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Static Nonsmooth Control Lyapunov Function Design via Dynamic Extension

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Abstract— **In this paper, we propose a method to obtain a control Lyapunov function (CLF) by the reduction of a CLF of an augmented system with a dynamic compensator. For asymptotically stabilizing control, dynamic compensators are not necessary in most of the cases. However, in some cases, we can easily design a stabilizing control law using a dynamic compensator. Therefore, a constructive design method using a static controller via a dynamic controller has advantages and is preferable in practice. In this paper, we assume that a CLF has been designed on an extended state space with a dynamic compensator, and show that taking minimum values of the CLF on the extended state space gives a nonsmooth CLF on the original state space. This method can be considered as an extension of the minimum projection method[1], [2]. We also show that the obtained CLF fulfills Lipschitz continuity and local semiconcavity if the original CLF on the extended state space is Lipschitz continuous and locally semiconcave. The effectiveness of the proposed method is demonstrated by an example.**

I. Introduction

In this paper, we propose a method obtaining a control Lyapunov function (CLF) by the reduction of a CLF of an augmented system with a dynamic compensator.

For asymptotic stabilization of nonlinear control systems, dynamic compensators are not necessary in most of the cases; i.e. every stabilizable system via a dynamic feedback can be also stabilized by a static state feedback in the sense of sampling solution[3]. However, in some cases where static controllers cannot be constructed straightforwardly, we can easily design stabilizing control laws using dynamic compensators. For instance, a decouplable system through a dynamic extension is easily linearizable with respect to state, using a dynamic compensator and a nonlinear coordinate transformation. It is well-known that systems that are not state-space linearizable via static feedback may be exactly linearized by using dynamic compensator[4]. In addition, using dynamic compensators, some systems on general manifolds are asymptotically stabilizable[5]. It is therefore natural to think whether a dynamic controller gives a static controller by a control-law reduction. There have been a series of studies of reduced-order control laws in linear systems. However, there were few studies of nonlinear systems. The model order of nonlinear systems can be reduced by the method of [6]. However, our method is distinguished from model-order reduction methods because we consider elimination of the unnecessary dynamics of the dynamic compensator.

In this study, we consider the problem of the order reduction of CLF, and employ the minimum projection method[1], [2], [7], [8] for the reduction. The minimum projection method is a nonsmooth CLF design method which is applicable to a general manifold that could be noncontractible. The original version[1] is a single-layer minimum projection method, and proposes a method of obtaining a nonsmooth CLF by projection of a minimum value of a CLF onto another simple manifold. The single-layer version is extended to the multilayer minimum projection method in [2]. The construction of a CLF by projection of the minimum value of CLFs onto an infinite number of layers is studied in [7]. Moreover, using the technique of desingularization[8], all singular points on the other manifolds, except the origin, are removed under a suitable condition.

In this paper, using the techniques in [5] and [6], we study the CLF order-reduction problem. We assume that a CLF has been designed on an extended state space with a dynamic compensator, and show that taking a minimum value of the CLF on the extended state space gives a nonsmooth CLF on the original state space. We also prove that the obtained CLF fulfills continuity, Lipschitz continuity, and local semiconcavity. Moreover, we demonstrate the effectiveness of the proposed method by an example.

II. Problem Statement

A. Control System on Original Manifold

We consider a nonlinear control system

$$\dot{x} = f(x,u),\ x \in X,\ u \in U = \mathbb{R}^\ell \tag{1}$$

defined on a differentiable manifold X, where $x \in X$ denotes a state variable, and $u \in U$ an input variable. The map $f : X \times U \to T_xX$ is assumed to be continuous with respect to x and u, where T_xX is a vector space called the tangent space to X at x, and an element of T_xX is called a tangent vector at x. We define 0 as the origin on X in this paper, and assume that $f(0,0) = 0$. We consider the CLF design problem for the global asymptotic stabilization of the origin $0 \in X$ of (1). In what follows, we call the differentiable manifold X the original manifold.

