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Two alternative approaches to stochastic discrete-time iterative learning control systems

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***Abstract*— This paper aims to address the robust convergence problem that arises from discrete-time iterative learning control (ILC) systems subject to random disturbances. Two alternative approaches are considered in order to achieve the perfect output tracking of the stochastic discrete-time ILC systems in the sense of both expectation and variance, which use the tracking error and the input error for analysis, respectively. It is shown that the convergence results of two approaches to ILC can be established by developing some statistical expressions in super-vector forms. Moreover, it is demonstrated that the convergence results of two approaches to ILC are not always equal, and they can keep the same only in the case where the controlled plants are square.**

***Index Terms*— Iterative learning control, discrete-time systems, random disturbances.**

I. Introduction

Iterative learning control (ILC) is developed as an effective technique aimed at the tracking performance improvement of systems that operate repetitively over a finite time interval. Its key feature is to use the information from previous operations to update the control signal for the current operation, in order to finally achieve the perfect tracking of any output reference trajectory. Due to its effectiveness, ILC has attracted considerable attention in many areas and applications over the past two decades, as claimed and demonstrated in, e.g., [1]-[4].

In the literature, the robustness of ILC has been considered a practically important issue with respect to, e.g., initial shifts [5]-[8] and model uncertainties [9]-[12]. For the past decade, the robustness issue arising from non-repetitive disturbances has attracted considerable attention in ILC. In [13], [14], the rejection of non-repetitive disturbances has been studied via adding some filters to ILC. From the stochastic point of view, the disturbance properties of ILC have been investigated, and some promising statistical expressions have been obtained for the control error in [15]-[20]. Using an H_∞ analysis approach to ILC (see, e.g., [21]-[23]), robust convergence results not only can be established against iteration-varying disturbances but also can be developed to derive formulas for the design of learning gains via the linear matrix inequality technique. In general, two alternative approaches have been employed to achieve such ILC developments, with one using the tracking error for analysis directly ([13]-[15] and [19]-[23]), and the other one using the input error for analysis and then inducing the convergence results of the tracking error indirectly ([16]-[18]). Under certain conditions, the zero output tracking can be obtained by both approaches. However, this result for two alternative approaches does not always hold, which in fact requires the controlled plants being square. This observation motivates the present study.

In this paper, the robust convergence of ILC is considered for a class of discrete-time systems with disturbances varying randomly from one iteration to the next, for which the two alternative approaches to ILC are applied, respectively. From the stochastic point of view, it demonstrates that although the convergence results can be developed through both direct and indirect approaches to ILC, they are equal only in the case where the input number and output number of the controlled plants are equal. In particular, this demonstration is given for stochastic discrete-time ILC in the sense of both expectation and variance.

Throughout this paper, $\mathrm{E}[\cdot]$ and $\mathrm{Var}[\cdot]$ represent the expectation operator and the variance operator with respect to the iteration domain, respectively, and q represents the forward shift operator along the time axis, e.g., $qx(t)=x(t+1)$ and $q^{-1}x(t)=x(t-1)$. Matrices are assumed to be compatible for algebraic operations if their dimensions are not stated explicitly. In particular, I and 0 denote the identity matrix and the zero matrix with the required dimensions, respectively. Additionally, $\|\alpha\|_2=\sqrt{\alpha^{\mathrm{T}}\alpha}$ is the 2-norm of a vector α; $\|\Omega\|_2=\bar{\sigma}(\Omega)$ is the 2-norm of a finite dimensional matrix Ω, where $\bar{\sigma}(\Omega)$ is the largest singular value of Ω; $\|\Omega\|_{\mathrm{F}}=\sqrt{\mathrm{tr}(\Omega^{\mathrm{T}}\Omega)}$ is the F-norm of Ω, where $\mathrm{tr}(\Omega^{\mathrm{T}}\Omega)$ is the trace of $\Omega^{\mathrm{T}}\Omega$.

