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# Reach Controllability of Single Input Affine Systems

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***Abstract*— We study the reach control problem (RCP) for a single input affine system with a simplicial state space. We extend previous results by exploring arbitrary triangulations of the state space; particularly allowing the set of possible equilibria to intersect the interior of simplices. In the studied setting, it is shown that closed-loop equilibria, nevertheless, only arise on the boundary of simplices. This allows to define a notion of reach controllability which quantifies the effect of the control input on boundary equilibria. Using reach controllability we obtain necessary and sufficient conditions for solvability of RCP by affine feedback.**

## I. Introduction

This paper studies the *reach control problem* (RCP) on simplices. The problem is for an affine system defined on a simplex $\mathcal{S}$ to reach a prespecified facet of the simplex in finite time without first leaving the simplex. The problem has been studied over a series of papers [3], [4], [5], [7], [2] due to its fundamental nature among reachability problems. The reader is referred to [1], [2], [3], [4], [5], [6], [7] for further motivations, including how the problem arises in reachability problems for hybrid systems. In [2] we studied RCP under the assumption that the state space was triangulated so that $\mathcal{O}$, the set of possible equilibria of the affine system, intersected with $\mathcal{S}$ was either the empty set or a face of $\mathcal{S}$. In this paper we assume $\mathcal{O}$ intersects the interior of $\mathcal{S}$, and we study only single input systems. Remarkably it emerges that if an equilibrium appears using an affine feedback to solve RCP, then the equilibrium is, nevertheless, on the boundary of $\mathcal{S}$. Using this fact, we propose a notion of *reach controllability* for determining if RCP is solvable by affine feedback. Simply put, an affine system is reach controllable on a simplex if each equilibrium can be "pushed off" the simplex boundary by an admissible affine feedback. Because the feedback is affine, the equilibrium is affected by the control input only through the control value applied at a vertex among those vertices whose convex hull contains the equilibrium. In sum, reach controllability measures the extent to which the control input can affect the dynamics on faces of the simplex. Using reach controllability, we obtain new necessary and sufficient conditions for solvability of RCP in the current setting.

*Notation.* Let $\mathcal{S} \subset \mathbb{R}^n$ be a set. The closure is $\overline{\mathcal{S}}$, and the interior is $\mathcal{S}^\circ$. The relative interior is denoted $\mathrm{ri}(\mathcal{S})$, the relative boundary of $\mathcal{S}$, denoted $\mathrm{rb}(\mathcal{S})$ is $\overline{\mathcal{S}} \setminus \mathrm{ri}(\mathcal{S})$, and $\partial\mathcal{S}$ is the boundary of $\mathcal{S}$. The symbol $\mathbb{U}$ represents a control class such as open-loop controls, continuous state feedback, affine feedback, etc. The notation $\mathbf{0}$ denotes the subset of $\mathbb{R}^n$ containing only the zero vector. The notation $\mathbf{1}$ stands for a vector with appropriate dimension whose entries are one. Notation $\mathrm{co}\{v_1, v_2, \ldots\}$ denotes the convex hull of a set of points $v_i \in \mathbb{R}^n$. The notation $\mathrm{ord}(M)$ denotes the order of the square matrix $M$.

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## II. Background

We consider an $n$-dimensional simplex $\mathcal{S} := \mathrm{co}\{v_0, v_1, \ldots, v_n\}$ with vertex set $V := \{v_0, v_1, \ldots, v_n\}$ and facets $\mathcal{F}_0, \ldots, \mathcal{F}_n$ (the facet is indexed by the vertex it does not contain). Without loss of generality (w.l.o.g.) we assume that $v_0 = 0$. Let $h_i$, $i = 0, \ldots, n$ be the unit normal vector to each facet $\mathcal{F}_i$ pointing outside of the simplex. Let $\mathcal{F}_0$ be the target set in $\mathcal{S}$. Define the index sets $I := \{1, \ldots, n\}$ and $I_i := I \setminus \{i\}$ (note $I_0 = I$).

Consider the affine control system defined on $\mathcal{S}$:

$$\dot{x} = Ax + Bu + a\,, \qquad x \in \mathcal{S}\,, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $a \in \mathbb{R}^n$, $B \in \mathbb{R}^{n\times m}$, and $\mathrm{rank}(B) = m = 1$. Let $\phi_u(t, x_0)$ denote the trajectory of (1) under a control $u(t)$ starting from $x_0 \in \mathcal{S}$ and evaluated at time $t$. We are interested in studying reachability of the target $\mathcal{F}_0$ from $\mathcal{S}$.

*Problem 1 (Reach Control Problem (RCP)):* Consider system (1) defined on $\mathcal{S}$. Find a feedback $u(x)$ such that: for every $x_0 \in \mathcal{S}$ there exist $T \geq 0$ and $\gamma > 0$ such that

(i) $\phi_u(t, x_0) \in \mathcal{S}$ for all $t \in [0, T]$,
(ii) $\phi_u(T, x_0) \in \mathcal{F}_0$, and
(iii) $\phi_u(t, x_0) \notin \mathcal{S}$ for all $t \in (T, T+\gamma)$.

RCP says that trajectories of (1) starting from initial conditions in $\mathcal{S}$ exit $\mathcal{S}$ through the target $\mathcal{F}_0$ in finite time, while not first leaving $\mathcal{S}$.

*Definition 1:* A point $x_0 \in \mathcal{S}$ can reach $\mathcal{F}_0$ with constraint in $\mathcal{S}$ with control class $\mathbb{U}$, denoted by $x_0 \xrightarrow{\mathcal{S}} \mathcal{F}_0$, if there exists a control $u$ of class $\mathbb{U}$ such that properties (i)-(iii) of Problem 1 hold. We write $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by control class $\mathbb{U}$ if for every $x_0 \in \mathcal{S}$, $x_0 \xrightarrow{\mathcal{S}} \mathcal{F}_0$ with control of class $\mathbb{U}$.

Define the closed, convex cones

$$\mathcal{C}_i := \left\{\ y \in \mathbb{R}^n\ :\ h_j \cdot y \leq 0,\ j \in I_i\ \right\}, i \in \{0, \ldots, n\}\,.$$

Also define $\mathrm{cone}(\mathcal{S}) := \mathcal{C}_0$. Note that $\mathrm{cone}(\mathcal{S})$ is the tangent cone to $\mathcal{S}$ at $v_0$.

