2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

$\mathcal{L}_1$ Adaptive Output Feedback Controller for a Class of Nonlinear Systems

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***Abstract*— This paper presents an extension of the $\mathcal{L}_1$ adaptive output feedback controller to a class of nonlinear systems where nonlinearities satisfy a semiglobal Lipschitz condition. The control algorithm consists of an output predictor which is designed to predict the system's output with arbitrary accuracy, and a control law which is used to control the predictor output instead of the actual system's output. It is shown that the adaptive output feedback controller ensures uniformly bounded output tracking for the system. The performance bound can be systematically improved by reducing the step size of integration. Numerical simulation results are provided to illustrate the algorithm's performance.**

I. Introduction

Output feedback control design for uncertain nonlinear systems is a challenging task. Most of the existing output feedback results impose restrictive assumptions on nonlinearities. For example, in [1] [2], nonlinearities can only depend on the measurement y; In [3], nonlinearities linearly depend on the unmeasured states; In [4], the nonlinear systems satisfy a global Lipschitz condition; In [5], it considers linear growth condition with a constant growth rate for nonlinear functions.

The separation principle based output feedback scheme is a popular approach to address output feedback control of nonlinear systems, especially those involve high gain observer design. The separation theorem for the output feedback control with high-gain observer was proved in [6] where it is shown that the trajectories of the state variables under output feedback come arbitrarily close to the ones under state feedback, as the observer gain becomes high enough. In contrast, [5] proposed a nonseparation principle paradigm for output feedback control by using a feedback domination design method to construct a linear output compensator. Another popular approach is the internal model based output feedback control scheme [7] which used to handle the output regulation problem with desired trajectories generated by an exosystem.

This paper extends the adaptive output feedback control design of [8] to a class of nonlinear systems in the presence of unknown state-dependent and time-varying nonlinearities. It considers that the nonlinear function satisfies a semiglobal Lipschitz condition. Based on this Lipschitz condition and additional assumptions, we prove that the difference between the output predictor and the actual system's output is bounded, and this bound can be arbitrarily small by reducing the step size of integration. In the classical separation principle based scheme, the control law is designed to control the actual system while $\mathcal{L}_1$ adaptive output feedback control law proposed in this paper is to control the output predictor.

The paper is organized as follows. Section II gives the problem formulation. In Section III, the novel $\mathcal{L}_1$ adaptive control architecture is presented. In Section IV, some preliminary results are developed towards the analysis of the $\mathcal{L}_1$ adaptive controller. Uniform performance bounds are presented in Section V. In Section VI, simulation results are presented, while Section VII concludes the paper. Unless otherwise mentioned, $||\cdot||$ will be used for the 2-norm of the vector.

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II. Problem Formulation

Consider the following single–input single–output (SISO) system

$$
\begin{aligned}
\dot{x}(t) &= A_m x(t) + b_m\Big(f(x,t) + a u(t)\Big) + \sigma(t)\,, \\
y(t) &= c_m^\top x(t)\,, \quad y(0) = y_0\,,
\end{aligned}
\tag{1}
$$

where $x(t) \in \mathbb{R}^n$ is the system state vector (unmeasurable), $u(t) \in \mathbb{R}$ is the input, $y(t) \in \mathbb{R}$ is the system output, A_m is a known $n \times n$ Hurwitz matrix, $b_m, c_m \in \mathbb{R}^n$ are known constant vectors, a is a positive constant, zeros of $c_m^\top(s\mathbb{I} - A_m)b_m$ lie in the open left-half s plane, $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is an unknown nonlinear function, and $\sigma(t) \in \mathbb{R}^n$ are unknown disturbances.

Assumption 1: [Semiglobal Lipschitz condition on x] For any $\delta > 0$, there exist $L(\delta) > 0$ and $B > 0$ such that

$$|f(x,t) - f(\bar{x},t)| \le L(\delta)\|x - \bar{x}\|_\infty\,, \quad |f(0,t)| \le B\,,$$

for all $\|x\|_\infty \le \delta$ and $\|\bar{x}\|_\infty \le \delta$ uniformly in u and t.

Assumption 2: There exist $B_\sigma > 0$ such that

$$||\sigma(t)|| \le B_\sigma$$

for all $t \ge 0$, where the numbers B_σ can be arbitrarily large. The control objective is to design an adaptive output feedback controller $u(t)$ such that the system output $y(t)$ tracks the reference system output $y_{des}(t)$ described by

$$
\begin{aligned}
\dot{x}_{des}(t) &= A_m x_{des}(t) + b_m \bar{k}_g r(t)\,, \\
y_{des}(t) &= c_m^\top x_{des}(t)\,,
\end{aligned}
\tag{2}
$$

where $\bar{k}_g = -(c_m^\top A_m^{-1} b_m)^{-1}$, $r(t)$ is a given bounded reference input signal with $r(t) \le \|r\|_{\mathcal{L}_\infty}$.

III. $\mathcal{L}_1$ ADAPTIVE OUTPUT FEEDBACK CONTROLLER

We consider the following output predictor

$$\begin{aligned}\dot{\hat{x}}(t) &= A_m\hat{x}(t) + b_m a u(t) + \hat{\sigma}(t)\,,\\ \hat{y}(t) &= c_m^\top \hat{x}(t)\,,\quad \hat{y}(0) = y_0\,,\end{aligned} \tag{3}$$

where $\hat{\sigma}(t) \in \mathbb{R}^n$ is the vector of adaptive parameters. We can find matrix $B_{um} \in \mathbb{R}^{n\times(n-1)}$ such that $b_m^\top B_{um} = 0$ and $rank([b_m \quad B_{um}]) = n$. Then, equation (3) can be written as

$$\begin{aligned}\dot{\hat{x}}(t) &= A_m\hat{x}(t) + b_m(au(t) + \hat{\sigma}_1) + B_{um}\hat{\sigma}_2\,,\\ \hat{y}(t) &= c_m^\top \hat{x}(t)\,,\quad \hat{y}(0) = y_0\,,\end{aligned} \tag{4}$$

where $\hat{\sigma}_1(t)$ represents the matched component of the uncertainties $\hat{\sigma}(t)$, and $\hat{\sigma}_2(t)$ represents the unmatched component.