II. Problem Formulation

A. System Description

Let us consider the following class of MIMO discrete-time systems over $0\le t\le T-1$, $k\ge 0$:

$$y_k(t)=G_p(q)u_k(t)+v_k(t) \tag{1}$$

where $y_k(t)$ is a $p\times 1$ output vector, $u_k(t)$ is an $m\times 1$ control input vector, $v_k(t)$ is a $p\times 1$ disturbance vector, and $G_p(q)$ is a $p\times m$ transfer function matrix. For disturbances of system (1), the following statistical assumptions are made.

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A1) Assume that $\{v_k(t)\}$ is a disturbance sequence which is white in the iteration domain such that $\mathrm{E}[v_k(t)] = 0$ and for $\forall l$,

$$\mathrm{E}\left[v_k(t)v_{k+i}^{\mathrm{T}}(l)\right] = \begin{cases} V_{tl}, & i = 0 \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

where $V_{tl} = V_{lt}^{\mathrm{T}}$, and V_{tt} is positive-definite.

Generally, the disturbance sequence is also considered to be white in the time domain, which results in $V_{tl} = 0$ for $\forall t \neq l$.

A2) It is assumed that $u_0(t)$ is bounded and uncorrelated with $v_k(t)$, i.e.,

$$\mathrm{E}\left[u_0(t)v_k^{\mathrm{T}}(l)\right] = 0, \quad \forall t, l. \tag{3}$$

In particular, the initial control input can be set to zero, i.e., $u_0(t) = 0$ is adopted for $\forall t$, to satisfy Assumption A2).

For system (1), our objective is such that the convergence of both expectation and variance of the tracking error between the system output $y_k(t)$ and the desired output trajectory $y_d(t)$ is achieved in the presence of random disturbances varying from one iteration to the next. Towards this end, the updating law considered in this paper is given by

$$u_{k+1}(t) = u_k(t) + L(q)e_k(t) \tag{4}$$

where $L(q)$ is an $m \times p$ polynomial gain operator, and $e_k(t) = y_d(t) - y_k(t)$ is the tracking error.

Remark 1: From the ILC system (1) and (4), it is easy to derive that

$$\begin{aligned} u_k(t) &= G_u(q)u_{k-1}(t) + L(q)\left[y_d(t) - v_k(t)\right] \\ &= G_u^k(q)u_0(t) + \sum_{i=0}^{k-1} G_u^{k-i-1}(q)L(q)\left[y_d(t) - v_i(t)\right] \end{aligned} \tag{5}$$

where $G_u(q) = I - L(q)G_p(q)$. Then for $i \geq 0$, combining (2) and (3) with (5) leads to

$$\mathrm{E}\left[u_k(t)v_{k+i}^{\mathrm{T}}(l)\right] = 0, \quad \forall t, l \tag{6}$$

i.e., Assumptions A1) and A2) imply that $u_k(t)$ is uncorrelated with $v_{k+i}(t)$ for $i \geq 0$.

B. System Representation

As demonstrated in, e.g., [3], [4], [19], [22], [23], a pure iteration-domain representation for linear discrete-time ILC systems can be established using the super-vector approach. With this approach and in a compact form, the system (1) and ILC law (4) can be described respectively as

$$\mathbf{y}_k = \mathbf{G_p}\mathbf{u}_k + \mathbf{v}_k \tag{7}$$

$$\mathbf{u}_{k+1} = \mathbf{u}_k + \mathbf{L}\mathbf{e}_k \tag{8}$$

where $\mathbf{y_k}$, $\mathbf{u}_k$, $\mathbf{v}_k$, $\mathbf{e}_k = \mathbf{y_d} - \mathbf{y}_k$, $\mathbf{y_d}$ are super-vectors lifted to contain T time points. For example, let r denote the relative degree of system (1) and then denote $G_p(q) = \sum_{i=r}^{\infty} H_i q^{-i}$ and $L(q) = \sum_{i=0}^{r} L_i q^i$, where H_i and L_i are matrices of appropriate dimensions. In this case, if the super-vectors are defined in the form of