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B. Augmented System on Extended Space

In this paper, we consider an augmented system of (1), defined on an extended state space $\tilde{X}$, as follows:

$$\begin{aligned}\dot{x} &= f(x, u), \\ \dot{p} &= v,\end{aligned} \tag{2}$$

where $p \in P = \mathbb{R}^m$ and $v = (v_1, \ldots, v_m)$ denote a state vector and an input vector of the dynamic compensator, respectively. We can choose the space P at will, so, for simplicity, we regard P as a Euclidean space. The extended state space is a direct product space $\tilde{X} := X \times P$, hence $\tilde{x} = (x, p)$ represents a state of (2) on $\tilde{X}$. In turn, $\tilde{u} = (u, v)$ represents an input of (2) on $\tilde{X}$. We define the origin on the extended state space as $(x, p) = (0, 0)$. Suppose that S_x is an atlas on X; then, $\{(x, p) \to (\varphi(x), p), x \in N, p \in P \mid (N, \varphi) \in S_x\}$ gives an atlas of $\tilde{X}$ since we set P as a Euclidean space. The Euclidean norm $d_x(x_1, x_2) = \|\varphi(x_1) - \varphi(x_2)\|$ gives a local distance between x_1 and x_2; this distance depends on selection of the local coordinate, where these two points are contained in a certain coordinate neighborhood of S_x. Then, $d_{\tilde{x}}((x_1, p_1), (x_2, p_2)) = (\|\varphi(x_1) - \varphi(x_2)\|^2 + \|p_1 - p_2\|^2)^{1/2}$ also gives a local distance between (x_1, p_1) and (x_2, p_2).

C. Projection Map

In this paper, we consider a projection map,

$$\pi_p : (x, p) \mapsto x, \tag{3}$$

which associates (x, p) on the augmented space $\tilde{X}$ with x on the original state space X.

D. Solution Concept

The method that will be proposed in the following section allows the obtained CLF to be nondifferentiable, and a nondifferentiable CLF often leads a discontinuous control law. Discontinuous differential equations do not ensure the existence and the uniqueness of the solutions, and these properties vary according to the definition of the solutions of the nondifferentiable ordinary differential equations. In this paper, instead of considering the definition of the solutions of (1) and (2), a solution concept is adopted as in [1]. In what follows, we have to guarantee consistency between the solution of the original system (1) and that of the augmented system (2).

For the control system (1), solutions corresponding to an initial state $x \in X$ and a control $u(\tau) : T \to U$ at time $t \in T$ are represented by $\psi(t, x; u)$, where $T = [0, t_{max})$ is the range of t, and the value of t_{max} depends on x and $u(\cdot)$. Similarly $\tilde{\psi}(t, \tilde{x}; \tilde{u})$ is a solution of the extended system (2) corresponding to an initial state $\tilde{x} \in \tilde{X}$ and a control $\tilde{u}(\tau) : T \to U \times \mathbb{R}^m$.

Definition 2.1: We call $\psi(\,\cdot\,)$ and $\tilde{\psi}(\,\cdot\,)$ the solutions of (1) and (2), respectively, if the following conditions are fulfilled.

1) For fixed (x, u), maps $\psi : T \to X$ and $\tilde{\psi}$: $T \to \tilde{X}$ are continuous.
2) For the input $\tilde{u}(\tau) = (u(\tau), v(\tau))$ $(\tau \in T)$ and the initial state $\tilde{x} = (x, p)$ of the augmented system,

$$\pi_p \circ \tilde{\psi}(t, \tilde{x}; \tilde{u}) = \psi(t, x; u). \tag{4}$$

3) For arbitrary $t, s \in T$, with $t > s$, $\tilde{\psi}(t - s, \tilde{\psi}(s, \tilde{x}; \tilde{u}); \tilde{u}(s + \cdot)) = \tilde{\psi}(t, \tilde{x}; \tilde{u})$.

E. Control Lyapunov Function(CLF)

In this section, we define a control Lyapunov function (CLF) using the definition in Sontag's textbook[9] and in Nakamura et al.[1]. This definition is weaker than the one used in Sontag's paper[10]. The definition of CLF in [10] requires the existence of u such that $\dot{V} < 0$ everywhere outside of the origin, but, in our definition, $V(x(t))$ should only be a function of t, which is decreasing with some property.

Definition 2.2: If a continuous function V satisfies the following conditions, then V is called a control Lyapunov function (CLF) for (1).

(A1) V is proper, i.e., the set

$$\{x \in X | V(x) \le L\} \quad (L > 0) \tag{5}$$

is compact.

(A2) V is positive definite, i.e.,

$$V(0) = 0 \text{ and } V(x) > 0 \quad \text{for each } x \in X,\ x \ne 0. \tag{6}$$

(A3) For each $x \ne 0$, there exist some time $\sigma > 0$, and some control $u(t)(0 \le t < \sigma)$ which is admissible for x, such that

$$\begin{aligned}&V(\psi(t, x; u)) \le V(x) \quad (t \in [0, \sigma)), \\ &V(\psi(\sigma, x; u)) < V(x).\end{aligned} \tag{7}$$

For the extended space $\tilde{X}$, we introduce definition of CLF that is even weaker than the above definition.