Letting $\tilde{y}(t) = \hat{y}(t) - y(t)$, the update law for $\hat{\sigma}(t)$ is given by

$$\begin{aligned}\hat{\sigma}(t) &= \hat{\sigma}(iT), \quad t \in [iT, (i+1)T)\,,\\ \hat{\sigma}(iT) &= -\Phi^{-1}(T)\mu(iT)\,, \quad i = 0, 1, 2, \cdots\,,\end{aligned} \tag{5}$$

where $\Phi(T) = \int_0^T \mathrm{e}^{\Lambda A_m \Lambda^{-1}(T-\tau)}\Lambda d\tau$ and

$$\mu(iT) = \mathrm{e}^{\Lambda A_m \Lambda^{-1} T}\mathbf{1}_1\tilde{y}(iT)\,, \quad i = 0, 1, 2, 3, \cdots\,. \tag{6}$$

The control signal is defined as follows

$$\begin{bmatrix}\hat{\sigma}_1(t)\\ \hat{\sigma}_2(t)\end{bmatrix} = \begin{bmatrix} b_m & B_{um}\end{bmatrix}^{-1}\hat{\sigma}(t)\,, \tag{7}$$

$$\begin{aligned}u(s) &= k_g r(s) - C_1(s)\frac{\hat{\sigma}_1(s)}{a}\\ &\quad -C_2(s)M(s)\hat{\sigma}_2(s)\,,\end{aligned} \tag{8}$$

where $r(s)$ is the Laplace transformation of the reference signal $r(t)$, $k_g = -\frac{1}{a}(c_m^\top A_m^{-1} b_m)^{-1}$, $M(s) = \frac{c_m^\top (s\mathbb{I}-A_m)^{-1}B_{um}}{c_m^\top(s\mathbb{I}-A_m)^{-1}b_m a}$, both $C_1(s)$ and $C_2(s)$ are low pass filters with unit DC gain, $C_2(s)$ needs to ensure that $\frac{C_2(s)c_m^\top(s\mathbb{I}-A_m)^{-1}B_{um}}{c_m^\top(s\mathbb{I}-A_m)^{-1}b_m}$ is a proper transfer function. $\hat{\sigma}_1(s)$ and $\hat{\sigma}_2(s)$ are Laplace transformations of matched uncertainties $\hat{\sigma}_1(t)$ and unmatched uncertainties $\hat{\sigma}_2(t)$ respectively. The $\mathcal{L}_1$ adaptive controller consists of (3), (5) and (8).

IV. PRELIMINARIES FOR THE MAIN RESULT

Since A_m is Hurwitz, there exists a positive-definite matrix $P = P^\top > 0$ that satisfies the following Lyapunov equation

$$A_m^\top P + PA_m = -Q,\ Q > 0\,.$$

From the properties of P, there exits a non-singular matrix $\sqrt{P}$ such that

$$P = (\sqrt{P})^\top\sqrt{P}\,.$$

Given the vector $c_m^\top(\sqrt{P})^{-1}$, let D be a $(n-1)\times n$ matrix that contains the null space of $c_m^\top(\sqrt{P})^{-1}$, i.e.,

$$D(c_m^\top(\sqrt{P})^{-1})^\top = 0\,. \tag{9}$$

Then we define

$$\Lambda = \begin{bmatrix} c_m^\top \\ D\sqrt{P}\end{bmatrix}. \tag{10}$$

and let

$$\alpha = \lambda_{max}(\Lambda^{-\top}P\Lambda^{-1})\Delta^2 \tag{11}$$

where $\Delta = \frac{2\|\Lambda^{-\top}Pb_m\|(L(\gamma_x)\gamma_x + B)}{\lambda_{min}(\Lambda^{-\top}Q\Lambda^{-1})} + \frac{2\|\Lambda^{-\top}P\|B_\sigma}{\lambda_{min}(\Lambda^{-\top}Q\Lambda^{-1})}$, γ_x is a positive constant, $L(\gamma_x)$ is a Lipschitz constant. Consider the inverse of Λ as

$$\Lambda^{-1} = \begin{bmatrix}\varrho_1 & \varrho_2\end{bmatrix}\,, \tag{12}$$

where ϱ_1 represents the first column of Λ^{-1}, and ϱ_2 represents the rest columns.

Further let

$$\begin{aligned}\varrho_3(s) &= C_1(s)\frac{1}{a}\mathbf{1}_1^\top\begin{bmatrix} b_m & B_{um}\end{bmatrix}^{-1}\\ &\quad +C_2(s)M(s)S\begin{bmatrix} b_m & B_{um}\end{bmatrix}^{-1}\end{aligned} \tag{13}$$

where $\mathbf{1}_1 \in \mathbb{R}^n$ is the basis vector with first element 1 and all other elements zero, $S = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & 0\\ 0 & 0 & 0 & \cdots & 1\end{bmatrix}_{(n-1)\times n}$.

