2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Adaptive Control Design for Piecewise-Linear Differential Inclusion with Parameter Uncertainty

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Abstract— **This paper presents Lyapunov-based adaptive control method for piecewise-linear differential inclusions {PWLDI} with parameter uncertainty. The basic idea of the proposed approach is to construct a piecewise control law and a parameter adaptive law for the PWLDI in a way such that a piecewise quadric Lyapunov function can be used to establish the global stability and H_∞ performance of the resulted closed-loop systems. All the synthesis conditions can be formulated as an optimization problem subjected to a set of Bilinear Matrix Inequalities{BMIs}. Finally we demonstrate the efficiency of the algorithm through an example.**

I. INTRODUCTION

Piecewise-linear systems{PLS} have been a subject of research in the systems and control community for some time; see for example, references [1-14]. PLS constitute a special class of hybrid systems [1] that often arise in practice when piecewise-linear components are encountered. These components include dead-zone, saturation, relays and hysteresis. In addition, many other classes of nonlinear systems can also be approximated by PLS [2]. Thus, PLS provide a useful framework for the analysis of and design for a large class of nonlinear systems.

A number of significant results have been obtained on controller design for PLS in the last few years. [3]–[6] presented results on stabilization of PLS based on common or piecewise Lyapunov function, which can be cast as optimization problem subjected to a set of Bilinear Matrix Inequalities{BMIs}. Furthermore, [7], [8] showed how BMIs can be replaced by Linear Matrix Inequalities {LMIs}, respectively. Recently, controller synthesis for uncertain PLS has attracted growing attention, [9]–[12] formulated the robust stabilization and H_∞ performance synthesis problem for uncertain PLS as a set of LMIs by using piecewise quadratic Lyapunov function. [13], [14] extended the piecewise quadratic Lyapunov function technique to the H_∞ output feedback control and optimal guaranteed cost control of uncertain PLS systems, which are cast as the feasibility of a set of BMIs.

On the other hand, PWLDI was introduced in [15] to express the uncertain PLS systems in the view of differential inclusion, which plays a very important role for analysis and synthesis of nonlinear systems. [16] presented a numerical method for computing PWLDI as an envelop of nonlinear systems, then controller synthesis method was proposed to realize the global stability and H_∞ performance for the obtained PWLDI by employing a common Lyapunov function, base on the envelop relationship, the obtained controller is also effective for original nonlinear systems. The similar idea was extended to establish the analysis and synthesis framework for uncertain nonlinear systems using PWLDI in [2]. Therefore, for establishing the synthesis framework for nonlinear systems with parameter uncertainty and uncertain nonlinear systems with both model and parameter uncertainties, it is natural to investigate the control design of PWLDI with parameter uncertainty(See Remark.1). However, as best of our knowledge, this topic was still blank.

Motivated by this observation, we address the problem of control design for PWLDI with parameter uncertainty satisfying matching condition. It is well known that the adaptive control is an effective scheme for control of uncertain systems with parameter uncertainty [17]–[21]. Therefore, a Lyapunov-based adaptive control scheme is adopted to realize global stability and H_∞ performance for PWLDI with parameter uncertainty for PWLDI with parameter uncertainty.

This paper is organized as follows: The system model is described and the control design problems are formulated in §2. In §3 we present an adaptive stabilization method for PWLDI with parameter uncertainty using piecewise quadratic Lyapunov function. Furthermore, §4 extend the synthesis method to realize H_∞ performance. A numerical example is shown in §5 and a remark for dealing with the case of sliding motion is presented §6. Finally, conclusions are drawn in §7.

Notation

- $I[k_1, k_2]$: For two integers $k_1\ k_2, k_1 < k_2, I[k_1, k_2] := \{k_1, k_1+1, \cdots, k_2\}$.
- $\overline{\text{co}}S$: The convex hull of a set S.
- $\succ$: The matrix Z$\succ$0 stands for the matrix Z has nonnegative entries.

II. PROBLEM STATEMENT

Consider the following PWLDI with parameter uncertainty and external disturbance,

$$\begin{aligned}&\dot{x} \in \overline{\text{co}}\{A_{ik}x + b_{ik} + B_i u + \varphi_i(x)\theta + D_{ik}\omega, k \in [1, N]\}\\ &z \in \overline{\text{co}}\{C_{ik}x + d_{ik}, k \in [1, N]\} \quad x \in R_i\end{aligned} \tag{1}$$

where $\{R_i\}, i \in I$ denotes a partition of the state space X into a number of closed polytopic regions, I is the index

This work is supported by National Natural Science Foundation (NNSF) of China under Grant 60736022.