Definition 2.3: If a continuous function $\tilde{V}$ satisfies the following conditions for (2), we say that $\tilde{V}$ is called a CLF almost everywhere:

(B1) $\tilde{V}$ is proper, i.e., the set

$$\{\tilde{x} \in \tilde{X} \mid \tilde{V}(x) \le L\} \quad (L > 0) \tag{8}$$

is compact.

(B2) V is positive definite, i.e.,

$$\tilde{V}(0) = 0 \text{ and } \tilde{V}(\tilde{x}) > 0 \quad \text{for each } \tilde{x} \in \tilde{X},\ \tilde{x} \ne 0. \tag{9}$$

(B3) There exists a set E with zero measure such that for each $\tilde{x} \in \tilde{X} \backslash E$, there exist some time $\sigma > 0$, and some control $\tilde{u}$ $(0 \le t < \sigma)$ which is admissible for $\tilde{x}$, satisfying

$$\begin{aligned}&\tilde{V}(\tilde{\psi}(t, \tilde{x}; \tilde{u})) \le \tilde{V}(\tilde{x}) \quad (t \in [0, \sigma)), \\ &\tilde{V}(\tilde{\psi}(\sigma, \tilde{x}; \tilde{u})) < \tilde{V}(\tilde{x}).\end{aligned} \tag{10}$$

In Definition 2.3, the condition (10) is not required at a point $\tilde{x} \in E$; this is the principal difference between Definitions 2.2 and 2.3.

In this paper, we assume the following for the augmented system.

Assumption 2.1: A continuous control Lyapunov function $\tilde{V}$ for (2) has been designed on $\tilde{X}$ almost everywhere except a set E, which has zero measure, in the sense of Definition 2.3. Moreover, for each $\tilde{x}_0 = (x_0, p_0) \in E, \tilde{x}_0 \neq (0,0)$, there exists p' such that

$$\tilde{V}(x_0, p') < \tilde{V}(x_0, p_0). \tag{11}$$

III. Main Theorem

The following is the main theorem in this study.

Theorem 3.1: Suppose Assumption 2.1 holds. Then, the function

$$V(x) := \min_p \tilde{V}(x, p). \tag{12}$$

is well-defined and becomes a continuous CLF on X. $\Diamond$

The following five lemmas lead to the proof of the theorem.

Lemma 3.1: $\tilde{V}$ takes a minimum value with respect to p for each fixed x, i.e.,

$$\min_p \tilde{V}(x, p) \tag{13}$$

is well-defined, for each $x \in X$. $\Diamond$

Proof: Given $x_0 \in X$, consider a set

$$A = \{(x,p) | \tilde{V}(x,p) \leq \tilde{V}(x_0, p_0)\}, \tag{14}$$

where p_0 is a arbitrary point in P. Since $\tilde{V}$ is proper, A is a compact subset. Let $\bar{A}$ be a complementary set of A. Because $\tilde{V}$ on $\bar{A}$ is greater than $\tilde{V}(x_0, p_0)$, $\tilde{V}(x_0, p)((x_0, p) \in \bar{A})$ is not the minimum value. Let F_{x_0} denote $\{(x,p) \,|\, x = x_0\}$. Then, $A \cap F_{x_0}$ is compact because the intersection of a compact subset and a closed set in topological space is always compact. A continuous function on a compact set takes a minimum value. Therefore, $\min_p \tilde{V}(x_0, p)$ is well-defined. ■

Lemma 3.2: If $\tilde{V}(x, p)$ is continuous, the function $V(x)$ defined by (12) is also continuous. $\Diamond$

Proof: We will show that $V(x)$ is continuous at each $x_0 \in X$. For arbitrary $\epsilon > 0$,

$$S_1 = \{(x,p) | \tilde{V}(x,p) < V(x_0) + \epsilon\} \tag{15}$$

is an open set since the image of an open set via the inverse map of a continuous map is always open. In addition, $\pi_p(S_1)$ is an open set because the projection map π_p is an open map. From the properness of $\tilde{V}$,

$$S_2 = \{(x,p) | \tilde{V}(x,p) \leq V(x_0) - \epsilon\} \tag{16}$$

is compact. The projection $\pi_p(S_2)$ is also compact since the image of a compact set via a continuous map is compact. Hence, $S_3 = \pi_p(S_1) \backslash \pi_p(S_2)$ is an open set. We will show that x_0 is contained in S_3. For $p_0 \in \mathrm{argmin}_p \tilde{V}(x_0, p)$,

$$\tilde{V}(x_0, p_0) = V(x_0) \leq V(x_0) + \epsilon, \tag{17}$$

is satisfied, so $\pi_p(S_1)$ contains x_0. If we assume x_0 is contained in $\pi_p(S_2)$, then $\tilde{V}(x_0, p_0) \leq V(x_0) - \epsilon$ must hold, but this contradicts $\tilde{V}(x_0, p_0) = V(x_0)$. Therefore, x_0 is not contained in $\pi_p(S_2)$. From this result, we can say that $x_0 \in S_3$, i.e., S_3 is an open neighborhood of x_0.