The norm of ϱ_4 is given by

$$\begin{aligned}\|\varrho_4\| &= \|(s\mathbb{I}-A_m)^{-1}b_m a\|_{\mathcal{L}_1}\|\varrho_3\|_{\mathcal{L}_1}\\ &\quad +\|(s\mathbb{I}-A_m)^{-1}\|_{\mathcal{L}_1}\,.\end{aligned} \tag{14}$$

Letting

$$\mathbf{1}_1^\top e^{\Lambda A_m\Lambda^{-1}t} = \begin{bmatrix}\eta_{y_0}(t) & \bar{\eta}_{y_0}^\top(t)\end{bmatrix}\,, \tag{15}$$

$$Se^{\Lambda A_m\Lambda^{-1}t} = \begin{bmatrix}\eta_{z_0}(t) & \bar{\eta}_{z_0}(t)\end{bmatrix}\,, \tag{16}$$

where $\eta_{y_0}(t) \in \mathbb{R}$ and $\bar{\eta}_{y_0}^\top \in \mathbb{R}^{n-1}$ contain the first and 2 to n elements of the row vector $\mathbf{1}_1^\top e^{\Lambda A_m \Lambda^{-1}t}$ respectively, $\eta_{z_0}(t) \in \mathbb{R}^{(n-1)\times 1}$ and $\bar{\eta}_{z_0} \in \mathbb{R}^{(n-1)\times(n-1)}$ contain the first and 2 to n columns of the matrix $Se^{\Lambda A_m\Lambda^{-1}t}$ respectively. We further introduce the following functions

$$\beta_{y_0}(T) = \max_{t\in[0,\ T]}|\eta_{y_0}(t)|,\ \bar{\beta}_{y_0}(T) = \max_{t\in[0,\ T]}\|\bar{\eta}_{y_0}(t)\|, \tag{17}$$

$$\beta_{z_0}(T) = \max_{t\in[0,\ T]}\|\eta_{z_0}(t)\|,\ \bar{\beta}_{z_0}(T) = \max_{t\in[0,\ T]}\|\bar{\eta}_{z_0}(t)\|. \tag{18}$$

Let

$$\eta_1(T) = \int_0^T \|\mathbf{1}_1^\top\varsigma(T-\tau)\|d\tau\,, \tag{19}$$

$$\eta_2(T) = \int_0^T |\mathbf{1}_1^\top\varsigma(T-\tau)b_m|d\tau\,, \tag{20}$$

where T is any positive constant, $\varsigma(T-\tau) = \mathrm{e}^{\Lambda A_m\Lambda^{-1}(T-\tau)}\Lambda$, and further define

$$\begin{aligned}\nu(T) &= \|\phi(T)\|\sqrt{\frac{\alpha}{\lambda_{max}(P_2)}} + \eta_1(T)B_\sigma\\ &\quad +\eta_2(T)(L(\gamma_x)\gamma_x + B)\,,\end{aligned} \tag{21}$$

where $\phi(T) \in \mathbb{R}^{n-1}$ is a vector, which consists of 2 to n elements of $\mathbf{1}_1^\top\mathrm{e}^{\Lambda A_m\Lambda^{-1}T}$, and P_2 is positive definite.

Let

$$\beta_1(T) = \max_{t\in[0,\ T]} \int_0^t \|\mathbf{1}_1^\top \varsigma(t-\tau)\| d\tau\,, \quad (22)$$

$$\beta_2(T) = \max_{t\in[0,\ T]} \int_0^t |\mathbf{1}_1^\top \varsigma(t-\tau) b_m| d\tau\,, \quad (23)$$

$$\beta_3(T) = \max_{t\in[0,\ T]} \int_0^t \|S\varsigma(t-\tau)\| d\tau\,, \quad (24)$$

$$\beta_4(T) = \max_{t\in[0,\ T]} \int_0^t \|S\varsigma(t-\tau) b_m\| d\tau\,, \quad (25)$$

$$\beta_5(T) = \max_{t\in[0,\ T]} \int_0^t |\mathbf{1}_1^\top \varsigma(t-\tau)\Phi(T)\varphi(T)\mathbf{1}_1| d\tau, \quad (26)$$

$$\beta_6(T) = \max_{t\in[0,\ T]} \int_0^t \|S\varsigma(t-\tau)\Phi(T)\varphi(T)\mathbf{1}_1\| d\tau, \quad (27)$$

where $\varphi(T) = \mathrm{e}^{\Lambda A_m \Lambda^{-1} T}$ and further define

$$\begin{aligned}\gamma_{\tilde{y}} &= \beta_{y_0}(T)\nu(T) + \bar{\beta}_{y_0}(T)\sqrt{\frac{\alpha}{\lambda_{max}(P_2)}} \\ &\quad + \beta_5(T)\nu(T) + \beta_1(T)B_\sigma \\ &\quad + \beta_2(T)(L(\gamma_x)\gamma_x + B)\,, \quad (28)\\ \gamma_{\tilde{z}} &= \beta_{z_0}(T)\nu(T) + \bar{\beta}_{z_0}(T)\sqrt{\frac{\alpha}{\lambda_{max}(P_2)}} \\ &\quad + \beta_6(T)\nu(T) + \beta_3(T)B_\sigma \\ &\quad + \beta_4(T)(L(\gamma_x)\gamma_x + B)\,. \quad (29)\end{aligned}$$