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set of regions. $x \in X \subseteq \mathbb{R}^{n_x}$, $u \in \mathbb{R}^{n_u}$, $\theta \in \Omega_\theta \subseteq \mathbb{R}^{n_\theta}$, $\omega \in \mathbb{R}^{n_\omega}$ and $z \in \mathbb{R}^{n_z}$ are respectively the state, input, unknown parameter vector, disturbance and output vector.

As shown in [5], each region is constructed as the intersection of a finite number of half-spaces

$$R_i = \{x \mid H_i x \le g_i\}, \quad i \in I \tag{2}$$

where H_i=$[h_{i1}\ h_{i2} \cdots h_{iq}]^T$,$g_i$=$[g_{i1}\ g_{i2} \cdots g_{iq}]^T$.

Any two regions R_i and R_j sharing a common boundary $l \in L$, which is contained in the hyperplane described by

$$\{x \in \mathbb{R}^n | E_l x - e_l = 0\}$$

another expression of the hyperplane is proposed in [8]

$$l = R_i \cap R_j \subseteq \{f_l + F_l s \mid s \in \mathbb{R}^{n-1}\} \tag{3}$$

In summary, a PWLDI can be described by (1-3).

Remark 1: From [16] and [22] we know, the proposed PWLDI (1-3) can be used as a envelop of a large class of nonlinear systems and uncertain nonlinear systems as below,

$$\begin{aligned} &\dot{x} = f(x) + Bu + D\omega + \varphi(x)\theta \\ &z = g(x) \end{aligned} \tag{4}$$

$$\begin{aligned} &\dot{x} \in \overline{\text{co}}\,\{f_k(x) + Bu + D_k\omega + \varphi(x)\theta, k \in [1, N]\} \\ &z \in \overline{\text{co}}\,\{g_k(x), k \in [1, N]\} \end{aligned} \tag{5}$$

Therefore, we can investigate the uncertain nonlinear systems (4) and (5) via proposed PWLDI. That is one of the main motivations of this work.

Asumption 1: For $i \in I_0, I_0 = \{i | 0 \in R_i\}$,

$$b_{ik} = 0,\ d_{ik} = 0, \varphi_i(0) = 0 \qquad k \in [1, N]$$

which guarantees that the origin is an equilibrium of autonomous PWLDI.

Asumption 2: The parameter uncertainty satisfies the matching condition [23], i.e. there exists functions $\psi_i(x)$, such that

$$B_i \psi_i(x) = \varphi_i(x) \quad \forall x \in R_i,\ i \in I$$

Our objectives are to design a piecewise control law,

$$u(t) = u_i(x, \hat{\theta}), \quad x \in R_i$$

and a piecewise parameter estimation law for the unknown parameter vector θ,

$$\dot{\hat{\theta}} = \upsilon_i(x, \hat{\theta}), \quad x \in R_i$$

such that, the closed-loop PWLDI is

1) asymptotically stable in the absence of external disturbance;

2) globally stable with disturbance attenuation γ, i.e.

$$\|z(t)\|_2 < \gamma \|\omega(t)\|_2, \quad \forall \omega(t) \in L_2^{n_\omega} \setminus 0.$$

III. Adaptive Stabilization of PWLDI

In this section, an adaptive stabilization scheme will be developed for the PWLDI (1-3). The piecewise control law consists of two parts given by

$$u_i(t) = u_{ia} + u_{is}, \quad i \in R_i \tag{6}$$

where u_{ia} represents the usual model compensation with the physical parameter estimates $\hat{\theta}$, which is updated by using an on-line adaptive algorithm. u_{is} represents the robust stabilization term which is used to quadratically stabilize the PWLDI in the absence of uncertain parameter vector θ.

Now, we present adaptive stabilization method as Theorem 1.

Theorem 1: Consider the PWLDI described by (1-3) with $\omega = 0$. If there is a solution for the following optimization problem, using the obtained piecewise control law

$$u = -\psi_i(x)\hat{\theta} + k_i x + m_i,\ x \in R_i \tag{7}$$

and piecewise on-line adaptive algorithm

$$\dot{\hat{\theta}} = x^T P_i \varphi_i(x),\ x \in R_i \tag{8}$$

then the closed-loop PWLDI is asymptotically stable in X.