By the construction of S_3, obviously,

$$V(x_0) - \epsilon < V(x) < V(x_0) + \epsilon, x \in S_3. \tag{18}$$

Hence,

$$|V(x) - V(x_0)| < \epsilon, x \in S_3. \tag{19}$$

Finally, we can conclude that for any $x_0 \in X$, there exists an open neighborhood S_3 of x_0 such that (19) holds for any $\epsilon > 0$. The continuity of $V(x)$ on X is therefore proven. ■

Lemma 3.3: V is proper, i.e., the set

$$\{x \in X | V(x) \leq L\} \quad (L > 0) \tag{20}$$

is compact. $\Diamond$

Proof: From the properness of $\tilde{V}$, any level set $A_L = \{\tilde{x} = (x,p) \in \tilde{X} | \tilde{V}(\tilde{x}) \leq L\}$ $(L > 0)$ is compact. Hence, from the continuity of π_p, $\pi_p(A_L) = \{x | V(x) \leq L\}$ is compact for any $L > 0$. Therefore, V is a proper function. ■

Lemma 3.4: V is positive definite, i.e.,

$$V(0) = 0 \text{ and } V(x) > 0 \quad \text{for each } x \in X, \ x \neq 0. \tag{21}$$

$\Diamond$

Proof: First of all, obviously, $V(0) = 0$ because $\min_p \tilde{V}(0, p) = V(0) = 0$. Moreover, the condition $\tilde{V}(\tilde{x}) > 0$ $(\tilde{x} \neq 0)$ leads to $V(x) > 0$ for each $x \in X, x \neq 0$ since the hypothesis $V(x) \leq 0$ $(x \neq 0)$ requires the existence of p such that $\tilde{V}(x,p) \leq 0$, which contradicts the positive definiteness of $\tilde{V}$. Consequently, $V(x)$ is positive definite. ■

Lemma 3.5: For each $x \neq 0$, there exist some time $\sigma > 0$, and some control $u(t)$ $(0 \leq t < \sigma)$ which is admissible for x, such that for the path $\psi(t, x; u)$,

$$\begin{aligned} &V(\psi(t,x;u)) \leq V(x)(t \in [0,\sigma)), \\ &V(\psi(\sigma,x;u)) < V(x). \end{aligned} \tag{22}$$

$\Diamond$

Proof: Let

$$L(x) = \operatorname*{argmin}_p \tilde{V}(x,p). \tag{23}$$

We regard $L(x)$ as a set because the p minimizing $\tilde{V}$ is not always unique. Suppose that $\tilde{x} = (x,p)$ satisfies $p \in L(x)$. By Assumption 2.1, $\tilde{x} \notin E$. Therefore $\tilde{V}$ satisfies (B3), so there exists $\sigma > 0$ such that

$$\begin{aligned} &\tilde{V}(\tilde{\psi}(t,\tilde{x};\tilde{u})) \leq \tilde{V}(\tilde{x}) \ (t \in [0,\sigma)), \\ &\tilde{V}(\tilde{\psi}(\sigma,\tilde{x};\tilde{u})) < \tilde{V}(\tilde{x}). \end{aligned} \tag{24}$$

Since $p \in L(x)$, $\tilde{V}(\tilde{x}) = V(x)$ holds. On the other hand, it is clear that

$$V(\psi(t,x;u)) \leq \tilde{V}(\tilde{\psi}(t,\tilde{x};\tilde{u})) \ (t \in [0,\sigma]), \tag{25}$$

from (4) and (12). Therefore, we obtain

$$\begin{aligned} &V(\psi(t,x;u)) \leq \tilde{V}(\tilde{\psi}(t,\tilde{x};\tilde{u})) \leq \tilde{V}(\tilde{x}) = V(x) \ (t \in [0,\sigma)), \\ &V(\psi(\sigma,x;u)) \leq \tilde{V}(\tilde{\psi}(\sigma,\tilde{x};\tilde{u})) < \tilde{V}(\tilde{x}) = V(x). \end{aligned} \tag{26}$$

Consequently, V satisfies (A3) of Definition 2.2, and this lemma has been proven. ■

Proof of Theorem 3.1 : By Lemmas 3.1 and 3.2, $V(x)$ is well-defined and continuous. Lemmas 3.3, 3.4, and 3.5 show that $V(x)$ defined by (12) satisfies the conditions (A1), (A2), and (A3) in Definition 2.2. Consequently, it is shown that Theorem 3.1 holds. ■

IV. Local Lipschitzness of CLF

The class of CLF satisfying the local Lipschitz condition is very useful for control design because there exist derivatives of a locally Lipschitz CLF in a weak sense, e.g., Dini lower/upper directional derivatives. We give the definition of local Lipschitzness.