*Definition 2:* We say the invariance conditions are solvable if there exist $u_0, \ldots, u_n \in \mathbb{R}^m$ such that $Av_i + a + Bu_i \in$

$\mathcal{C}_i$ for $i = 0, \ldots, n$. Equivalently,

$$h_j \cdot (Av_i + a + Bu_i) \leq 0\,,\ i \in \{0, \ldots, n\},\ j \in I_i\,. \quad (2)$$

The inequalities (2) are called *invariance conditions*. These Nagumo-like conditions guarantee that trajectories cannot exit through the restricted facets $\mathcal{F}_1, \ldots, \mathcal{F}_n$ and are used to construct affine feedbacks [4]. For general state feedbacks (particularly those not satisfying convexity), stronger conditions are needed to ensure that trajectories do not exit restricted facets. To that end, for $x \in \mathcal{S}$, define $J(x) = \{j \in I \mid x \in \mathcal{F}_j\}$. Define the closed, convex cone

$$\mathcal{C}(x) := \{\ y \in \mathbb{R}^n\ :\ h_j \cdot y \leq 0,\ j \in J(x)\ \}.$$

*Definition 3:* We say a state feedback $u(x)$ satisfies the invariance conditions if $Ax + Bu(x) + a \in \mathcal{C}(x)$. Equivalently, for all $x \in \mathcal{S}$ and $j \in J(x)$,

$$h_j \cdot (Ax + Bu(x) + a) \leq 0\,. \quad (3)$$

Given $x \in \mathcal{S}$, let $I(x)$ be the minimal index set such that $x \in \text{co}\{v_i \mid i \in I(x)\}$. A form of (3) we will often employ is as follows. Suppose $x \in \text{co}\{v_i \mid i \in I(x)\}$. Using the properties of the simplex [2], one can show this implies $x \in \mathcal{F}_j$, for $j \in I \setminus I(x)$. Then (3) becomes

$$h_j \cdot (Ax + Bu(x) + a) \leq 0\,,\ j \in I \setminus I(x)\,.$$

For Problem 1 the following necessary and sufficient conditions have been established for the case of affine feedback.

*Theorem 4:* [5], [7] Given the system (1) and an affine feedback $u(x) = Kx + g$, where $K \in \mathbb{R}^{m \times n}$, $g \in \mathbb{R}^m$, and $u_0 = u(v_0), \ldots, u_n = u(v_n)$, the closed-loop system satisfies $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ if and only if (a) the invariance conditions (2) hold, and (b) there is no equilibrium in $\mathcal{S}$.

Let $\mathcal{B} = \text{Im}(B)$, the image of $B$. Define the set of *possible equilibrium points*

$$\mathcal{O} := \{\ x \in \mathbb{R}^n\ :\ Ax + a \in \mathcal{B}\}\,.$$

One can show that either $\mathcal{O} = \emptyset$ or $\mathcal{O}$ is an affine space with $m \leq \dim(\mathcal{O}) \leq n$. Notice that the vector field $Ax + Bu + a$ on $\mathcal{O}$ can vanish for an appropriate choice of $u$, so $\mathcal{O}$ is the set of all possible equilibrium points of the system. Also define the set of *open-loop equilibrium points*

$$\mathcal{E} := \{\ x \in \mathbb{R}^n\ :\ Ax + a = 0\}\,.$$

Define $\mathcal{O}_\mathcal{S} := \mathcal{S} \cap \mathcal{O}$ and $\mathcal{E}_\mathcal{S} := \mathcal{S} \cap \mathcal{E}$. Clearly $\mathcal{E} \subset \mathcal{O}$ and $\mathcal{E}_\mathcal{S} \subset \mathcal{O}_\mathcal{S}$. The following result was proved in [2] for the case when the state space is triangulated so that $\mathcal{O}_\mathcal{S}$ is a $\kappa$-dimensional face of $\mathcal{S}$. Here we generalize to arbitrary triangulations.

*Theorem 5:* If the invariance conditions (2) are solvable and $\mathcal{B} \cap \text{cone}(\mathcal{S}) \neq \mathbf{0}$, then $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by affine feedback.

## III. Necessary Conditions

The goal of this paper is to obtain new necessary and sufficient conditions for solvability of RCP by affine feedback; unlike the conditions of Theorem 4, we seek conditions that lead to a synthesis of the controller. We begin with necessary conditions for solvability. Suppose

$$\mathcal{O}_\mathcal{S} = \text{co}\{o_1, \ldots, o_{\kappa+1}\}$$

and define $I_{\mathcal{O}_\mathcal{S}} := \{1, \ldots, \kappa + 1\}$. Also define

$$\text{cone}(\mathcal{O}_\mathcal{S}) := \bigcap_{i \in I_{\mathcal{O}_\mathcal{S}}} \mathcal{C}(o_i)\,.$$

This cone consists of all vectors that simultaneously satisfy all invariance conditions at all vertices $o_i$, $i \in I_{\mathcal{O}_\mathcal{S}}$. In the following two results, no assumption is made on the placement of $\mathcal{O}_\mathcal{S}$ with respect to $\mathcal{S}$.

*Lemma 6 ([4]):* If $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by a continuous state feedback $u(x)$, then $u(x)$ satisfies the invariance conditions (3).

*Theorem 7:* Suppose $\mathcal{O}_\mathcal{S} \neq \emptyset$. If $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by continuous state feedback $u(x)$, then $\mathcal{B} \cap \text{cone}(\mathcal{O}_\mathcal{S}) \neq \mathbf{0}$.

## IV. Preliminaries

In this section we present preliminary technical results that will enable us to characterize (in Section V) useful geometric properties of $\mathcal{O}_\mathcal{S}$ and $\mathcal{E}_\mathcal{S}$. We begin by posing the main assumptions for the rest of the paper. In [2] we assumed that if $\mathcal{O}_\mathcal{S} \neq \emptyset$, then $\mathcal{O}_\mathcal{S}$ is a $\kappa$-dimensional face of $\mathcal{S}$, where $0 \leq \kappa \leq n$. More generally, if the intersection is arbitrary, then $\mathcal{O}_\mathcal{S}$ is a convex polytope. In the present paper we assume $\mathcal{O}_\mathcal{S}$ is a simplex that intersects the interior of $\mathcal{S}$. Finally, we restrict $\mathcal{O}_\mathcal{S}$ so that it does not touch $\mathcal{F}_0$. The latter is an extra restriction on the geometry that must be addressed in future work.

*Assumption 8:*

(A1) $\mathcal{O}_\mathcal{S} = \text{co}\{o_1, \ldots, o_{\kappa+1}\}$, a $\kappa$-dimensional simplex with $m \leq \kappa < n$.

(A2) If $\mathcal{E}_\mathcal{S} \neq \emptyset$, then $\mathcal{E}_\mathcal{S} = \text{co}\{\epsilon_1, \ldots, \epsilon_{\kappa_0+1}\}$, a $\kappa_0$-dimensional simplex with $0 \leq \kappa_0 \leq \kappa$.

(A3) $\mathcal{O}_\mathcal{S} \cap \mathcal{S}^\circ \neq \emptyset$.

(A4) $\mathcal{O}_\mathcal{S} \cap \mathcal{F}_0 = \emptyset$.

The following basic properties of $\mathcal{O}_\mathcal{S}$ and $\mathcal{E}_\mathcal{S}$ derive from the fact that they are formed as intersections of affine spaces and a simplex.

*Lemma 9:* Suppose Assumptions (A1)-(A3) hold. If $\dim(\mathcal{O}_\mathcal{S}) \geq 1$, then $\text{rb}(\mathcal{O}_\mathcal{S}) \subset \partial\mathcal{S}$. If $\dim(\mathcal{E}_\mathcal{S}) \geq 1$, then $\text{rb}(\mathcal{E}_\mathcal{S}) \subset \text{rb}(\mathcal{O}_\mathcal{S}) \subset \partial\mathcal{S}$.