The optimization problem:

$$\begin{aligned} &\max(\min_i a_i)\ s.t.\ (9-12) \\ &\text{variables} : k_i, m_i, r_i, P_i; Z_i, W_i \succ 0; \end{aligned}$$

$$m_i = 0,\ r_i = 0, \quad i \in I_0 \tag{9}$$

$$\bar{F}_l^T(\bar{P}_i - \bar{P}_j)\bar{F}_l = 0, \quad l = \bar{R}_i \cap \bar{R}_j \tag{10}$$

$$\begin{bmatrix} P_i - H_i^T Z_i H_i & H_i^T Z_i g_i \\ (\bullet)^T & r_i - g_i^T Z_i g_i \end{bmatrix} > 0 \tag{11}$$

$$\begin{bmatrix} \bar{A}_{ik}^T P_i + P_i \bar{A}_{ik} + a_i P_i \\ +H_i^T W_i H_i & P_i \bar{b}_{ik} - H_i^T Z_i g_i \\ (\bullet)^T & g_i^T Z_i g_i + a_i r_i \end{bmatrix} < 0 \tag{12}$$

where

$$\bar{A}_{ik} = A_{ik} + B_{ik}k_i, \bar{b}_{ik} = b_{ik} + B_{ik}m_i$$

$$\bar{P}_i = \begin{bmatrix} P_i & 0 \\ 0 & r_i \end{bmatrix}, \bar{F}_l = [F_l, f_l]$$

Proof.

It is shown by constrain (9) that the origin is an equilibrium of closed-loop PWLDI. Therefore, we choose a piecewise quadratic Lyapunov function $V(x, \tilde{\theta})$ as below:

$$\begin{aligned} &V(x, \tilde{\theta}) = \sum_{i \in I} \alpha_i V_i + V_{\tilde{\theta}} \\ &V_i(x) = x^T P_i x + r_i,\ V_{\tilde{\theta}}(\widetilde{\theta}) = \widetilde{\theta}^T \widetilde{\theta} \end{aligned}$$

where

$$\alpha_i = \begin{cases} 1 & x \in R_i \\ 0 & \text{others} \end{cases}, \quad \tilde{\theta} = \hat{\theta} - \theta$$

From [8], it can be shown that, for all matrices Q_i with compatible dimensions and nonnegative entries,

$$x \in R_i \implies \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} H_i^T Q_i H_i & -H_i^T Q_i g_i \\ -g_i^T Q_i H_i & g_i^T Q_i g_i \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} > 0 \tag{13}$$

Then we can obtain the following conclusions:

1) $V(x, \tilde{\theta}) > 0, \quad \forall x \in R_i \backslash 0$

With (11) and (13), it can be shown by S-procedure [24],

$$\begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} P_i & 0 \\ 0 & r_i \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} > 0, \quad \forall x \in R_i \backslash 0$$

i.e.,

$$V_i(x) > 0, \quad \forall x \in R_i \backslash 0$$

as $\widetilde{\theta}^T \widetilde{\theta} \geq 0$, it implies,

$$V(x) = V_i(x) + \widetilde{\theta}^T \widetilde{\theta} > 0, \quad \forall x \in R_i \backslash 0$$

2) $\frac{dV(x,\tilde{\theta})}{dt} < 0 \quad \forall x \in R_i$

From the definition of convex hull, for any $x \in R_i$, there exists $\mu_k(x) \geq 0, \sum_{k=1}^N \mu_k(x) = 1$, such that,

$$\begin{aligned} \Pi = \frac{dV(x,\tilde{\theta})}{dt} &= \frac{dV_i(x)}{dt} + \frac{dV_\theta(x)}{dt} \\ &= \sum_{k=1}^N \mu_k(x) \{ (A_{ik}x + b_{ik} + B_i u + \varphi_i \theta)^T P_i x \\ &+ x^T P_i (A_{ik}x + b_{ik} + B_i u + \varphi_i \theta) \} + \dot{\tilde{\theta}}^T \tilde{\theta} + \tilde{\theta}^T \dot{\tilde{\theta}} \end{aligned}$$

with the piecewise control law (7) and on-line adaptive algorithm (8), it can be obtained,