Definition 4.1: We say a function $f : X \to \mathbb{R}$ is locally Lipschitz at $x_0 \in X$, if there exist $M > 0$, which is called the locally Lipschitz constant, and a sufficiently small neighborhood Ω of x_0, such that

$$\|f(x_0) - f(y)\| \leq M d_x(x_0, y) = M\|\varphi(x_0) - \varphi(y)\|, \quad (27)$$

for all $y \in \Omega \subset N \subset \mathbb{R}^n$, where (N, φ) is a local chart of X at x_0.

Remark : The existence of a locally Lipschitz constant does not depend on the choice of local chart, but the value of the locally Lipschitz constant depends on the choice of local chart. Hence, we only prove that V satisfies the locally Lipschitz condition on a certain local chart to say that V satisfies the locally Lipschitz condition.

Theorem 4.1: If $\tilde{V}$ is a locally Lipschitz continuous CLF on $\tilde{X}$, then V is a locally Lipschitz continuous CLF on X. $\diamondsuit$

Proof of Theorem 4.1 : Let us consider an open neighborhood N around $x_0 \in X$, which is a domain of the local chart φ, and consider Ω as a closed neighborhood containing x_0 and contained in N. We can easily show that the two sets

$$\begin{aligned} U_1 &= \{(x,p) \in \Omega \times P | \tilde{V}(x,p) \leq V(x_0) + 2a\}, \\ U_2 &= \{(x,p) \in \Omega \times P | \tilde{V}(x,p) \leq V(x_0) + a\} \end{aligned} \quad (28)$$

are compact, where a is some positive constant. From the assumption of the theorem, for any $(x_1, p_1), (x_2, p_2) \in U_1 \cap \Omega$,

$$|\tilde{V}(x_1,p_1) - \tilde{V}(x_2,p_2)| \leq C_1 d_{\tilde{x}}((x_1,p_1),(x_2,p_2)) \quad (29)$$

is satisfied, where C_1 is the Lipschitz constant of $\tilde{V}$ on $U_1 \cap \Omega$. Note that the local metric in $\tilde{X} := X \times P$ was previously given in section II-B. Let us define a cylindrical neighborhood centered at x_0 as

$$B_\epsilon(x_0) = \{(x,p) | \, d_x(x,x_0) \leq \epsilon\}, \quad 0 < \epsilon \leq M_1, \quad (30)$$

where $M_1 < a/C_1$ and $\pi_p \circ B_{M_1}(x_0) \subset \Omega$. Then, we consider the following sets $S_{1\epsilon}$ and $S_{2\epsilon}$ such that

$$\begin{aligned} S_{1\epsilon} &= \{p | (\exists x, p) \in B_\epsilon(x_0) \cap U_2\}, \\ S_{2\epsilon} &= \{p | (\exists x, p) \in B_\epsilon(x_0) \cap \text{cl}\, \bar{U}_1\}. \end{aligned} \quad (31)$$

We will show that $S_{1\epsilon} \cap S_{2\epsilon}$ is empty for an ϵ that is small enough. Let us consider a positive monotonically decreasing sequence ϵ_1, ϵ_2,..., where $\epsilon_1 < M_1$ and $\epsilon_k \to 0$ $(k \to \infty)$. For this sequence, there exist inclusive sequences

$$\begin{aligned} S_{1\epsilon_1} &\supset S_{1\epsilon_2} \supset ..., \\ S_{2\epsilon_1} &\supset S_{2\epsilon_2} \supset ..., \end{aligned} \quad (32)$$

and it is obvious from the continuity of $\tilde{V}$ that

$$\begin{aligned} S_{1\epsilon_k} &\to \{p | \tilde{V}(x_0,p) \leq V(x_0) + a\}, \\ S_{2\epsilon_k} &\to \{p | \tilde{V}(x_0,p) \geq V(x_0) + 2a\}, \end{aligned} \quad (33)$$

as k tends to infinity. Therefore, there exists $\epsilon_0 > 0$ such that $S_{1\epsilon_0} \cap S_{2\epsilon_0}$ is empty. Then, for an arbitrary x such that $d_x(x,x_0) \leq \epsilon < \epsilon_0$,

$$\tilde{V}(x_0,p) - C_1\epsilon \leq \tilde{V}(x,p) \leq \tilde{V}(x_0,p) + C_1\epsilon, \; p \in S_{1\epsilon_0}. \quad (34)$$