For the proof of stability and uniform performance bounds, the choices of $C_1(s)$, $C_2(s)$, and integration step T together with system dynamics need to ensure that their exists γ_x such that

$$\begin{aligned}&\|\varrho_1\|\gamma_{\tilde{y}}(T) + \|\varrho_2\|\gamma_{\tilde{z}} + \|\varrho_4\|\|A_m\Lambda^{-1}\mathbf{1}_1\|\nu(T) \\ &+ \|\varrho_4\|(\|b_m\|(L(\gamma_x)\gamma_x + B) + B_\sigma) \\ &+ \|\varrho_4\|\|A_m\Lambda^{-1}\|\sqrt{\frac{\alpha}{\lambda_{max}(P_2)}} + \|\bar{r}_{t'}\|_{\mathcal{L}_\infty} < \gamma_x \quad (30)\end{aligned}$$

V. Analysis of $\mathcal{L}_1$ Adaptive Controller

In this section, we analyze the performance bounds of the $\mathcal{L}_1$ adaptive controller. Let $\tilde{x}(t) = \hat{x}(t) - x(t)$. The error dynamics between (1) and (3) are

$$\dot{\tilde{x}}(t) = A_m\tilde{x}(t) + \hat{\sigma}(t) - b_m f(x,t) - \sigma(t)\,, \quad (31)$$

$$\tilde{y}(t) = c_m^\top \tilde{x}(t), \quad \tilde{y}(0) = 0\,. \quad (32)$$

Considering the following state transformation

$$\tilde{\xi} = \Lambda\tilde{x}\,, \quad (33)$$

it follows from (32) that

$$\begin{aligned}\dot{\tilde{\xi}}(t) &= \Lambda A_m \Lambda^{-1}\tilde{\xi}(t) + \Lambda\hat{\sigma}(t) - \Lambda b_m f(x,t) \\ &\quad - \Lambda\sigma(t)\,, \quad (34)\\ \tilde{y}(t) &= \tilde{\xi}_1(t)\,, \quad (35)\end{aligned}$$

where $\tilde{\xi}_1(t)$ is the first element of $\tilde{\xi}(t)$.

Theorem 1: Given the system in (1) and the $\mathcal{L}_1$ adaptive controller in (3), (5) and (8) subject to (30), if $x(0) < \gamma_x$, and $\hat{x}(0)$ in the output predictor is chosen such that $\tilde{z}^\top(0)P_2\tilde{z}(0) \le \alpha$, then

$$\|\tilde{y}\|_{\mathcal{L}_\infty} \le \gamma_{\tilde{y}}(T)\,, \quad (36)$$

$$\|\tilde{z}\|_{\mathcal{L}_\infty} \le \gamma_{\tilde{z}}\,, \quad (37)$$

$$\|x\|_{\mathcal{L}_\infty} < \gamma_x\,, \quad (38)$$

$$\|u\|_{\mathcal{L}_\infty} < \gamma_u\,, \quad (39)$$

$\bar{\gamma}_{\tilde{y}}(T)$ and $\gamma_{\tilde{z}}$ are introduced in (28) and (29) respectively, γ_x is a positive constant, and

$$\begin{aligned}\gamma_u &= \|\varrho_3\|_{\mathcal{L}_1}\|A_m\Lambda^{-1}\mathbf{1}_1\|\nu(T) + \|k_g r_{t'}\|_{\mathcal{L}_\infty} \\ &\quad + \|\varrho_3\|_{\mathcal{L}_1}\|A_m\Lambda^{-1}\|\sqrt{\frac{\alpha}{\lambda_{max}(P_2)}} \\ &\quad + \|\varrho_3\|_{\mathcal{L}_1}\|b_m\|(L(\gamma_x)\gamma_x + B) + \|\varrho_3\|_{\mathcal{L}_1}B_\sigma \quad (40)\end{aligned}$$

Proof. Since $x(0) < \gamma_x$ and $x(t)$ is continuous, then assuming the opposite implies that there exist t' such that

$$x(t') = \gamma_x\,, \quad (41)$$

while

$$\|x_{t'}\|_{\mathcal{L}_\infty} \le \gamma_x\,. \quad (42)$$

At first, we prove that for all $iT < t'$ one has

$$|\tilde{y}(iT)| \le \nu(T)\,, \quad (43)$$

$$\tilde{z}^\top(iT)P_2\tilde{z}(iT) \le \alpha\,. \quad (44)$$

We prove the bounds in (43) and (44) by induction. At the beginning, when $t = 0$ we have

$$\tilde{y}(0) = 0 \le \nu(T)\,, \quad (45)$$

$$\tilde{z}^\top(0)P_2\tilde{z}(0) \le \alpha\,. \quad (46)$$

where P_2 is positive definite. In the next step, we will prove that if (43) and (44) hold at time jT, then they also hold at time $(j+1)T$.

It follows from (34) that

$$\begin{aligned}\tilde{\xi}((j+1)T) &= \mathrm{e}^{\Lambda A_m\Lambda^{-1}T}\tilde{\xi}(jT) \\ &\quad + \int_0^T \varsigma(T-\tau)\hat{\sigma}(jT)d\tau \\ &\quad - \int_0^T \varsigma(T-\tau)b_m f(x, jT+\tau)d\tau \\ &\quad - \int_0^T \varsigma(T-\tau)\sigma(jT+\tau)d\tau. \quad (47)\end{aligned}$$