$$\begin{aligned} \Pi &= \sum_{k=1}^N \mu_k(x) [(\bar{A}_{ik}x + \bar{b}_{ik} - \varphi_i \tilde{\theta})^T P_i x \\ &+ x^T P_i (\bar{A}_{ik}x + \bar{b}_{ik} - \varphi_i \tilde{\theta})] + \tilde{\theta}^T \varphi_i^T P_i x + x^T P_i \varphi_i \tilde{\theta} \\ &= \sum_{k=1}^N \mu_k(x) [x^T (\bar{A}_{ik}^T P_i + P_i \bar{A}_{ik}) x + x^T P_i \bar{b}_{ik} + \bar{b}_{ik}^T P_i x] \\ &= \sum_{k=1}^N \mu_k(x) \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} \bar{A}_{ik}^T P_i + P_i \bar{A}_{ik} & P_i \bar{b}_{ik} \\ \bar{b}_{ik}^T P_i & 0 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}. \end{aligned}$$

Then according to constrains (12) and (13), it can be shown by S-procedure,

$$\frac{dV(x,\tilde{\theta})}{dt} \leq 0, \quad \forall x \in R_i \tag{14}$$

and '=' holds if and only if $x = 0$.

3) $V(x, \tilde{\theta})$ is continuous on the boundary $l \in L$

It can be seen from (3) and (10),

$$V_i(x) = V_j(x), \quad \forall x \in l = \bar{R}_i \cap \bar{R}_j$$

Consequently,

$$V_i(x) + \tilde{\theta}^T \tilde{\theta} = V_j(x) + \tilde{\theta}^T \tilde{\theta}, \quad \forall x \in l$$

Thus, the continuity of the Lyapunov function $V(x, \tilde{\theta})$ is guaranteed.

In summary, the positive definite Lyapunov function $V(x, \hat{\theta})$ decreases within the region R_i, and keeps invariant when crossing the boundary $l \in L$. Therefore, in the absence of sliding motion at the boundary, the solution $x(t)$ of closed-loop PWLDI is globally bounded, and $\hat{\theta}$ is bounded too because of (14).

Moreover, consider the augmented system (1),(7),(8), let

$$E = \{(x, \hat{\theta}) \in (X, R^{n_\theta}) | \dot{V}(x, \hat{\theta}) = 0\}$$

then

$$E = \{(0, \theta_1) | \theta_1 \in R^{n_\theta}\}$$

and E is a invariant set, therefore, using LaSalle's theorem [25], the solution of the augmented system $(x(t), \hat{\theta}(t))$ converges to the set E, that is,

$$\lim_{t \to \infty} x(t) = 0$$

Therefore, the PWLDI is asymptotically stabilized by the proposed control law and adaptive law.

This completes the proof.

Remark 2: It is worth to note that, the theorem do not consider the possibility of sliding motion on the boundary between the polytopic regions, which plays a important role in the stability analysis by piecewise Lyapunov function, therefore, an additional condition is presented in Section 6 to guarantee the asymptotic stability when considering the possibility of sliding motion.

Remark 3: It can be observed from the proof of theorem 1, the estimation of uncertain parameter vector $\hat{\theta}$ may not converges to the real parameter vector θ, which is because the proposed PWLDI do not satisfies PE condition [20],

$$\int_t^{t+T} |\varphi_i(0)|^2 dt = 0, \quad \forall t \geq 0$$

Remark 4: This optimization problem is not convex because there are terms involving products of unknowns, such as P_i and $B_i k_i$ in the constraint (12) . However, the constraints are BMIs, which can be solved effectively using the existing software YALMIP [26].

IV. Adaptive H_∞ Control of PWLDI

In this section, we extend the adaptive control approach discussed in the previous section to the case of H_∞ disturbance attenuation performance.

Rewrite the PWLDI as:

$$\begin{aligned} \dot{\bar{x}} &\in \overline{\text{co}} \left\{ \widetilde{A}_{ik} \bar{x} + \widetilde{B}_i u + \widetilde{\varphi}_i(x) \theta + \widetilde{D}_{ik} \omega \right\} \\ z &\in \overline{\text{co}} \left\{ \widetilde{C}_{ik} \bar{x} \right\} \end{aligned} \tag{15}$$

where

$$\begin{aligned} \bar{x} &= \begin{bmatrix} x \\ 1 \end{bmatrix}, \widetilde{A}_{ik} = \begin{bmatrix} A_{ik} & b_{ik} \\ 0 & 0 \end{bmatrix}, \widetilde{B}_i = \begin{bmatrix} B_i \\ 0 \end{bmatrix} \\ \widetilde{\varphi}_i &= \begin{bmatrix} \varphi_i \\ 0 \end{bmatrix}, \widetilde{D}_{ik} = \begin{bmatrix} D_{ik} \\ 0 \end{bmatrix}, \widetilde{C}_{ik} = \begin{bmatrix} C_{ik} & d_{ik} \end{bmatrix} \end{aligned} \tag{16}$$

Then, we are ready to present the following theorem.