By the definition of $S_{1\epsilon_0}$, we obtain

$$V(x_0) + a \leq \tilde{V}(x,p), \quad p \in \bar{S}_{1\epsilon_0}. \quad (35)$$

which shows that $L(x_0) \subset S_{1\epsilon_0}$. For $p_0 \in L(x_0)$ specifically, (34) becomes

$$V(x_0) - C_1\epsilon \leq \tilde{V}(x,p_0) \leq V(x_0) + C_1\epsilon, \quad (36)$$

and this leads to

$$V(x) \leq \tilde{V}(x,p_0) \leq V(x_0) + C_1\epsilon. \quad (37)$$

Moreover, for $p_1 \in L(x)$, (34) becomes

$$\tilde{V}(x_0,p_1) - C_1\epsilon \leq V(x) \leq \tilde{V}(x_0,p_1) + C_1\epsilon, \quad (38)$$

so we obtain

$$V(x) - C_1\epsilon \leq \tilde{V}(x_0,p_1) - C_1\epsilon \leq V(x). \quad (39)$$

Therefore, for any ϵ $(< \epsilon_0)$,

$$|V(x) - V(x_0)| \leq C_1\epsilon, \quad d_x(\forall x, x_0) \leq \epsilon. \quad (40)$$

is established. Specifically, for x such that $d_x(x,x_0) = \epsilon$, the above inequality becomes

$$|V(x) - V(x_0)| \leq C_1 d_x(x,x_0). \quad (41)$$

Therefore, we can say that for all x in $\Omega' = \{x \,|\, d_x(x,x_0) < \epsilon_0\}$, which is a neighborhood of x_0, (41) is satisfied. This consequence holds for each x_0, hence $V(x)$ is locally Lipschitz. ■

V. Local Semiconcavity of CLF

Locally semiconcave CLFs represent an important class of CLFs[13]. We will show the local semiconcavity condition of the obtained CLF V.

Definition 5.1: We say a continuous function f is locally semiconcave at $x_0 \in X$, if there exist $C > 0$ and a sufficiently small neighborhood Ω of x_0 such that

$$\begin{aligned} & f(x_0) + f(y) - 2f\left(\varphi^{-1}\left(\frac{\varphi(x_0) + \varphi(y)}{2}\right)\right) \\ & \leq C d_x(x_0,y)^2 = C\|\varphi(x_0) - \varphi(y)\|^2, \\ & \forall y \in \Omega \subset N, \end{aligned} \quad (42)$$

where (N, φ) is a local chart around x_0.

Theorem 5.1: If $\tilde{V}$ is a locally semiconcave function on $\tilde{X}$, then V is a locally semiconcave function on X. $\diamondsuit$

Proof of Theorem 5.1 : There exists a compact set $\Omega \subset X$ such that $x_0 \in$ interior of Ω, $\Omega \subset N$, where (N, φ) is a local chart around x_0, and $\varphi(\Omega)$ is convex. Moreover, we consider a compact set $S = \{(x,p) \in \Omega \times P \,|\, \tilde{V}(x,p) \leq$

$a + \epsilon\} \subset \tilde{X}$, where $a = \sup_{x\in\Omega} V(x)$, and $\epsilon > 0$. Now, since $\tilde{V}(x,p)$ is semiconcave on S, there exists C such that

$$\begin{aligned}&\tilde{V}(x,p_x) + \tilde{V}(y,p_y) - 2\tilde{V}\left(\frac{x+y}{2}, \frac{p_x+p_y}{2}\right)\\ &\le Cd_{\tilde{x}}((x,p_x),(y,p_y))^2, \\ &\forall (x,p_x),(y,p_y), \left(\frac{x+y}{2}, \frac{p_x+p_y}{2}\right) \in S,\end{aligned} \tag{43}$$

where $(x+y)/2$ denotes $\varphi^{-1}((\varphi(x)+\varphi(y))/2)$. Let p_0 be an element of $\mathrm{argmin}_p \tilde{V}((x+y)/2, p)$, and x, y are arbitrary points in Ω. Then, $V((x+y)/2) = \tilde{V}((x+y)/2, p_0)$ holds. Note that $V(x) \le \tilde{V}(x,p_0)$ and $V(y) \le \tilde{V}(y,p_0)$. Hence,

$$\begin{aligned}&V(x) + V(y) - 2V((x+y)/2)\\ &\le \tilde{V}(x,p_0) + \tilde{V}(y,p_0) - 2\tilde{V}((x+y)/2,p_0)\\ &\le Cd_{\tilde{x}}((x,p_0),(y,p_0))^2\end{aligned} \tag{44}$$