Recall the index set $I_{\mathcal{O}_\mathcal{S}} := \{1, \ldots, \kappa + 1\}$ and define the index set $I_{\mathcal{E}_\mathcal{S}} := \{1, \ldots, \kappa_0 + 1\}$. First we examine an implication of the fact that $\mathcal{O}_\mathcal{S} \cap \mathcal{S}^\circ \neq \emptyset$ on the index sets $I(o_k)$ and $I(\epsilon_k)$.

*Lemma 10:* Suppose Assumptions (A1), (A3), and (A4) hold. Then each set $I(o_k)$, $k \in I_{\mathcal{O}_\mathcal{S}}$, has a nonzero exclusive member. That is, there exists $e_k \in I(o_k)$, $e_k \neq 0$ and $e_k \notin I(o_j)$, for all $j \neq k$.

*Lemma 11:* Suppose Assumptions (A2)-(A4) hold. Then either $\mathcal{E}_\mathcal{S} \cap \mathcal{S}^\circ = \emptyset$ or each set $I(\epsilon_k)$, $k \in I_{\mathcal{E}_\mathcal{S}}$, has a nonzero exclusive member. That is, there exists $e_k \in I(\epsilon_k), e_k \neq 0$ and $e_k \notin I(\epsilon_j)$, for all $j \neq k$.

Suppose Assumptions (A1), (A3), and (A4) hold, and suppose we reorder indices $\{0, \ldots, n\}$ so that indices that belong to more than one set $I(o_k)$, $k \in I_{\mathcal{O}_\mathcal{S}}$, are listed first. These are the shared indices

$$\bigcup_{1\le i,j\le \kappa+1,\ i\ne j} I(o_i)\cap I(o_j)\,. \tag{4}$$

In light of (A4), assume w.l.o.g. this list begins with index 0. Next, we list indices that correspond to exclusive members of $I(o_1),\dots,I(o_{\kappa+1})$, respectively, and in this order. By Lemma 10 the exclusive member lists are non-empty. Also by (A3), all elements of $\{0,\dots,n\}$ are included in the new ordering since $\cup_{j=1}^{\kappa+1} I(o_j) = \{0,\dots,n\}$. In the sequel we call this an ordering according to exclusive members of $\{I(o_k)\}$.

We now turn to an algebraic characterization of points in $\mathcal{E}_\mathcal{S}$. Define the matrices

$$H := [h_1\ \dots\ h_n]\,,\quad Y := [Av_1\ \dots Av_n]$$

and

$$\Gamma := H^T Y\,, \qquad \gamma := H^T a\,. \tag{5}$$

Suppose $\mathcal{E}_\mathcal{S} \ne \emptyset$. Assume that $x \in \mathcal{E}_\mathcal{S}$ and $x = \sum_{i=0}^n \beta_i v_i$ for some $\beta_i \in [0,1]$, $\sum_i \beta_i = 1$. By the properties of the simplex (Lemma 4.4, [2]), one can show that $H$ is nonsingular. Hence, we have

$$\begin{aligned} Ax + a = 0 &\iff H^T(Ax+a) = 0 \\ &\iff \sum_{i=1}^n \beta_i H^T A v_i + H^T a = 0 \\ &\iff \Gamma\boldsymbol{\beta} + \gamma = 0 \end{aligned} \tag{6}$$

where $\boldsymbol{\beta} = (\beta_1,\dots,\beta_n)$. Note that the derivation uses the fact that $v_0 = 0$. In the sequel, points in $\mathcal{E}_\mathcal{S}$ will be characterized using (6). Using (6) we can relate geometric properties of $\mathcal{E}_\mathcal{S}$ and $\mathcal{O}_\mathcal{S}$ to certain restrictions on the form of matrices $\Gamma$ and $\gamma$. There are several distinct cases.

*Lemma 12:* Suppose $\dim(\mathcal{E}_\mathcal{S}) = \kappa_0$ with $\kappa_0 \ge 0$ and assume that $\mathcal{E}_\mathcal{S} \cap \mathcal{S}^\circ \ne \emptyset$. Then, $\Gamma$ and $\gamma$ cannot have the form

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \dots & \Gamma_{1(p+2)} \\ 0 & \Gamma_{22} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \Gamma_{(p+2)(p+2)} \end{bmatrix},\ \gamma = \begin{bmatrix} \gamma_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{7}$$

where $p \ge \kappa_0$ and $\mathrm{ord}(\Gamma_{ii}) \ge 1,\ i = 2,\dots,p+2$. Vector $\gamma$ is partitioned corresponding to the partition of $\Gamma$.

Lemma 12 gives the the algebraic consequences of the statement $\mathcal{E}_\mathcal{S} \cap \mathcal{S}^\circ \ne \emptyset$. The next result gives the analogous algebraic consequences when $\mathcal{E}_\mathcal{S}$ is empty or is a face of $\mathcal{O}_\mathcal{S}$.

*Lemma 13:* Suppose Assumption 8 holds. In addition, suppose that either $\mathcal{E}_\mathcal{S} = \emptyset$ or $\mathcal{E}_\mathcal{S} = \mathrm{co}\{o_1,\dots,o_{\kappa_0+1}\}$ with $0 \le \kappa_0 \le \kappa$. Then the following cannot hold simultaneously:

$$h_j \cdot Av_i = 0\,,\quad h_j \cdot a = 0\,, \tag{8}$$

where $i \in I(o_k)$, $j \in I \setminus I(o_k)$, and $k \in I_{\mathcal{O}_\mathcal{S}}$.

*Proof:* [Proof of Lemma 13] Suppose by way of contradiction that constraints (8) hold simultaneously. First suppose $\mathcal{E}_\mathcal{S} = \mathrm{co}\{o_1,\dots,o_{\kappa_0+1}\}$ with $0 \le \kappa_0 \le \kappa$. Let $x \in \mathcal{E}_\mathcal{S}$ and $x = \sum_{i=0}^n \beta_i v_i$ for some $\beta_i \in [0,1]$, $\sum_i \beta_i = 1$. Using (5), (6), and (8), and ordering indices by exclusive members of $\{I(o_k)\}$, we obtain

$$\begin{gathered} \Gamma_{11}\boldsymbol{\beta_1} + \dots + \Gamma_{1(\kappa+2)}\boldsymbol{\beta_{(\kappa+2)}} + \hat{\gamma}_1 = 0, \\ \Gamma_{22}\boldsymbol{\beta_2} = 0, \\ \vdots \\ \Gamma_{(\kappa+2)(\kappa+2)}\boldsymbol{\beta_{(\kappa+2)}} = 0\,. \end{gathered} \tag{9}$$