Theorem 2: Consider the PWLDI described by (15-16), if there is a solution to the following optimization problem, such that

$$\bar{P}_i = \begin{bmatrix} P_i & 0 \\ 0 & r_i \end{bmatrix}, E_{1i} = \begin{bmatrix} H_i^T Z_i H_i & -H_i^T Z_i g_i \\ -g_i^T Z_i H_i & g_i^T Z_i g_i \end{bmatrix}$$
$$E_{2i} = \begin{bmatrix} H_i^T W_i H_i & -H_i^T W_i g_i \\ -g_i^T W_i H_i & g_i^T W_i g_i \end{bmatrix}$$

then using the obtained piecewise adaptive control law

$$u = -\psi_i(x)\hat{\theta} + k_i x + m_i, \quad x \in R_i \tag{17}$$

and piecewise on-line adaptive algorithm

$$\dot{\hat{\theta}} = x^T P_i \varphi_i(x), \quad x \in R_i \tag{18}$$

then the closed-loop PWLDI is globally stable with disturbance attenuation γ_0.
The optimization problem is formulated as below.

$$\gamma_0 = \min \gamma \quad s.t. \quad (19-22)$$
$$variables: \; k_i, m_i, P_i, r_i; Z_i, W_i \succ 0$$
$$m_i = 0, r_i = 0 \quad i \in I_0 \tag{19}$$
$$\bar{F}_l^T(\bar{P}_i - \bar{P}_j)\bar{F}_l = 0, \quad l = R_i \cap R_j \tag{20}$$
$$\bar{P}_i - E_{1i} > 0 \tag{21}$$
$$\begin{bmatrix} \widehat{A}_{ik_1}^T \bar{P}_i + \bar{P}_i \widehat{A}_{ik_1} + \widetilde{C}_{ik_2}^T \widetilde{C}_{ik_2} & (\bullet)^T \\ \gamma^{-1}\widetilde{D}_{ik_1}^T \bar{P}_i^T & -1 \end{bmatrix} < 0 \tag{22}$$

where

$$\widehat{A}_{ik_1} = \widetilde{A}_{ik_1} + \widetilde{B}_i K_i,$$

Proof. It is shown by constrain (19) the origin is an equilibrium of the closed-loop PWLDI. We choose the same expression of piecewise quadratic Lyapunov function $V(x,\tilde{\theta})$ as the previous section.

It can be shown by Schur Complement Lemma [24], constraint (22) is equivalent to,

$$\widehat{A}_{ik_1}^T \bar{P}_i + \bar{P}_i \widehat{A}_{ik_1} + \gamma^{-2}\bar{P}_i \widetilde{D}_{ik_1}\widetilde{D}_{ik_1}^T \bar{P}_i^T + E_{2i}$$
$$+ \widetilde{C}_{ik_2}^T \widetilde{C}_{ik_2} < 0, \quad \forall k_1, k_2 \in [1, N] \tag{23}$$

Then we can obtain the following conclusions:

1) Global stability

Consider PWLDI (15-16) with $\omega(t) = 0$, it just needs to note that

$$\gamma^{-2}\bar{P}_i \widetilde{D}_{ik_1}\widetilde{D}_{ik_1}^T \bar{P}_i^T \geq 0, \widetilde{C}_{ik_1}^T \widetilde{C}_{ik_1} \geq 0$$

then it can be implied by constraint (23),

$$\widetilde{A}_{ik_1}^T \bar{P}_i + \bar{P}_i \widetilde{A}_{ik_1} + E_{2i} < 0,$$

Thus, by Theorem 1, we can conclude that the closed-loop PWLDI is globally stable in X.

2) H_∞ Performance

For given initial state $x(0)$, disturbance $\omega(t)$, parameter vector θ, control law and parameter adaptive law, let $\{t_j\}_1^{N_t}$ denote the switch times, i.e., at each time point t_j, the state trajectory transfers from polytopic region R_i to R_j.