Since

$$\tilde{\xi}(jT) = \begin{bmatrix}\tilde{y}(jT)\\ 0\end{bmatrix} + \begin{bmatrix}0\\ \tilde{z}(jT)\end{bmatrix}\,, \quad (48)$$

equation (47) can be written as

$$\tilde{\xi}((j+1)T) = \chi((j+1)T) + \zeta((j+1)T)\,, \quad (49)$$

where

$$\chi((j+1)T) = e^{\Lambda A_m \Lambda^{-1} T} \begin{bmatrix} \tilde{y}(jT) \\ 0 \end{bmatrix} + \int_0^T \varsigma(T-\tau)\hat{\sigma}(jT)d\tau \,, \quad (50)$$

$$\begin{aligned} \zeta((j+1)T) &= e^{\Lambda A_m \Lambda^{-1} T} \begin{bmatrix} 0 \\ \tilde{z}(jT) \end{bmatrix} \\ &- \int_0^T \varsigma(T-\tau)\sigma(jT+\tau)d\tau \\ &- \int_0^T \varsigma(T-\tau)b_m f(x, jT+\tau)d\tau \quad (51) \end{aligned}$$

Substitution of the adaptive law (5) into (50) results in

$$\chi((j+1)T) = 0 \,. \quad (52)$$

Following from (51), consider $\zeta(t)$ as the solution of the following dynamics

$$\dot{\zeta}(t) = \Lambda A_m \Lambda^{-1}\zeta(t) - \Lambda b_m f(x,t) - \Lambda\sigma(t) \,, \quad (53)$$

$$\zeta(jT) = \begin{bmatrix} 0 \\ \tilde{z}(jT) \end{bmatrix}, \quad t \in [\ jT, \quad (j+1)T \ ] \,. \quad (54)$$

Consider the following function

$$V(t) = \zeta^\top(t)\Lambda^{-\top}P\Lambda^{-1}\zeta(t) \quad (55)$$

over $t \in [jT, \ (j+1)T]$. Note that Λ is non-singular and P is positive definite, $\Lambda^{-\top}P\Lambda^{-1}$ is positive definite, and therefore $V(t)$ is a positive definite function. It follows from Lemma 2 in [8] and (54) that

$$V(\zeta(jT)) = \tilde{z}^\top(jT)P_2\tilde{z}(jT) \leq \alpha \,. \quad (56)$$

Following from (53) over $t \in [jT, \ (j+1)T]$, we obtain the derivative of $V(t)$

$$\begin{aligned} \dot{V}(t) &= -\zeta^\top(t)\Lambda^{-\top}Q\Lambda^{-1}\zeta(t) \\ &\quad - 2\zeta^\top(t)\Lambda^{-\top}Pb_m f(x,t) \\ &\quad - 2\zeta^\top(t)\Lambda^{-\top}P\sigma(t) \,. \quad (57) \end{aligned}$$

It follows from Assumption 1 and (42) that

$$|f(x,t)| \leq L(\gamma_x)\gamma_x + B \,. \quad (58)$$

From Assumption 2 and (58), we can further derive the upper bound of $\dot{V}(t)$ over $t \in [jT, \ (j+1)T]$

$$\begin{aligned} \dot{V}(t) &\leq -\lambda_{min}(\Lambda^{-\top}Q\Lambda^{-1})\|\zeta(t)\|^2 \\ &\quad +2\,\|\zeta(t)\|\|\Lambda^{-\top}Pb_m\|(L(\gamma_x)\gamma_x + B) \\ &\quad +2\,\|\zeta(t)\|\|\Lambda^{-\top}P\|B_\sigma \,. \quad (59) \end{aligned}$$

If

$$V(t) \geq \alpha \,, \quad (60)$$

then, from(55) and the definition of α we have

$$\begin{aligned} \|\zeta(t)\| &\geq \sqrt{\frac{\alpha}{\lambda_{max}(\Lambda^{-\top}P\Lambda^{-1})}} \\ &\geq \frac{2\|\Lambda^{-\top}Pb_m\|(L(\gamma_x)\gamma_x + B)}{\lambda_{min}(\Lambda^{-\top}Q\Lambda^{-1})} \\ &\quad + \frac{2\|\Lambda^{-\top}P\|B_\sigma}{\lambda_{min}(\Lambda^{-\top}Q\Lambda^{-1})} \,, \quad (61) \end{aligned}$$

which together with (59) yields

$$\dot{V}(t) \leq 0 \,. \quad (62)$$

It follows from (56), (60) and (62) that

$$V(t) \leq \alpha \,, \quad \forall t \in [jT, \ (j+1)T] \,. \quad (63)$$

Using the result of Lemma 2 in [8] together with (63), one can derive that

$$\tilde{z}^\top((j+1)T)P_2\tilde{z}((j+1)T) \leq \alpha \,, \quad (64)$$

which implies that (44) holds for $(j+1)T$.

It follows from (48), (49), (51) and (52) that

$$\begin{aligned} \tilde{y}((j+1)T) &= \mathbf{1}_1^\top \zeta((j+1)T) \\ &= \mathbf{1}_1^\top e^{\Lambda A_m \Lambda^{-1} T} \begin{bmatrix} 0 \\ \tilde{z}(jT) \end{bmatrix} \\ &\quad -\mathbf{1}_1^\top \int_0^T \varsigma(T-\tau)b_m f(x, jT+\tau)d\tau \\ &\quad -\mathbf{1}_1^\top \int_0^T \varsigma(T-\tau)\sigma(jT+\tau)d\tau \,, \quad (65) \end{aligned}$$

By using definitions in (19), (20) and (21), we arrive at the following upper bound

$$\begin{aligned} |\tilde{y}((j+1)T)| &\leq \|\phi(T)\|\|\tilde{z}(jT)\| + \eta_1(T)B_\sigma \\ &\quad +\eta_2(T)(L(\gamma_x)\gamma_x + B) \\ &\leq \nu(T) \,, \quad (66) \end{aligned}$$

This confirms the upper bound in (43) holds for $(j+1)T$. Hence, (43) and (44) hold for all $iT \leq t'$.