$$\begin{aligned} \mathbf{y}_k &= \left[y_k^{\mathrm{T}}(r), \ \cdots, \ y_k^{\mathrm{T}}(T-1+r)\right]^{\mathrm{T}} \\ \mathbf{u}_k &= \left[u_k^{\mathrm{T}}(0), \ \cdots, \ u_k^{\mathrm{T}}(T-1)\right]^{\mathrm{T}} \\ \mathbf{v}_k &= \left[v_k^{\mathrm{T}}(r), \ \cdots, \ v_k^{\mathrm{T}}(T-1+r)\right]^{\mathrm{T}} \end{aligned}$$

then (7) and (8) hold with matrices $\mathbf{G_p}$ and $\mathbf{L}$ given by

$$\mathbf{G_p} = \begin{bmatrix} H_r & 0 & \cdots & 0 \\ H_{r+1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ H_{r+T-1} & \cdots & H_{r+1} & H_r \end{bmatrix}$$

$$\mathbf{L} = \begin{bmatrix} L_r & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ L_0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & L_0 & \cdots & L_r \end{bmatrix}.$$

Clearly, $\mathbf{G_p}$ is the corresponding $pT \times mT$ lower-triangular block Toeplitz matrix whose elements are the pulse response coefficients (or Markov parameters), and $\mathbf{L}$ is also a lower-triangular block Toeplitz matrix which is of $mT \times pT$ dimensions and results from the polynomial operator $L(q)$.

Remark 2: For $\{\mathbf{v}_k\}$, Assumption A1) implies $\mathrm{E}[\mathbf{v}_k] = 0$, $\mathrm{E}\left[\mathbf{v}_k\mathbf{v}_{k+i}^{\mathrm{T}}\right] = 0$, $i \neq 0$ and $\mathrm{E}\left[\mathbf{v}_k\mathbf{v}_k^{\mathrm{T}}\right] = \mathbf{V}$, where $\mathbf{V}$ is a positive-definite block matrix formed by V_{tl}. Using Assumptions A1) and A2), it leads to (6) which gives $\mathrm{E}\left[\mathbf{u}_k\mathbf{v}_{k+i}^{\mathrm{T}}\right] = 0$, $i \geq 0$.

In ILC, the convergence objective is generally carried out in two alternatives: either directly using the tracking error for analysis or using the control input error for analysis and then indirectly inducing convergence of the tracking error. This paper considers two alternatives for analysis separately, discusses differences between the results obtained with them, and presents some conditions under which these differences can be eliminated. In order to present the convergence result, the following lemma will be used.

Lemma 1: Let X and Y be two real matrices of appropriate dimensions. Then

(1) [24, p. 298]: $\lim_{k\to\infty} X^k = 0$ if and only if $\rho(X) < 1$, i.e., the spectral radius of X is strictly less than one;

(2) [24, p. 300]: $\sum_{k=0}^{\infty} X^k$ is well defined if and only if $\rho(X) < 1$;

(3) [24, p. 313]: $\|XY\|_{\mathrm{F}} \leq \min\{\|X\|_2 \|Y\|_{\mathrm{F}}, \|X\|_{\mathrm{F}} \|Y\|_2\}$.

III. Direct Approach

Let us consider the convergence analysis of ILC using the tracking error directly. First of all, it is easy to show that the expected tracking error satisfies (see [22], [23])

$$\begin{aligned} \mathrm{E}[\mathbf{e}_k] &= \mathbf{G_e}\mathrm{E}[\mathbf{e}_{k-1}] + \mathrm{E}[\mathbf{v}_{k-1}] - \mathrm{E}[\mathbf{v}_k] \\ &= \mathbf{G_e}\mathrm{E}[\mathbf{e}_{k-1}] \\ &= \mathbf{G_e}^k\mathrm{E}[\mathbf{e}_0] \end{aligned} \tag{9}$$

where $\mathbf{G_e} = I - \mathbf{G_p}\mathbf{L}$. Next, denote $\hat{\mathbf{e}}_k = \mathbf{e}_k - \mathrm{E}[\mathbf{e}_k]$. Then the variance matrix of the tracking error $\mathrm{Var}[\mathbf{e}_k]$ is considered of the form

$$\mathrm{Var}[\mathbf{e}_k] = \mathrm{E}\left[\hat{\mathbf{e}}_k\hat{\mathbf{e}}_k^{\mathrm{T}}\right]. \tag{10}$$