Now, we consider the integral of $V(x,\tilde{\theta})$ from zero to infinity,

$$\Pi = \int_0^\infty \frac{d}{dt}(V(x,\tilde{\theta}))dt$$
$$= \sum_{j=1}^{N_t+1} \int_{t_{j-1}}^{t_j} \frac{d}{dt}(\bar{x}^T \bar{P}_{i_j}\bar{x} + \tilde{\theta}^T\tilde{\theta})dt$$
$$= \sum_{j=1}^{N_t+1}\sum_{k=1}^{N} \int_{t_{j-1}}^{t_j} \{\mu_k(x)[\bar{x}^T \bar{P}_{i_j}(\widehat{A}_{i_jk}^T + \widetilde{B}_{i_j}u + \widetilde{\varphi}_i\theta$$
$$+ \widetilde{D}_{i_jk}\omega) + (\widehat{A}_{i_jk}x + \widetilde{B}_{i_j}u + \widetilde{\varphi}_i\theta + \widetilde{D}_{i_jk}\omega)^T \bar{P}_{i_j}\bar{x}]$$
$$+ \dot{\hat{\theta}}^T\tilde{\theta} + \tilde{\theta}^T\dot{\hat{\theta}}\}dt$$

with the piecewise control law (17) and on-line adaptive algorithm (18), it can be shown that,

$$\Pi = \sum_{j=1}^{N_t+1}\sum_{k=1}^{N} \int_{t_{j-1}}^{t_j} \mu_k(x)[\bar{x}^T(\widehat{A}_{i_jk}^T \bar{P}_{i_j} + \bar{P}_{i_j}\widehat{A}_{i_jk})\bar{x}$$
$$+ \omega^T \widetilde{D}_{i_jk}^T \bar{P}_{i_j}\bar{x} + \bar{x}\bar{P}_{i_j}\widetilde{D}_{i_jk}\omega]dt$$

then, with the help of constrain (23), we obtain,

$$\Pi < \sum_{j=1}^{N_t+1}\sum_{k=1}^{N} \int_{t_{j-1}}^{t_j} \{\mu_k(x)[-\bar{x}^T\gamma^{-2}\bar{P}_{i_j}\widetilde{D}_{i_jk}\widetilde{D}_{i_jk}^T \bar{P}_{i_j}^T\bar{x}$$
$$- \bar{x}^T E_{2i_j}\bar{x} + \omega^T \widetilde{D}_{i_jk}^T \bar{P}_{i_j}\bar{x} + \bar{x}\bar{P}_{i_j}\widetilde{D}_{i_jk}\omega]$$
$$- \lambda_k(x)\bar{x}^T \widehat{C}_{i_jk}^T \widehat{C}_{i_jk}\bar{x}\}dt$$
$$< \sum_{j=1}^{N_t+1}\sum_{k=1}^{N} \int_{t_{j-1}}^{t_j} \{\mu_k(x)[-\bar{x}^T\gamma^{-2}\bar{P}_{i_j}\widetilde{D}_{i_jk}\widetilde{D}_{i_jk}^T \bar{P}_{i_j}^T\bar{x}$$
$$- \bar{x}^T E_{2i_j}\bar{x} + \gamma^2\omega^T\omega + \gamma^{-2}\bar{x}^T\bar{P}_{i_j}\widetilde{D}_{i_jk}\widetilde{D}_{i_jk}^T\bar{P}_{i_j}\bar{x}]$$
$$- z^T z\}dt$$
$$\leq \int_0^\infty [-z^T z + \gamma^2\omega^T\omega]dt.$$

Therefore,

$$\Pi = V(x(\infty)) - V(x(0)) \leq \int_0^\infty [-z^T z + \gamma^2\omega^T\omega]dt$$

which implies that with $x(0) = 0$,

$$\|z(t)\|_2 < \gamma\|\omega(t)\|_2.$$

Thus, the proof is completed.

V. Numerical Example

In this section, we will demonstrate the proposed PWLDI synthesis framework through a numerical example. The two control objectives are to design controllers under the constraints $\|k_i\|_\infty \leq 4$, $\|m_i\|_\infty \leq 4$ respectively, such that the closed-loop system is 1) asymptotically stable; 2) globally stable with minimal disturbance attenuation.