In addition, we get

$$Cd_{\tilde{x}}((x,p_0),(y,p_0))^2 = Cd_x(x,y)^2, \tag{45}$$

where the local norms have the relationship stated in section II-B. From (44) and (45), we obtain

$$V(x) + V(y) - 2V((x+y)/2) \le Cd_x(x,y)^2. \tag{46}$$

Consequently, the semiconcavity of V on any Ω for any x_0 is established, and the theorem is proven. ■

VI. Example

In this section, we apply the proposed method to the example which is a decouplable system by the dynamic extension method. We consider the system

$$\begin{aligned}\dot{x}_1 &= u_1,\\ \dot{x}_2 &= u_2,\\ \dot{x}_3 &= 2x_2 - x_2\frac{u_1}{1+u_1^2}.\end{aligned} \tag{47}$$

The system is not an affine system, so finding a stabilizing static feedback for (47) is nontrivial. However, this system is easily stabilized by a dynamic feedback. Let us consider the dynamic compensator

$$\dot{u}_1 = v, \tag{48}$$

where u_1 is regarded as a component of the state of the augmented system (47), (48). The augmented system is exactly state-space linearizable. Actually, under the state coordinate transformation $z = (x_1, u_1, x_3, 2x_2 - x_2u_1/(1+u_1^2))^T$ and the feedback

$$\begin{pmatrix} v \\ u_2 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ -\dfrac{x_2(1-u_1^2)}{(1+u_1^2)^2} & 2 - \dfrac{u_1}{1+u_1^2}\end{bmatrix}^{-1} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \tag{49}$$

the system can be converted into

$$\dot{z} = \begin{pmatrix} 0&1&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{pmatrix} z + \begin{pmatrix} 0&0\\1&0\\0&0\\0&1\end{pmatrix}\begin{pmatrix} w_1 \\ w_2\end{pmatrix}. \tag{50}$$

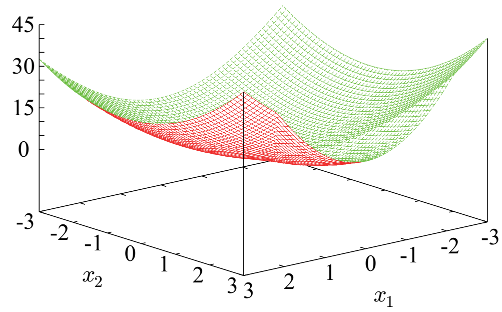

Fig. 1. The obtained CLF on the surface $x_3 = 0$.

We can obtain a CLF

$$\begin{aligned}\tilde{V}'(z) &= z^T P_0 z\\ P_0 &= \begin{pmatrix} 2&1&0&0\\1&1&0&0\\0&0&2&1\\0&0&1&1\end{pmatrix}\end{aligned} \tag{51}$$

for the augmented system (47), (48). Therefore, $\tilde{V}(\tilde{x}) = z'(\tilde{x})^T P_0 z'(\tilde{x})$, $z'(\tilde{x}) = (x_1, u_1, x_3, 2x_2 - x_2p/(1+p^2))^T$ is a CLF for the system (47) with

$$\dot{p} = v, \tag{52}$$

where $(x_1, x_2, x_3, p)^T$ is the state vector and $(u_1, u_2, v)^T$ is the input vector, because, by choosing the input u_1 as $u_1 = p$, the system (47), (52) coincides with (47), (48).

By taking a minimum value of $\tilde{V}$, we obtain a nonsmooth CLF V on the original manifold. Fig. 1 shows the obtained CLF on the surface $x_3 = 0$; it indicates that V is nondifferentiable. To obtain the value of p minimizing $\tilde{V}$, we have to solve a seventh-order polynomial equation numerically. However, if we obtain the value of p minimizing $\tilde{V}$ uniquely, we can calculate the derivative of V as $(\partial V/\partial x)(x) = (\partial \tilde{V}/\partial x)(x,p)$ because $\tilde{V}$ is differentiable in this case and $\partial\tilde{V}/\partial p = 0$ for $p \in L(x)$. Generally in this example, the directional derivative of $V(x)$ can be obtained as

$$\nabla_f V(x) = \min_{p\in L(x)} \nabla_{f'} \tilde{V}(x,p), \quad f' = [f^T, 0]^T. \tag{53}$$