For all $iT + t \leq t'$, where $0 \leq t \leq T$, it follows from (34) that

$$\begin{aligned} \tilde{y}(iT+t) &= \mathbf{1}_1^\top e^{\Lambda A_m \Lambda^{-1} t}\tilde{\xi}(iT) \\ &\quad +\mathbf{1}_1^\top \int_0^t \varsigma(t-\tau)\hat{\sigma}(iT)d\tau \\ &\quad -\mathbf{1}_1^\top \int_0^t \varsigma(t-\tau)\sigma(iT+\tau)d\tau \\ &\quad -\mathbf{1}_1^\top \int_0^t \varsigma(t-\tau)b_m f(x, iT+\tau)d\tau \,, \quad (67) \end{aligned}$$

$$\begin{aligned} \tilde{z}(iT+t) &= Se^{\Lambda A_m \Lambda^{-1} t}\tilde{\xi}(iT) + S\int_0^t \varsigma(t-\tau)\hat{\sigma}(iT)d\tau \\ &\quad -S\int_0^t \varsigma(t-\tau)\sigma(iT+\tau)d\tau \\ &\quad -S\int_0^t \varsigma(t-\tau)b_m f(x, iT+\tau)d\tau \,. \quad (68) \end{aligned}$$

Considering (43)-(44) and recalling the definitions of $\beta_{y_0}(T)$, $\bar{\beta}_{y_0}(T)$, $\beta_1(T)$, $\beta_2(T)$, and $\beta_5(T)$ in (17), (22), (23) and (26), we arrive at the following upper bound

$$\begin{aligned} |\tilde{y}(iT+t)| &\leq \beta_{y_0}(T)\nu(T) + \bar{\beta}_{y_0}(T)\sqrt{\frac{\alpha}{\lambda_{max}(P_2)}} \\ &\quad + \beta_5(T)\nu(T) + \beta_1(T)B_\sigma \\ &\quad + \beta_2(T)(L(\gamma_x)\gamma_x + B) \,. \quad (69) \end{aligned}$$

Similarly, by introducing (18), (24), (25), and (27), we have

$$\begin{aligned}\|\tilde{z}(iT+t)\| &\leq \beta_{z_0}(T)\nu(T)+\bar{\beta}_{z_0}(T)\sqrt{\frac{\alpha}{\lambda_{max}(P_2)}}\\ &\quad+\beta_6(T)\nu(T)+\beta_3(T)B_\sigma\\ &\quad+\beta_4(T)(L(\gamma_x)\gamma_x+B)\,. \qquad (70)\end{aligned}$$

Then, for all $t\in[0,\ t']$, it follows from (69) - (70) and definitions of $\gamma_{\tilde{y}}(T)$ in (28) and $\gamma_{\tilde{z}}$ in (29) that

$$|\tilde{y}(t)| \leq \gamma_{\tilde{y}}(T)\,, \qquad (71)$$

$$\|\tilde{z}(t)\| \leq \gamma_{\tilde{z}}\,. \qquad (72)$$

Since $x(t)=\hat{x}(t)-\tilde{x}(t)$, we have

$$\|x(t)\|\leq\|\hat{x}(t)\|+\|\tilde{x}(t)\|\,. \qquad (73)$$

It follows from (33) that

$$\tilde{x}(t)=\Lambda^{-1}\tilde{\xi}(t)=\Lambda^{-1}\begin{bmatrix}\tilde{y}(t)\\ \tilde{z}(t)\end{bmatrix}\,. \qquad (74)$$

Then, the upper bound of $\tilde{x}(t)$ is written as

$$\|\tilde{x}(t)\| \leq \|\varrho_1\|\,|\tilde{y}(t)|+\|\varrho_2\|\,\|\tilde{z}(t)\|\,, \qquad (75)$$

where ϱ_1 and ϱ_2 are introduced in (12).

Furthermore, it follows from (3) that

$$\begin{aligned}\hat{x}(s) &= (s\mathbb{I}-A_m)^{-1}b_m a u(s)+(s\mathbb{I}-A_m)^{-1}\hat{\sigma}(s)\\ &\quad+(s\mathbb{I}-A_m)^{-1}x_0\,. \qquad (76)\end{aligned}$$

Then, we arrive at following upper bound of

$$\begin{aligned}\|\hat{x}_{t'}\|_{\mathcal{L}_\infty} &\leq \|(s\mathbb{I}-A_m)^{-1}b_m a\|_{\mathcal{L}_1}\|u_{t'}\|_{\mathcal{L}_\infty}\\ &\quad+\|(s\mathbb{I}-A_m)^{-1}\|_{\mathcal{L}_1}\|\hat{\sigma}_{t'}\|_{\mathcal{L}_\infty}+\|r_0\|_{\mathcal{L}_\infty} \qquad (77)\end{aligned}$$

over $t\in[0,\ t']$,where $r_0(s)=(s\mathbb{I}-A_m)^{-1}x_0$.