Here $\hat{\gamma}_1 := \gamma_1$. Also $\boldsymbol{\beta_i}$, $i = 2,\dots,\kappa+2$ correspond to the exclusive members of $I(o_1),\dots,I(o_{\kappa+1})$, respectively, and $\boldsymbol{\beta_1}$ corresponds to the indices in (4). Note that by Lemma 10, $\dim(\boldsymbol{\beta_i}) \ge 1$ $i = 2,\dots,\kappa+2$. Suppose that $\mathrm{ord}(\Gamma_{ii}) = p_i$ $i = 2,\dots,\kappa+2$. From the second equation of (9), we deduce that if $\mathrm{rank}(\Gamma_{22}) = p_2$, then $\boldsymbol{\beta_2} = 0$ for all $x \in \mathcal{E}_\mathcal{S}$. This means that no exclusive members of $I(o_1)$ appear in $I(x)$, for any $x \in \mathcal{E}_\mathcal{S}$. In particular, from Lemma 10, $o_1 \notin \mathcal{E}_\mathcal{S}$, a contradiction. Thus, $\mathrm{rank}(\Gamma_{22}) < p_2$. Similarly $\mathrm{rank}(\Gamma_{ii}) < p_i,\ i = 2,\dots,\kappa_0+2$. This means that (9) provides at most $n-(\kappa_0+1)$ independent constraints to define $\mathcal{E}_\mathcal{S}$. Hence, $\dim(\mathcal{E}_\mathcal{S}) \ge \kappa_0+1$, a contradiction.

Second, suppose $\mathcal{E}_\mathcal{S} = \emptyset$. For each $k \in I_{\mathcal{O}_\mathcal{S}}$, $Ao_k + a := \lambda_k B$ and $\lambda_k \ne 0$ since $\mathcal{E}_\mathcal{S} = \emptyset$. Using (8), for each $k \in I_{\mathcal{O}_\mathcal{S}}$,

$$h_j \cdot (Ao_k + a) = h_j \cdot \big(\sum_{i\in I(o_k)} \alpha_i^k (Av_i + a)\big) = 0\,, j \in I \setminus I(o_k)$$

where $\sum_{i\in I(o_k)} \alpha_i^k = 1$ and $\alpha_i^k > 0$ for $i \in I(o_k)$. Combining with (8), this gives

$$h_j \cdot Av_i = 0\,, h_j \cdot a = 0\,, h_j \cdot B = 0\,, \tag{10}$$

where $i \in I(o_k)$, $j \in I \setminus I(o_k)$, and $k \in I_{\mathcal{O}_\mathcal{S}}$. Now suppose $x \in \mathcal{O}_\mathcal{S}$ and $x = \sum_{i=0}^n \beta_i v_i$ for some $\beta_i \in [0,1]$, $\sum_i \beta_i = 1$. Let $u(x)$ be any affine feedback that satisfies $Ax + a + Bu(x) = 0$. The same reasoning that yields (6) gives a formula

$$\Gamma\boldsymbol{\beta} + \gamma + H^T B u(x) = 0\,, \tag{11}$$

where $\boldsymbol{\beta} = (\beta_1,\dots,\beta_n)$. Using (5), (11), and (10), and ordering indices by exclusive members of $\{I(o_k)\}$, we obtain (9) with $\hat{\gamma}_1 := \gamma + H^T B \sum_{i=1}^n \beta_i u(v_i)$. Note that we use the fact that $u(x)$ is affine. Suppose that $\mathrm{ord}(\Gamma_{ii}) = p_i$, $i = 2,\dots,\kappa+2$. From the second equation of (9), we deduce that if $\mathrm{rank}(\Gamma_{22}) = p_2$, then $\boldsymbol{\beta_2} = 0$ for all $x \in \mathcal{O}_\mathcal{S}$, so $x \in \partial\mathcal{S}$. This contradicts Assumption (A3). Thus, $\mathrm{rank}(\Gamma_{22}) < p_2$. Similarly $\mathrm{rank}(\Gamma_{ii}) < p_i,\ i = 2,\dots,\kappa+2$. This means that (9) provides at most $n-(\kappa+1)$ independent constraints to define $\mathcal{O}_\mathcal{S}$. Hence, $\dim(\mathcal{O}_\mathcal{S}) \ge \kappa+1$, a contradiction. ∎

## V. Properties of Equilibrium Set

In this section we exploit the algebraic properties discovered in the previous section, and particularly we examine their geometric consequences. The most important result is that equilibria cannot appear in the interior of $\mathcal{S}$ when the necessary conditions for solvability of RCP are satisfied. First we present a technical lemma that links the appearance of an equilibrium with algebraic constraints of the type studied in the previous section.

*Lemma 14:* Suppose that $Av_i+a \in \mathcal{C}_i$ for $i \in \{0,\ldots,n\}$. Suppose there exists $x \in \mathcal{O}_\mathcal{S}$ and $j \in I \setminus I(x)$ such that $0 \in I(x)$ and $h_j \cdot (Ax+a) = 0$. Then

$$h_j \cdot Av_i = 0\,, \quad h_j \cdot a = 0\,, \quad i \in I(x)\,.$$

*Proof:* Let $x \in \mathcal{O}_\mathcal{S}$ as above and suppose $x = \sum_{i\in I(x)} \alpha_i v_i$, where $\sum_{i\in I(x)} \alpha_i = 1$ and $\alpha_i > 0$. By assumption, there exists $j \in I \setminus I(x)$

$$h_j \cdot (Ax+a) = h_j \cdot \sum_{i\in I(x)} \alpha_i (Av_i + a) = 0\,.$$

Also by assumption, $h_j \cdot (Av_i + a) \leq 0$, $i \in I(x)$. Since $\alpha_i > 0$ it follows

$$h_j \cdot (Av_i + a) = 0\,, \quad i \in I(x)\,. \tag{12}$$

Since $0 \in I(x)$ and $v_0 = 0$ we obtain $h_j \cdot Av_i = 0$, $h_j \cdot a = 0$, $i \in I(x)$. ■

The previous algebraic results lead to a remarkable property on the placement of equilibria in $\mathcal{S}$: under the assumption that the necessary conditions of Lemma 6 and Theorem 7 hold, open-loop equilibria can only appear on the boundary of $\mathcal{S}$.

*Theorem 15:* Suppose that Assumption 8 holds. Also suppose $Av_i+a \in \mathcal{C}_i$ for $i \in \{0,\ldots,n\}$ and $\mathcal{B} \cap \text{cone}(\mathcal{O}_\mathcal{S}) \neq \mathbf{0}$. If $\mathcal{E}_\mathcal{S} \neq \emptyset$, then $\mathcal{E}_\mathcal{S} \subset \text{rb}(\mathcal{O}_\mathcal{S}) \subset \partial\mathcal{S}$.

*Proof:* Suppose by way of contradiction there is $\overline{x} \in \mathcal{S}^\circ$ such that $A\overline{x} + a = 0$. By Lemma 9, $\overline{x} \in \text{ri}(\mathcal{O}_\mathcal{S})$.