We make a simulation where the original plant (47) is controlled via the obtained CLF V. The control stabilizing (47) is chosen as $u_1 = \alpha_1(x) = \mathrm{argmin}_p \tilde{V}$, $u_2 = \alpha_2(x)$, where α_2 is the Sontag-type control law

$$\alpha_2(x) = -\frac{L_{\tilde{f}}V + \sqrt{L_{\tilde{f}}V^2 + L_{\tilde{g}}V^4}}{L_{\tilde{g}}V} \tag{54}$$

$$\tilde{f}(x) = (\alpha_1(x), 0, 2x_2 - x_2\alpha_1(x)/(1+\alpha_1(x)^2))^T \tag{55}$$

$$\tilde{g}(x) = (0,1,0)^T. \tag{56}$$

At the nondifferentiable point of V, $\dot{V}$ is not determined uniquely because $\alpha_1(x)$ and $\alpha_2(x)$ are set-valued functions. Therefore, $\dot{V}$ under the feedback $u_1 = \alpha_1(x)$, $u_2 = \alpha_2(x)$ is a function of x and $p(\in L(x))$. In the actual controller, we choose p as $\mathrm{argmin}_{p\in L(x)} \dot{V}(x,p)$, and then $\alpha_1(x)$ and

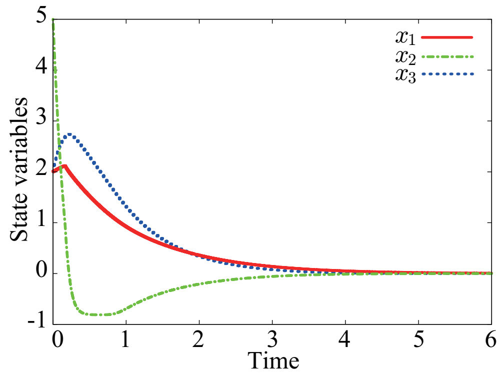

Fig. 2. Time responses of the state variables.

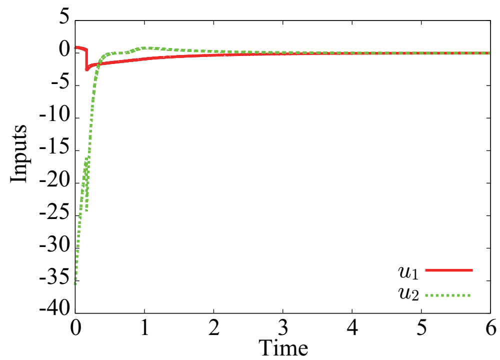

Fig. 3. Time responses of the inputs.

$\alpha_2(x)$ can be determined uniquely. Fig. 2 shows the time responses of the state variables, and Fig. 3 shows the time responses of the inputs. We can see that the state of the controlled system tends to the origin, and the control inputs are discontinuous with respect to time.

Remark : In this example, $\alpha_1(x)$ and $\alpha_2(x)$ stabilize the system (44). However, in general, the definition of CLF is weaker than one of [10], so it is hard to give a general method to construct a stabilizing control law based on the obtained CLF. It is expected that for the general controller construction some additional conditions are necessary. However, the construction method of the controller is out of range of this paper's discussion, and it is our future work.

VII. Conclusions

In this paper, we propose a construction method for a CLF by reduction of a CLF designed on an extended space. We show that if we construct a continuous CLF almost everywhere on an extended space with the property (11), taking minimum values of the CLF gives a continuous CLF on the original manifold. Moreover, if the CLF on the extended space is locally Lipschitz, the obtained CLF on the original manifold is also locally Lipschitz, and the local semiconcavity of the CLF on the extended space gives the local semiconcavity of the CLF on the original manifold. The proposed method enables asymptotic stabilization for a class which is linearizable by a dynamic extension using static state feedback. Furthermore, this method is applicable to stabilization problems on noncontractible manifolds via the CLF designed for the augmented system as described in [5].

The proposed method is similar to the multilayer minimum projection method with infinite layers[7], but our method is different to that of [7] in the following two points. First, we assume a CLF has been designed on an etale bundle (specifically a fiber bundle in our case) in this method, but in [7] a CLF is designed on each layer. Secondly, we do not suppose compactness of the variable which represents the index of layers, but assume properness of the CLF on the extended space. In addition to these points, the topological structure of the extended space is nontrivial in [7]. In this regard, [7] addresses more general topological spaces.

Extension of the proposed method to the problem of the construction of a strict CLF[2], [8] and control law synthesis[11] based on a nonsmooth CLF are our future works.

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