains the origin, it induces a norm $\|x\|_{\mathcal{O}_\infty^\lambda} := \min\{\mu \geq 0 : x \in \mu\mathcal{O}_\infty^\lambda\}$ (or the Minkowski distance function of $\mathcal{O}_\infty^\lambda$). Let $V(x(t)) := \|x(t)\|_{\mathcal{O}_\infty^\lambda}$. Unlike conventional Lyapunov functions, $V(x(t))$ does not decreases at every step, but decreases at every τ time. Contractivity of $\mathcal{O}_\infty^\lambda$ ensures that $V(x(t_{s_{k+1}})) \leq \lambda V(x(t_{s_k}))$ where t_{s_k} and $t_{s_{k+1}}$ are consecutive switching instants. Hence, the sequence of $V(x(t_{s_k}))$ with respect to index k is a decreasing sequence that converges to zero. This also means that $V(t)$ may increase in between switching instants, see example in Section V.

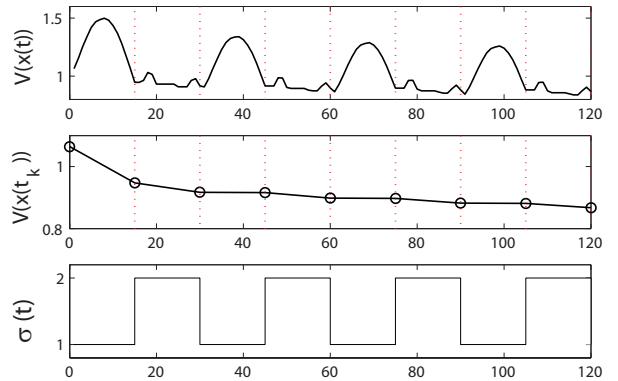
V. NUMERICAL EXAMPLE

The numerical example is on a switching system with $\mathcal{A} = \{A_1, A_2\}$, $A_1 = \begin{bmatrix} -1 & 0.1 \\ -0.2 & 0.9 \end{bmatrix}$, $A_2 = \begin{bmatrix} -1 & 0.1 \\ -0.9 & 0.9 \end{bmatrix}$ with state constraints $\mathcal{X} = \{x \in \mathbb{R}^2 : \|x\|_\infty \leq 1\}$. The intention is to determine the minimum dwell time and the maximal constraint admissible domain of attraction under dwell time switching for this system. It is worthy to note that existing techniques [9], [10] meant for systems under arbitrary switching is not applicable for this example.

Using the approach of [8] discussed in section II and equation (3), $\lambda_m = 0.995$, $h_1 = 1.1861$, $h_2 = 3.0531$ and $h_m = 3.0531$ and an upper bound on τ , $\tau^Z := 233$ is obtained for this example. Using the piece-wise quadric DT-contractive function of Remark 5, it is observed that the smallest τ for which LMI (10) admits a solution is at $\tau^{LMI} := 16$. Algorithm 2 yields a minimum dwell time of $\tau_{\min} = 15$. Figure 3(a) shows the $\mathcal{O}_\infty^\lambda$ (with $\lambda = 0.999$) set and a state trajectory under a periodic switching sequence where $t_{s_{k+1}} - t_{s_k} = 15$ for all k and $\sigma(0) = 1$. That the state moves out of $\mathcal{O}_\infty^\lambda$ is clear but it comes back in no more than 15 steps. Moreover, $x(t) \in \mathcal{X}$ at all times. The “generalized” Lyapunov function of $V(x(t)) = \|x(t)\|_{\mathcal{O}_\infty^\lambda}$ for this trajectory is shown in Figure 3(b). Again, $V(t)$ is not monotonically decreasing with respect to t but is monotonically decreasing with k and $V(t) \rightarrow 0$ as $t \rightarrow \infty$.



(a) State trajectory in $x_1 - x_2$ space;



(b) Lyapunov function $V(x(t)) = \|x(t)\|_{\mathcal{O}_\infty^\lambda}$

Fig. 3. Illustration of CADT-contractive set $\mathcal{O}_\infty^\lambda$ for $\tau_{\min} = 15$.

The example above shows that there is a significant improvement on stability conditions in terms of dwell time calculations when compared to the results available in the literature to date; see [8]. Moreover, constraint admissible domain of attraction of dwell time switching systems is obtained, which is appeared to be the first of its kind.

VI. CONCLUSIONS

Definitions of a DT-invariant set and a CADT-invariant set are given for discrete-time switching systems under dwell-time switching. The characterization of a DT-invariant set with dwell time τ corresponds to the satisfaction of t -step invariance for $t = \tau, \dots, 2\tau - 1$ for the set. This characterization allows for a numerical algorithm for the computation of the maximal CADT-invariant set. Using this algorithm as a sub-algorithm, a procedure for the computations of the minimal dwell time needed for stability of the origin of the switching system is obtained.

VII. ACKNOWLEDGMENTS

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APPENDIX

Proof of Theorem 2:

(i) (\Leftarrow) We show that asymptotic stability of (6) implies asymptotic stability of (1a) with dwell time τ . It is well-known [24], [23] that (6) is asymptotically stable iff a polyhedral contractive set exists w.r.t. (6). This implies (6) is asymptotically stable iff a polyhedral set S and a $\lambda \in (0, 1)$ exist such that $\hat{A}x \in \lambda S$ for every $x \in S$ and for every $\hat{A} \in \{A_i^\tau, A_i^{\tau+1}, \dots, A_i^{2\tau-1}$ for all $i \in \mathcal{I}_N\}$. Now consider an admissible switching sequence of the form (4) and $x(0) \in S$, it follows that

$$x(t) = A_{\mathcal{S}_\tau(t)} x(0) = (A_{s_n}^{k_n} \dots A_{s_1}^{k_1} A_{s_0}^{k_0}) x(0) \in \lambda^{\bar{k}} S \quad (11)$$

where $\bar{k} := \lfloor k_0/\tau \rfloor + \lfloor k_1/\tau \rfloor + \dots + \lfloor k_n/\tau \rfloor$. The rightmost condition of (11) follows from the fact that all $k_j \geq \tau$ and that $x(0) \in S$ implies $A_i^t x(0) \in \lambda S$ for all $i \in \mathcal{I}_N$ and for all $\tau \leq t \leq 2\tau - 1$. Since $\bar{k} \rightarrow \infty$ as $t \rightarrow \infty$, asymptotic stability of (1a) follows.

(ii) (\Rightarrow) We show that asymptotic stability of (1a) implies asymptotic stability of (6) with arbitrary switching. Proof of this part is by contradiction. Suppose that (1a) is asymptotically stable but (6) is not. This means there exist an arbitrary switching sequence w.r.t. (6) that is not converging to the origin. Clearly this switching sequence is an admissible switching sequence that satisfies the dwell time condition w.r.t. (1a) and hence it violates the asymptotic stability of (1a), which is a contradiction.

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