First, suppose $\dim(\mathcal{E}_\mathcal{S}) = 0$ and let $b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_\mathcal{S})$. Since $\dim(\mathcal{O}_\mathcal{S}) \geq 1$, w.l.o.g., at least one pair of vertices of $\mathcal{O}_\mathcal{S}$, say $(o_1, o_2)$, satisfy $Ao_1 + a = \eta_1 b$ and $Ao_2 + a = \eta_2 b$ with $\eta_1 < 0$ and $\eta_2 > 0$. Since $b \in \mathcal{C}(o_1)$,

$$h_j \cdot b \leq 0\,, \qquad j \in I \setminus I(o_1)\,.$$

By assumption

$$h_j \cdot (Ao_1 + a) = h_j \cdot (\eta_1 b) \leq 0\,, \quad j \in I \setminus I(o_1)\,.$$

Since $\eta_1 < 0$, the previous two inequalities imply $h_j \cdot b = 0$, $j \in I \setminus I(o_1)$. Equivalently we get

$$h_j \cdot (Ao_1 + a) = 0\,,\ j \in I \setminus I(o_1)\,.$$

Then by Lemma 14 we get

$$h_j \cdot Av_i = 0\,, h_j \cdot a = 0\,, \quad i \in I(o_1)\,, j \in I \setminus I(o_1)\,. \tag{13}$$

Suppose w.l.o.g. $I(o_1) = \{0,1,\ldots,q\}$ for some $1 \leq q \leq n-1$ (note that $0 \in I(o_1)$ by (A4); $q < n$ since $\dim(\mathcal{O}_\mathcal{S}) \geq 1$ by (A3); and $q \geq 1$, otherwise $o_1 = v_0$ and $\overline{x} \in \mathcal{S}^\circ$ together imply $\mathcal{O}_\mathcal{S} \cap \mathcal{F}_0 \neq \emptyset$, a contradiction to (A4)). Now write (13) using (5). This yields (7) with two diagonal blocks $A_{11} \in \mathbb{R}^{q\times q}$ and $A_{22} \in \mathbb{R}^{(n-q)\times(n-q)}$. Thus, $p = 0$ in (7). This contradicts Lemma 12.

Second, suppose $\dim(\mathcal{E}_\mathcal{S}) = \kappa_0$ with $\kappa_0 > 0$. Then

$$h_j \cdot (A\epsilon_k + a) = 0\,, \quad j \in I \setminus I(\epsilon_k)\,, k \in I_{\mathcal{E}_\mathcal{S}}\,,$$

where $I \setminus I(\epsilon_k) \neq \emptyset$ by Lemma 9. By Lemma 14 we have

$$h_j \cdot Av_i = 0\,, h_j \cdot a = 0\,, \quad i \in I(\epsilon_k)\,,\ j \in I \setminus I(\epsilon_k)\,, k \in I_{\mathcal{E}_\mathcal{S}}\,. \tag{14}$$

Suppose we order $\{0,\ldots,n\}$ according to exclusive members of $\{I(\epsilon_j)\}$. Now write (14) using (5). This yields (7) with $p = \kappa_0$ and $\text{ord}(\Gamma_{ii}) \geq 1$, $i = 2,\ldots,\kappa_0+2$. This contradicts Lemma 12. ■

*Remark 16:* Theorem 15 extends to the case when an affine feedback $u = Kx+g$ is applied to the system (1). For then we obtain the closed-loop system $\dot{x} = (A+BK)x + a + Bg = \tilde{A}x + \tilde{a}$, and the analysis can be repeated for the sets $\tilde{\mathcal{O}}$ and $\tilde{\mathcal{E}}$. We conclude that using any affine feedback that solves the invariance conditions and under Assumption 8, closed-loop equilibria can only appear on the boundary of $\mathcal{S}$.

*Corollary 17:* Suppose that Assumption 8 holds. Also suppose $Av_i+a \in \mathcal{C}_i$ for $i \in \{0,\ldots,n\}$ and $\mathcal{B} \cap \text{cone}(\mathcal{O}_\mathcal{S}) \neq \mathbf{0}$. Then $\dim(\mathcal{E}) \leq \dim(\mathcal{O}) - 1$.

In Theorem 15 we showed that the set of equilibria $\mathcal{E}_\mathcal{S}$ lies in the relative boundary of $\mathcal{O}_\mathcal{S}$. In the following we show further that $\mathcal{E}_\mathcal{S}$ is indeed a face of $\mathcal{O}_\mathcal{S}$.

*Theorem 18:* Suppose that Assumption 8 holds. Also suppose $Av_i+a \in \mathcal{C}_i$ for $i \in \{0,\ldots,n\}$ and $\mathcal{B} \cap \text{cone}(\mathcal{O}_\mathcal{S}) \neq \mathbf{0}$. If $\mathcal{E}_\mathcal{S} \neq \emptyset$, then $\mathcal{E}_\mathcal{S} = \text{co}\{o_1,\ldots,o_{\kappa_0+1}\}$, a $\kappa_0$-dimensional face of $\mathcal{O}_\mathcal{S}$, where $0 \leq \kappa_0 < \kappa$.

*Proof:* Suppose $\mathcal{E}_\mathcal{S} \neq \emptyset$ but is not a face of $\mathcal{O}_\mathcal{S}$. By Theorem 15, $\mathcal{E}_\mathcal{S} \subset \text{rb}(\mathcal{O}_\mathcal{S})$. Hence, $\mathcal{E}_\mathcal{S}$ can be expressed as

$$\mathcal{E}_\mathcal{S} = \text{co}\{\epsilon_1,\ldots,\epsilon_{\kappa_0+1}\} \subset \text{co}\{o_1,\ldots,o_p\}$$

where $2 \leq p < \kappa+1$. Define $\hat{I}(\mathcal{E}_\mathcal{S}) := \{1,\ldots,p\}$ as the minimal index set such that for all $x \in \mathcal{E}_\mathcal{S}$, $x \in \text{co}\{o_i \mid i \in \hat{I}(\mathcal{E}_\mathcal{S})\}$. Since $\mathcal{E}_\mathcal{S}$ is on a face but not an entire face of $\mathcal{O}_\mathcal{S}$ and since the faces of $\mathcal{O}_\mathcal{S}$ are simplices, at least one of the vertices of $\mathcal{E}_\mathcal{S}$, say $\epsilon_1$, is not a vertex of $\mathcal{O}_\mathcal{S}$. Hence, there exist $2 \leq q \leq p$ and $\alpha_i \in (0,1)$ with $\sum_{i=1}^q \alpha_i = 1$ such that $\epsilon_1 = \sum_{i=1}^q \alpha_i o_i$. Let $y(o_i) := Ao_i + a = \lambda_i B$ with $\lambda_i \in \mathbb{R}$, $i \in I_{\mathcal{O}_\mathcal{S}}$. Then,

$$y(\epsilon_1) = 0 = \sum_{i=1}^q \alpha_i y(o_i) = \left(\sum_i \alpha_i \lambda_i\right) B\,.$$