From (7) and (8), we arrive at the following upper bound

$$\|u_{t'}\|_{\mathcal{L}_\infty}\leq\|\varrho_3\|_{\mathcal{L}_1}\,\|\hat{\sigma}_{t'}\|_{\mathcal{L}_\infty}+\|k_g r_{t'}\|_{\mathcal{L}_\infty}\,, \qquad (78)$$

where ϱ_3 is defined in (13). Substitution of (78) into (77) yields

$$\|\hat{x}_{t'}\|_{\mathcal{L}_\infty} \leq \|\varrho_4\|\|\hat{\sigma}_{t'}\|_{\mathcal{L}_\infty}+\|\bar{r}_{t'}\|_{\mathcal{L}_\infty}\,, \qquad (79)$$

where $\|\varrho_4\|$ is defined in (14), and $\|\bar{r}_t\|_{\mathcal{L}_\infty}=\|(s\mathbb{I}-A_m)^{-1}b_m a k_g\|_{\mathcal{L}_1}\|r_t\|_{\mathcal{L}_\infty}+\|r_0\|_{\mathcal{L}_\infty}$.

From the adaptive law in (5) and (6), we have

$$\int_0^T\varsigma(T-\tau)\hat{\sigma}(iT)d\tau+\mathrm{e}^{\Lambda A_m\Lambda^{-1}T}\mathbf{1}_1\tilde{y}(iT)=0\,. \qquad (80)$$

We further obtain that

$$\begin{aligned}\tilde{y}(iT) &= (1-\mathbf{1}_1^\top\mathrm{e}^{\Lambda A_m\Lambda^{-1}T}\mathbf{1}_1)\tilde{y}(iT)\\ &\quad-\int_0^T\mathbf{1}_1^\top\varsigma(T-\tau)\hat{\sigma}(iT)d\tau\\ &= -\int_0^T\mathbf{1}_1^\top\varsigma(T-\tau)A_m\Lambda^{-1}\mathbf{1}_1\tilde{y}(iT)d\tau\\ &\quad-\int_0^T\mathbf{1}_1^\top\varsigma(T-\tau)\hat{\sigma}(iT)d\tau\,. \qquad (81)\end{aligned}$$

It follows from (65) that

$$\begin{aligned}\tilde{y}(iT) &= \int_0^T\mathbf{1}_1^\top\varsigma(T-\tau)A_m\Lambda^{-1}\begin{bmatrix}0\\ \tilde{z}((i-1)T)\end{bmatrix}d\tau\\ &\quad-\int_0^T\mathbf{1}_1^\top\varsigma(T-\tau)b_m f(x,(i-1)T+\tau)d\tau\\ &\quad-\int_0^T\mathbf{1}_1^\top\varsigma(T-\tau)\sigma((i-1)T+\tau)d\tau\,. \qquad (82)\end{aligned}$$

Following from the relation between (81) and (82), (58) and Assumption 2, we arrive at the following upper bound

$$\begin{aligned}\|\hat{\sigma}(iT)\| &\leq \|A_m\Lambda^{-1}\mathbf{1}_1\||\tilde{y}(iT)|\\ &\quad+\|A_m\Lambda^{-1}\|\|\tilde{z}((i-1)T)\|\\ &\quad+\|b_m\|(L(\gamma_x)\gamma_x+B)+B_\sigma\,. \qquad (83)\end{aligned}$$

Note that $\hat{\sigma}(t)$ is piece-wise continuous, and following from (43) and (44), we obtain

$$\begin{aligned}\|\hat{\sigma}_{t'}\|_{\mathcal{L}_\infty} &\leq \|A_m\Lambda^{-1}\mathbf{1}_1\|\nu(T)\\ &\quad+\|A_m\Lambda^{-1}\|\sqrt{\frac{\alpha}{\lambda_{max}(P_2)}}\\ &\quad+\|b_m\|(L(\gamma_x)\gamma_x+B)+B_\sigma\,. \qquad (84)\end{aligned}$$

Substitution of (84) into (79) yields

$$\begin{aligned}\|\hat{x}_{t'}\|_{\mathcal{L}_\infty} &\leq \|\varrho_4\|\|A_m\Lambda^{-1}\mathbf{1}_1\|\nu(T)+\|\bar{r}_{t'}\|_{\mathcal{L}_\infty}\\ &\quad+\|\varrho_4\|\|A_m\Lambda^{-1}\|\sqrt{\frac{\alpha}{\lambda_{max}(P_2)}}\\ &\quad+\|\varrho_4\|(\|b_m\|(L(\gamma_x)\gamma_x+B)+B_\sigma)\,. \qquad (85)\end{aligned}$$

Finally, following from (73), (75) and (85), we obtain the upper bound of $x(t)$

$$\begin{aligned}\|x_{t'}\|_{\mathcal{L}_\infty} &\leq \|\varrho_1\|\gamma_{\tilde{y}}(T)+\|\varrho_2\|\gamma_{\tilde{z}}+\|\bar{r}_{t'}\|_{\mathcal{L}_\infty}\\ &\quad+\|\varrho_4\|\|A_m\Lambda^{-1}\mathbf{1}_1\|\nu(T)\\ &\quad+\|\varrho_4\|\|A_m\Lambda^{-1}\|\sqrt{\frac{\alpha}{\lambda_{max}(P_2)}}\\ &\quad+\|\varrho_4\|(\|b_m\|(L(\gamma_x)\gamma_x+B)+B_\sigma)\,. \qquad (86)\end{aligned}$$

By considering stability condition, (86) becomes

$$\|x_{t'}\|_{\mathcal{L}_\infty}<\gamma_x\,, \qquad (87)$$

which contradicts (42) and proves (38). Following from (38), (71) and (72), we further obtain results (36) and (37). It follows from (38), (78) and (84) that

$$\begin{aligned}\|u\|_{\mathcal{L}_\infty} &< \|\varrho_3\|_{\mathcal{L}_1}\|A_m\Lambda^{-1}\mathbf{1}_1\|\nu(T)+\|k_g r_{t'}\|_{\mathcal{L}_\infty}\\ &\quad+\|\varrho_3\|_{\mathcal{L}_1}\|A_m\Lambda^{-1}\|\sqrt{\frac{\alpha}{\lambda_{max}(P_2)}}\\ &\quad+\|\varrho_3\|_{\mathcal{L}_1}\|b_m\|(L(\gamma_x)\gamma_x+B)+\|\varrho_3\|_{\mathcal{L}_1}B_\sigma\\ &< \gamma_u \qquad (88)\end{aligned}$$

which proves (39) and concludes the proof.