Example 1 Consider the following uncertain nonlinear system in $X = (-2, 2) \times (-4, 4)$,

$$\dot{x} \in \overline{\text{co}}\{f_k(x) + Ax + Bu + D\omega + \varphi(x)\theta, k = 1, 2\}$$
$$z \in \overline{\text{co}}\{C_k x, k = 1, 2\}$$

where

$$f_1(x) = \begin{bmatrix} 0.5x_1^2 - x_2 \\ |x_1| \end{bmatrix}, f_2(x) = \begin{bmatrix} 0.5|x_1|^3 - x_2 \\ |x_1| \end{bmatrix}$$
$$B = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, D = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \varphi(x) = \begin{bmatrix} -2x_1^2 \\ x_1^2 \end{bmatrix}$$
$$C_1 = \begin{bmatrix} 0.8 & 0 \end{bmatrix}, C_2 = \begin{bmatrix} 1.2 & 0 \end{bmatrix}, \theta \in (-0.2, 0.2)$$

With the four regions partition,

$$R_1 = (-2, -1) \times (-4, 4), R_2 = (-1, 0) \times (-4, 4),$$
$$R_3 = (0, 1) \times (-4, 4), R_4 = (1, 2) \times (-4, 4)$$

using the computing algorithm proposed in [22], the following PWLDI can be obtained as an envelop of the above uncertain nonlinear system,

$$\dot{x} \in \overline{\text{co}}\{A_{ik}x + b_{ik} + Bu + D\omega + \varphi(x)\theta, k = 1, 2\}$$
$$z \in \overline{\text{co}}\{C_k x, k = 1, 2\}$$

where

$$A_{11} = \begin{bmatrix} -3 & -1 \\ -1 & 0 \end{bmatrix}, A_{12} = \begin{bmatrix} -1.5 & -1 \\ -1 & 0 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} -0.5 & -1 \\ -1 & 0 \end{bmatrix}, A_{22} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$
$$A_{31} = \begin{bmatrix} 0.5 & 1 \\ -1 & 0 \end{bmatrix}, A_{32} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$
$$A_{41} = \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}, A_{42} = \begin{bmatrix} 1.5 & -1 \\ 1 & 0 \end{bmatrix}$$
$$b_{11} = b_{14} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}, b_{21} = b_{24} = \begin{bmatrix} -1.125 \\ 0 \end{bmatrix}$$

Note that the obtained PWLDI satisfies the Assumption 2, and $\psi(x)$=x_1^2, therefore, the following control law and on-line adaptative algorithm are employed,

$$u = -x_1^2\hat{\theta} + k_i x + m_i, \; x \in R_i$$
$$\dot{\hat{\theta}} = x^T P_i \varphi(x), \; x \in R_i$$

where k_i, m_i, P_i can be obtained via solving the proposed optimization problems in Theorem 1 and Theorem 2. The suboptimal solutions are obtained by YALMIP as below:

1) Adaptive Stabilization

$\alpha = 0.8463, m_1 = -3.8938, m_4 = 3.9964$

$k_1 = \begin{bmatrix} 3.9543 & -0.7016 \end{bmatrix}, k_2 = \begin{bmatrix} 3.9879 & -0.5860 \end{bmatrix}$

$k_3 = \begin{bmatrix} 3.9719 & -0.2531 \end{bmatrix}, k_4 = \begin{bmatrix} 3.9971 & -0.1019 \end{bmatrix}$

$$P_1 = \begin{bmatrix} 0.9660 & 0.1949 \\ 0.1949 & 0.4496 \end{bmatrix}, P_2 = \begin{bmatrix} 0.9967 & 0.1949 \\ 0.1949 & 0.4496 \end{bmatrix}$$
$$P_3 = \begin{bmatrix} 0.9514 & 0.2558 \\ 0.2558 & 0.4496 \end{bmatrix}, P_4 = \begin{bmatrix} 0.9977 & 0.2558 \\ 0.2558 & 0.4496 \end{bmatrix}$$