Thus $\sum_{i=1}^q \alpha_i \lambda_i = 0$. Since $\alpha_i > 0$, either $\lambda_i = 0$ for all $i \in \{1,\ldots,q\}$, or there exists at least one pair of vertices of $\mathcal{O}_\mathcal{S}$, say $(o_1, o_2)$, such that $\lambda_1 < 0$ and $\lambda_2 > 0$. For the first case, define $\widehat{\mathcal{O}_\mathcal{S}} = \text{co}\{o_1,\ldots,o_q\}$. Then $\widehat{\mathcal{O}_\mathcal{S}} \subset \mathcal{E}_\mathcal{S}$. This means $\epsilon_1$, a vertex of $\mathcal{E}_\mathcal{S}$, is expressible as a convex combination of points in $\mathcal{E}_\mathcal{S}$, a contradiction. For the second case, we have $\lambda_1 < 0$ and $\lambda_2 > 0$. If $\lambda_{p+1} = 0$, then $o_{p+1} \in \mathcal{E}_\mathcal{S}$, and $p+1 \in \widehat{I}(\mathcal{E}_\mathcal{S})$, a contradiction. Therefore, assume w.l.o.g. that $\lambda_{p+1} > 0$. Then there exists $x \in \text{co}\{o_1, o_{p+1}\}$ s.t. $Ax + a = 0$. Hence, $p+1 \in \widehat{I}(\mathcal{E}_\mathcal{S})$, a contradiction. ■

## VI. Reach Controllability

In this section we define the notion of reach controllability. Simply put, this notion describes the condition when a velocity vector $0 \neq b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_\mathcal{S})$ can be injected into the system at vertices of $\mathcal{S}$ that contribute to the generation of equilibria on $\mathcal{O}_\mathcal{S}$. For the rest of the paper and without loss of generality we make the following assumption.

*Assumption 19:* If the invariance conditions (2) are solvable, then they hold for (1) with $u = 0$.

This assumption is made to avoid complexity of the notations only. Indeed, by Lemma 6, solvability of the invariance conditions is a necessary condition for solvability of RCP by continuous state feedback. To achieve Assumption 19 one applies an affine feedback transformation $u = Kx + g + w$ such that $(A + BK)v_i + (Bg + a) \in \mathcal{C}_i$ for $i = 0, \ldots, n$ and $w$ is the new exogenous input. In this manner, there is no loss of generality in assuming that the invariance conditions already hold for the presented system (1) with $u = 0$.

*Definition 20:* Suppose $\mathcal{B} \cap \text{cone}(\mathcal{O}_\mathcal{S}) \neq \mathbf{0}$. We say the triple $(A, B, a)$ is *reach controllable* if either $\mathcal{E}_\mathcal{S} = \emptyset$; or $\mathcal{E}_\mathcal{S} = \text{co}\{o_1, \ldots, o_{\kappa_0+1}\}$ with $0 \leq \kappa_0 < \kappa$, and for each $k \in I_{\mathcal{E}_\mathcal{S}}$, there exists $i \in I(o_k)$ and $u_i > 0$ such that $Av_i + Bu_i + a \in \mathcal{C}_i$.

We now explain reach controllability in informal terms. Consider the open-loop system $\dot{x} = Ax + a$ whose equilibria are given by $\mathcal{E}_\mathcal{S}$. By Theorem 15 we know these equilibria lie on a face of $\mathcal{S}$. In this situation, RCP is solvable if we are able to "push" the equilibria off the face of $\mathcal{S}$ by help of an affine feedback. Thus, for any single equilibrium $\overline{x}$, one necessary condition, as we will later show, is to be able to inject a non-zero velocity component $b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_\mathcal{S})$ in at least one of the vertices of $\mathcal{S}$ whose convex hull contains $\overline{x}$; that is, one of the vertices $v_i$ with $i \in I(\overline{x})$. At the same time, the injection of this $b$ component should not induce a violation of the invariance conditions at $v_i$. By convexity of affine feedbacks, a $b$ component will appear in the velocity vector at $\overline{x}$. This in turn has the effect to eliminate the equilibrium at $\overline{x}$. Of course, other equilibria may appear. The restriction that one must use $b \in \mathcal{B} \cap \mathcal{O}_\mathcal{S}$ is a consequence of Lemma 21 below, and this guarantees that no further equilibria appear as a result of applying the newly made feedback to the open-loop system. In sum, the notion of reach controllability captures that there exists an affine feedback that "pushes" all equilibria of the open-loop system off $\mathcal{S}$ while also preserving the invariance conditions.

We now present two properties of reach controllability. First, we show that reach controllability is intrinsic in the sense that it is not affected by affine feedback transformations that preserve the invariance conditions. Second, we relate reach controllability to the existence of a coordinate transformation that decomposes the dynamics to those that contribute to open-loop equilibria and quotient dynamics. First, we need two technical results that provide insight on the allowable velocity vectors at vertices of $\mathcal{O}_\mathcal{S}$.

*Lemma 21:* Suppose that Assumption 8 holds. Also suppose $Av_i + a \in \mathcal{C}_i$ for $i \in \{0, \ldots, n\}$. If $\exists\ 0 \neq b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_\mathcal{S})$, then for each $k \in I_{\mathcal{O}_\mathcal{S}}$,

$$Ao_k + a = \lambda_k b \,,$$

where $\lambda_k \geq 0$.

*Proof:* By Theorem 18, if $\mathcal{E}_\mathcal{S} \neq \emptyset$, then $\mathcal{E}_\mathcal{S} = \text{co}\{o_1, \ldots, o_{\kappa_0+1}\}$. Then the result is obviously true for vertices of $\mathcal{O}_\mathcal{S}$ also in $\mathcal{E}_\mathcal{S}$ because $Ao_k + a = 0$ for $k \in I_{\mathcal{E}_\mathcal{S}}$. Second, consider vertices of $\mathcal{O}_\mathcal{S}$ that are not vertices of $\mathcal{E}_\mathcal{S}$ (including the case when $\mathcal{E}_\mathcal{S} = \emptyset$). For these the coefficients $\lambda_k$, $k \in I_{\mathcal{O}_\mathcal{S}} \setminus I_{\mathcal{E}_\mathcal{S}}$ must all have the same sign; otherwise, by convexity there is $x \in \text{co}\{o_k \mid k \in I_{\mathcal{O}_\mathcal{S}} \setminus I_{\mathcal{E}_\mathcal{S}}\}$ such that $Ax + a = 0$, which implies $x \in \mathcal{E}_\mathcal{S}$, a contradiction. Now if each $\lambda_k > 0$ for $k \in I_{\mathcal{O}_\mathcal{S}} \setminus I_{\mathcal{E}_\mathcal{S}}$, we are done. Suppose instead $\lambda_k < 0$ for $k \in I_{\mathcal{O}_\mathcal{S}} \setminus I_{\mathcal{E}_\mathcal{S}}$. By assumption

$$h_j \cdot b \leq 0 \,, \qquad j \in I \setminus I(o_k),\ k \in I_{\mathcal{O}_\mathcal{S}} .$$

Also

$$h_j \cdot (Ao_k + a) = h_j \cdot (\lambda_k b) \leq 0 \,, \quad j \in I \setminus I(o_k),\ k \in I_{\mathcal{O}_\mathcal{S}} \setminus I_{\mathcal{E}_\mathcal{S}} .$$