Remark 1: By making the bandwidth of low pass filters $C_1(s)$ and $C_2(s)$ large enough, the control law (8) can ensure that $\lim_{s\to 0}\hat{y}(s)=y_{des}(s)$. The result (36) in Theorem 1 together with the control law guarantee that the difference between $y(t)$ and $y_{des}(t)$ is bounded, and the bound can be reduced by decreasing the integration time step T. In practice, we can always make the bandwidth of low pass filters compatible with the control channel specifications.

VI. Simulations

Consider the system in (1) with

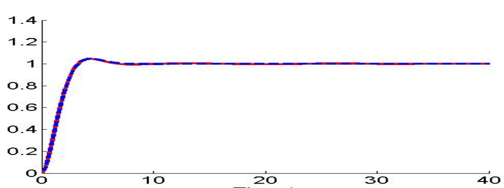
$$A_m = \begin{bmatrix} 0 & 1 \\ -1 & -1.4 \end{bmatrix},\ \sigma(t) = \begin{bmatrix} \sin(0.4t) \\ \sin(0.2t) \end{bmatrix}, \quad (89)$$

$$b_m = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ c_m = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad (90)$$

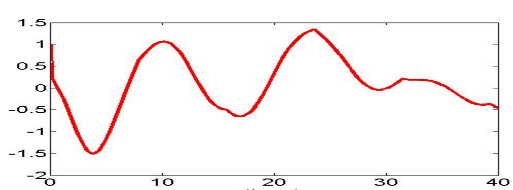
$$f(x,u,t) = 2u + (\sin(x_1)\cos(x_2))u + e^{-t}x_2 + \sin(t)x_1 + \ln(1+|x_1x_2|), \quad (91)$$

The control objective is to design $u(t)$ to achieve tracking of the desire system output $y_{des}(t)$ in (2) with a bounded reference input $r(t)$. In the implementation of the $\mathcal{L}_1$ adaptive controller, set integration step $T = 1e-4$, $Q = \mathbb{I}$, and hence $P = \begin{bmatrix} 1.4143 & 0.5 \\ 0.5 & 0.7143 \end{bmatrix}$, and $\Lambda = \begin{bmatrix} 1.0000 & 0 \\ -0.5916 & -0.8452 \end{bmatrix}$. Two low pass filters are designed as $C_1(s) = \dfrac{40}{s+40}$, $C_2(s) = \dfrac{70}{s+70}$.

The simulation results of the $\mathcal{L}_1$ adaptive controller for the constant reference input $r(t) = 1$ are shown in Figs. 1(a)-1(b). Next, we change the unknown function to $f(x,u,t) = 3u + (\sin(x_1^2)\cos(x_2))u + e^{-t}x_2 + \sin(t)x_1 + x_1\sin(x_2^2) + u^2x_2$, unknown disturbances to $\sigma(t) = \begin{bmatrix} \sin(0.4t) \\ 1+\sin(8t) \end{bmatrix}$ with the new reference input $r(t) = 0.5sin(0.3t)$, and apply the same controller without re-tuning. The system response and control signal are plotted in Figs. 2(a)-2(b).



(a) $y(t)$ (solid) and $y_{des}(t)$ (dashed)

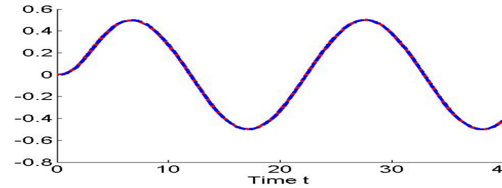


(b) Time-history of $u(t)$

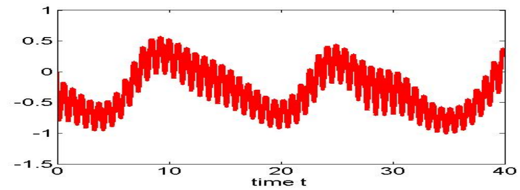
Fig. 1. Performance for $r(t) = 1$ and $f(x,u,t) = 2u + (\sin(x_1)\cos(x_2))u + e^{-t}x_2 + \sin(t)x_1 + \ln(1+|x_1x_2|)$.

VII. Conclusions

This paper presents an extension of the $\mathcal{L}_1$ adaptive output feedback controller to a class of nonlinear systems where nonlinearities satisfy a semiglobal Lipschitz condition. The algorithm contains an output predictor and a predictor based feedback control law. It is proven that the difference between the predicted output and the actual system's output is bounded, which can be systematically improved by reducing the step size of integration. The simulation results show uniformly bounded tracking performance by using this adaptive output feedback control design.



(a) $y(t)$ (solid) and $y_{des}(t)$ (dashed)



(b) Time-history of $u(t)$

Fig. 2. Performance for $r(t) = 0.5sin(0.3t)$ and $f(x,u,t) = 3u + (\sin(x_1^2)\cos(x_2))u + e^{-t}x_2 + \sin(t)x_1 + x_1\sin(x_2^2) + u^2x_2$.

VIII. Acknowledgements

The authors gratefully acknowledge comments from Jennifer Hacker and Ali Elahidoost.

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