2) Adaptive H_∞ Control

$\gamma = 0.1625$

$K_1 = \begin{bmatrix} 3.5863 & 0.1972 & -3.1626 \end{bmatrix}$

$K_2 = \begin{bmatrix} 3.8581 & 0.2183 & 0 \end{bmatrix}$

$K_3 = \begin{bmatrix} 4.0000 & 0.8578 & 0 \end{bmatrix}$

$K_4 = \begin{bmatrix} 3.9993 & 0.7711 & 3.9988 \end{bmatrix}$

$$P_1 = \begin{bmatrix} 0.3323 & 0.0358 \\ 0.0358 & 0.1000 \end{bmatrix}, P_2 = \begin{bmatrix} 0.2134 & 0.0358 \\ 0.0358 & 0.1000 \end{bmatrix}$$
$$P_3 = \begin{bmatrix} 0.2002 & 0.0483 \\ 0.0483 & 0.1000 \end{bmatrix}, P_4 = \begin{bmatrix} 0.3284 & 0.0483 \\ 0.0483 & 0.1000 \end{bmatrix}$$

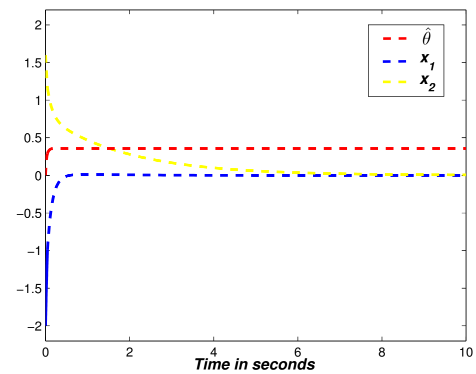

Fig. 1: System Response of original nonlinear system with obtained stabilization controller

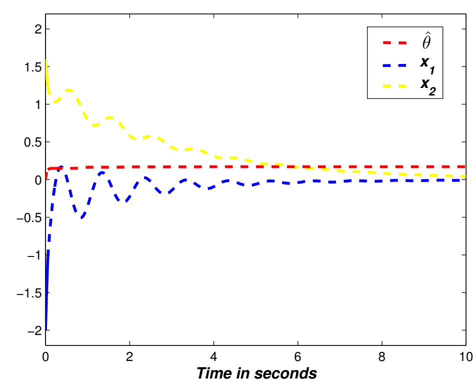

Fig. 2: System Response of original nonlinear system with obtained H_∞ controller

To simulate the performance of the obtained controllers for original uncertain nonlinear systems, we set

$$f(x) = \frac{1}{2}(f_1(x) + f_2(x)), \; C = \frac{1}{2}(C_1 + C_2), \; \theta = 0.1$$

then simulations(see Fig.1) have been carried out with initial condition x(0)=(-2,1.6) and $\hat{\theta}(0)$=0, which illustrates $\lim_{t\to\infty}\hat{\theta}(t) = 0.3588$, and the system trajectory of closed-loop nonlinear systems converges to origin in the absence of disturbance.

Furthermore, assume the system is subjected to a timevarying disturbance given by

$$w(t) = 5e^{-0.6t} \sin(2\pi t)$$

which has finite L_2 norm. Fig. 2 shows the response of the closed-loop nonlinear system. It can be seen that the disturbance is attenuated, $\lim_{t\to\infty} \hat{\theta}(t) = 0.1703$ and the system state converges to the origin as the disturbance converges to origin.

VI. A Remark on Sliding Motion

Actually, by extending the idea of [27], if there exists a quadratic Lyapunov function $V_l(x, \tilde{\theta}) \geq 0$ for each boundary $l \in L$, such that for all $x \in l = R_i \cap R_j$,

$$V_l(x, \tilde{\theta}) \geq 0 \tag{24}$$

$$\frac{\partial V_l}{\partial x} \bar{co}\{A_{\tau k}x + B_\tau u + \varphi_\tau(x)\theta\} + \frac{\partial V_l}{\partial \tilde{\theta}} \dot{\tilde{\theta}} < 0, \tau = i, j \tag{25}$$

the asymptotical stability can be guaranteed, although the sliding motion may happen.

Note that for all $x \in l$,

$$\bar{E}_l \bar{x} = 0, \quad \bar{E}_l = [E_l, e_l]$$

then using S-procedure, the constraints (24-25) can be expressed as BMI too.

VII. Conclusion

In this paper, a Lyapunov-based adaptive control method is developed for PWLDI with parameter uncertainty, where the piecewise quadric Lyapunov function is employed to establish the global stability and H_∞ performance, the piecewise control law and parameter adaptive algorithm can be obtained by solving a optimization problem subject to a set of BMIs. Moreover, the possibility of sliding motion at the boundary between regions is considered, which makes the design method rigorous.

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