Since $\lambda_k < 0$, the previous two inequalities imply

$$h_j \cdot b = h_j \cdot (Ao_k + a) = 0 \,,\ j \in I \setminus I(o_k),\ k \in I_{\mathcal{O}_\mathcal{S}} \setminus I_{\mathcal{E}_\mathcal{S}} . \tag{15}$$

Also $h_j \cdot (Ao_k + a) = 0$, $j \in I \setminus I(o_k)$, $k \in I_{\mathcal{E}_\mathcal{S}}$. Then by Lemma 14,

$$h_j \cdot Av_i = 0 \,,\ h_j \cdot a = 0 \,,\ i \in I(o_k) \,,\ j \in I \setminus I(o_k) \,,\ k \in I_{\mathcal{O}_\mathcal{S}} . \tag{16}$$

By Lemma 13, this is a contradiction. ■

*Corollary 22:* Suppose that Assumption 8 holds. Also suppose $Av_i + a \in \mathcal{C}_i$ for $i \in \{0, \ldots, n\}$. If $\exists\ 0 \neq b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_\mathcal{S})$, then for all $x \in \mathcal{O}$, $Ax + a = \lambda(x) b$, where $\lambda(x) \geq 0$.

*Theorem 23:* Suppose that Assumptions 8 and 19 hold and $\exists 0 \neq b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_\mathcal{S})$. Then, reach controllability is invariant under affine feedback transformations which preserve the invariance conditions.

Now we explore the second property of reach controllability: that it suggests a decomposition of the dynamics into those contributing to open-loop equilibria and quotient dynamics. It is noted that a complete geometric characterization of reach controllability has not yet been obtained, but the following result gives a first evidence that one may exist.

*Lemma 24:* Suppose $Av_i + a \in \mathcal{C}_i$ for $i \in \{0, \ldots, n\}$. Also suppose there exists $\overline{x} \in \mathcal{E}_\mathcal{S}$ such that $\overline{x} \in \text{co}\{v_0, \ldots, v_q\}$. Then there exists a coordinate transformation $z = T^{-1}x$ such that the transformed system has the form

$$\dot{z} = \begin{bmatrix} A_1 & \star \\ 0 & A_2 \end{bmatrix} z + \begin{bmatrix} a_1 \\ 0 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u \,, \tag{17}$$

where $A_1 \in \mathbb{R}^{q \times q}$, $a_1 \in \mathbb{R}^q$, $b_1 \in \mathbb{R}^q$, $A_2 \in \mathbb{R}^{(n-q)\times(n-q)}$, and $b_2 \in \mathbb{R}^{n-q}$ for $q > 0$.

## VII. Main Result

The following result provides constructive necessary and sufficient conditions for solvability of RCP in the studied setting.

*Theorem 25:* Consider the system (1) and suppose Assumption 8 and 19 hold. We have $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by affine feedback if and only if

(i) The invariance conditions (2) are solvable.
(ii) $\mathcal{B} \cap \text{cone}(\mathcal{O}_\mathcal{S}) \neq \mathbf{0}$.
(iii) $(A, B, a)$ is reach controllable.

To prove the theorem we first require a technical lemma on the selection of $b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_\mathcal{S})$.

*Lemma 26:* Suppose that Assumption 8 holds. Also suppose $Av_i + a \in \mathcal{C}_i$ for $i \in \{0, \ldots, n\}$. If $\exists\, 0 \neq b \in \mathcal{B} \cap \mathrm{cone}(\mathcal{O}_\mathcal{S})$, then $-b \notin \mathcal{B} \cap \mathrm{cone}(\mathcal{O}_\mathcal{S})$.

*Proof:* Suppose not. Then for all $k \in I_{\mathcal{O}_\mathcal{S}}$

$$\begin{aligned} h_j \cdot b &\leq 0\,, \quad j \in I \setminus I(o_k) \\ h_j \cdot (-b) &\leq 0\,, \quad j \in I \setminus I(o_k)\,. \end{aligned}$$

This implies $h_j \cdot b = h_j \cdot (Ao_k + a) = 0$, $j \in I \setminus I(o_k)$, $k \in I_{\mathcal{O}_\mathcal{S}}$. By Lemma 14,

$$h_j \cdot Av_i = 0\,, h_j \cdot a = 0\,, i \in I(o_k)\,, j \in I \setminus I(o_k)\,, k \in I_{\mathcal{O}_\mathcal{S}}\,.$$

By Lemma 13 this is a contradiction. ∎

*Proof:* [Proof of Theorem 25] ($\Longrightarrow$) Since the invariance conditions are solvable, by Assumption 19, they are solvable using $u = 0$. Now if $\mathcal{E}_\mathcal{S} = \emptyset$, by Theorem 4, $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by affine feedback $u(x) = 0$. Alternatively, if $\mathcal{E}_\mathcal{S} \neq \emptyset$, then by Theorem 18, $\mathcal{E}_\mathcal{S} = \mathrm{co}\{o_1, \ldots, o_{\kappa_0+1}\}$ with $0 \leq \kappa_0 < \kappa$. Following Lemma 26, w.l.o.g. we can take $B = b \in \mathcal{B} \cap \mathrm{cone}(\mathcal{O}_\mathcal{S})$. By reach controllability, for each $k \in I_{\mathcal{E}_\mathcal{S}}$, there exist $i_k \in I(o_k)$ and $u_{i_k} > 0$ such that

$$Av_{i_k} + Bu_{i_k} + a \in \mathcal{C}_{i_k}\,. \tag{18}$$

Select $i_k \in I(o_k)$ and $u_{i_k} > 0$ as above. Set $u_i = 0$ for the remaining vertices of $\mathcal{S}$. Form the associated affine feedback $u(x) = Kx + g$ and let $y(x) := Ax + Bu(x) + a$. Consider any $x \in \mathcal{E}_\mathcal{S}$. There exist $\xi_k > 0$ with $\sum_k \xi_k = 1$ such that $x = \sum_k \xi_k o_k$. Also for each $o_k$, there exist $\alpha_{j_k} > 0$ with $\sum_{j_k \in I(o_k)} \alpha_{j_k} = 1$ such that $o_k = \sum_{j_k \in I(o_k)} \alpha_{j_k} v_{j_k}$. By construction, for each $k \in I_{\mathcal{E}_\mathcal{S}}$ there exists $i_k \in I(o_k)$ such that $u(v_{i_k}) > 0$ and the remaining controls are zero. Then by convexity

$$y(x) = \sum_k \xi_k (Bu(o_k)) = \sum_k \sum_{j_k \in I(o_k)} \xi_k B \alpha_{j_k} u(v_{j_k}) =: \epsilon b\,,$$

where $\epsilon > 0$. Thus, $y(x) \neq 0$ for all $x \in \mathcal{E}_\mathcal{S}$. Next consider $\mathcal{O}_\mathcal{S} \backslash \mathcal{E}_\mathcal{S}$. We claim $y(x) \neq 0$ for all $x \in \mathcal{O}_\mathcal{S} \backslash \mathcal{E}_\mathcal{S}$. Suppose not. Then there is $\overline{x} = \sum_{i \in I(\overline{x})} \alpha_i v_i$ with $\alpha_i > 0$ and $\sum_i \alpha_i = 1$ such that $y(\overline{x}) = A\overline{x} + Bu(\overline{x}) + a = 0$. That is $A\overline{x} + a = -Bu(\overline{x}) = -b \sum_{i \in I(\overline{x})} \alpha_i u_i =: -\gamma b$. Note that $\gamma \neq 0$, otherwise $\overline{x} \in \mathcal{E}_\mathcal{S}$, a contradiction. Also, $\gamma$ cannot be negative since $\alpha_i > 0$ and $u_i \geq 0$ by construction. Finally, suppose $\gamma > 0$. Then there must be $i \in I_{\mathcal{O}_\mathcal{S}}$ such that $Ao_i + a = \lambda_i b$ with $\lambda_i < 0$. This contradicts Lemma 21. We conclude $y(x) \neq 0$ for all $x \in \mathcal{O}_\mathcal{S}$. Finally, by (18) and the fact that $u_i = 0$ for the remaining vertices, the invariance conditions hold with $u(x)$. By Theorem 4, $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ using $u(x)$.

($\Longleftarrow$) Suppose $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by affine feedback $u(x) = Kx + g$. By Theorem 7, $\mathcal{B} \cap \mathrm{cone}(\mathcal{O}_\mathcal{S}) \neq \mathbf{0}$. Also, by Theorem 4, $Av_i + Bu(v_i) + a \in \mathcal{C}_i$, $i \in \{0, \ldots, n\}$. Hence, by Assumption 19, $u = 0$ solves the invariance conditions. If $\mathcal{E}_\mathcal{S} = \emptyset$, then $(A, B, a)$ is reach controllable. Alternatively, if $\mathcal{E}_\mathcal{S} \neq \emptyset$, then by Theorem 18 $\mathcal{E}_\mathcal{S} = \mathrm{co}\{o_1, \ldots, o_{\kappa_0+1}\}$ where $0 \leq \kappa_0 < \kappa$. Following Lemma 26, w.l.o.g. we can take $B = b \in \mathcal{B} \cap \mathrm{cone}(\mathcal{O}_\mathcal{S})$. Suppose $(A, B, a)$ is not reach controllable. Then there exists $k_0 \in I_{\mathcal{E}_\mathcal{S}}$ such that for all $i \in I(o_{k_0})$, $Av_i + Bu(v_i) + a \in \mathcal{C}_i$ implies $u(v_i) \leq 0$. Since $u(o_{k_0}) = \sum_{i \in I(o_{k_0})} \alpha_i u(v_i)$ for some $\alpha_i \in (0, 1)$, we obtain $u(o_{k_0}) \leq 0$. Thus, $Ao_{k_0} + Bu(o_{k_0}) + a = \xi_{k_0} b$ with $\xi_{k_0} < 0$. It follows that $Ao_k + Bu(o_k) + a = \xi_k b$ with $\xi_k < 0$ for all $k \in I_{\mathcal{O}_\mathcal{S}}$ (for otherwise by convexity there is $x \in \mathrm{co}\{o_k \mid k \in I_{\mathcal{O}_\mathcal{S}}\}$ such that $Ax + Bu(x) + a = 0$, a contradiction. Because $Ao_k + Bu(o_k) + a \in \mathcal{C}(o_k)$, we get $-b \in \mathrm{cone}(\mathcal{O}_\mathcal{S})$, a contradiction with Lemma 26. ∎

We present an example of Theorem 25 where reach controllability fails.

*Example 27:* Consider a simplex $\mathcal{S} = \mathrm{co}\{v_0, \ldots, v_4\}$, where $v_0 = 0$ and $v_i = e_i$, the $i$th Euclidean coordinate. Consider the following affine system

$$\dot{x} = \begin{bmatrix} -1 & 0 & 1 & 0 \\ -3 & -6 & -3 & -2 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} x + \begin{bmatrix} -3 \\ -5 \\ 8 \\ 4 \end{bmatrix} u + \begin{bmatrix} 0 \\ 3 \\ 1 \\ 0 \end{bmatrix} \tag{19}$$

Let $b := (-3, -5, 8, 4)$. We make several observations. First, $\mathcal{B} \cap \mathrm{cone}\mathcal{S} = \mathbf{0}$ because $h_1 \cdot b = 3 > 0$ and $h_3 \cdot (-b) = 8 > 0$, so Theorem 5 cannot be applied. Second, it can be verified that $\mathcal{O} := \left\{ x \in \mathbb{R}^n \mid x_1 = x_2 = x_4 + \frac{1}{4}, x_3 = -2x_4 + \frac{1}{4} \right\}$. Setting $x_4 = 0$ in the defining equations for $\mathcal{O}$, we get $o_1 := \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0\right)$. Setting $x_3 = 0$, we get $o_2 := \left(\frac{3}{8}, \frac{3}{8}, 0, \frac{1}{8}\right)$. Thus, $\mathcal{O}_\mathcal{S} = \mathrm{co}\{o_1, o_2\}$ where $o_1 = \frac{1}{4}v_0 + \frac{1}{4}v_1 + \frac{1}{4}v_2 + \frac{1}{4}v_3 \in \mathcal{F}_4$ and $o_2 = \frac{1}{8}v_0 + \frac{3}{8}v_1 + \frac{3}{8}v_2 + \frac{1}{8}v_4 \in \mathcal{F}_3$. Also we have that $Ao_1 + a = 0$ and $Ao_2 + a \neq 0$, so $\mathcal{E}_\mathcal{S} = \{o_1\}$. We observe that $\dim(\mathcal{O}_\mathcal{S}) = 1$, $\dim(\mathcal{E}_\mathcal{S}) = 0$, $\mathcal{O}_\mathcal{S} \cap \mathcal{F}_0 = \emptyset$, and $\mathcal{O}_\mathcal{S} \cap \mathcal{S}^\circ \neq \emptyset$. Because $o_1 \in \mathcal{F}_4$ and $o_2 \in \mathcal{F}_3$, we have $\mathrm{cone}(\mathcal{O}_\mathcal{S}) = \left\{ y \in \mathbb{R}^n \mid h_3 \cdot y \leq 0,\ h_4 \cdot y \leq 0 \right\}$. Clearly $b \in \mathcal{B} \cap \mathrm{cone}(\mathcal{O}_\mathcal{S})$, so solvability of RCP by continuous state feedback cannot be ruled out by Theorem 7. Also it can be verified that the invariance conditions (3) are satisfied when $u = 0$, so solvability of RCP by continuous state feedback cannot be ruled out by Lemma 6. Nevertheless, for the given simplex $\mathcal{S}$ and system (19), RCP is not solvable by affine feedback. This is due to the fact that $(A, B, a)$ is not reach controllable according to the Definition 20. Indeed $Av_i + a + Bu_i \in \mathcal{C}_i$ results in $u_i = 0$ for $\forall\ i \in I(o_1)$.

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