From [22], [23], an expression of $\mathrm{Var}[\mathbf{e}_k]$ can be given as

$$\mathrm{Var}[\mathbf{e}_k] = \mathbf{G_e}\mathrm{Var}[\mathbf{e}_{k-1}]\mathbf{G_e}^{\mathrm{T}} + \mathbf{G_p}\mathbf{L}\mathbf{V} + \mathbf{V}(\mathbf{G_p}\mathbf{L})^{\mathrm{T}} \tag{11}$$

and the stationary value of the variance matrix can be found as the solution to the Lyapunov equation

$$\Pi = \mathbf{G_e}\Pi\mathbf{G_e}^{\mathrm{T}} + \mathbf{G_p}\mathbf{L}\mathbf{V} + \mathbf{V}\left(\mathbf{G_p}\mathbf{L}\right)^{\mathrm{T}}. \tag{12}$$

Using (11) and (12) can lead to

$$\begin{aligned}\mathrm{Var}\left[\mathbf{e}_k\right] - \Pi &= \mathbf{G_e}\left(\mathrm{Var}\left[\mathbf{e}_{k-1}\right] - \Pi\right)\mathbf{G_e}^{\mathrm{T}} \\ &= \mathbf{G_e}^{k}\left(\mathrm{Var}\left[\mathbf{e}_0\right] - \Pi\right)\mathbf{G_e}^{k\mathrm{T}}.\end{aligned} \tag{13}$$

If two conditions related to $\mathbf{G_e} = I - \mathbf{G_p}\mathbf{L}$ are presented as

$$\rho\left(\mathbf{G_e}\right) < 1 \tag{14}$$

$$\left\|\mathbf{G_e}\right\|_2 < 1 \tag{15}$$

then the following convergence results can be derived based on (9) and (13).

Proposition 1: Consider system (1) satisfying Assumptions A1) and A2). If the ILC law (4) is applied, then

i) $\lim_{k\to\infty}\mathrm{E}\left[\mathbf{e}_k\right] = 0$ if and only if (14) holds, and this convergence is monotonic in the sense of the 2-norm if (15) holds;
ii) $\lim_{k\to\infty}\mathrm{Var}\left[\mathbf{e}_k\right] = \Pi$ if and only if (14) holds, and this convergence is monotonic in the sense of the F-norm if (15) holds;

where Π is the solution to the equation (12).

Proof: See [22], [23] for the details. ■

Remark 3: Clearly, the zero-mean disturbance in MIMO systems offers nothing to the expected value of the tracking error, which coincides with and, thus, extends the result obtained for SISO systems (see, e.g., [20]). Since fulfillment of (15) implies that of (14), (15) provides a sufficient but not necessary condition for convergence of $\mathrm{E}\left[\mathbf{e}_k\right]$.

Remark 4: For convergence of the variance of the tracking error, (15) presents also a sufficient but not necessary condition. For its stationary value Π, (14) can guarantee that it is the unique solution to equation (12). Otherwise, assume that $\widehat{\Pi}$ is another solution to (12). Then it follows from (12) immediately that Π and $\widehat{\Pi}$ satisfy

$$\Pi - \widehat{\Pi} = \mathbf{G_e}\left(\Pi - \widehat{\Pi}\right)\mathbf{G_e}^{\mathrm{T}} = \mathbf{G_e}^{n}\left(\Pi - \widehat{\Pi}\right)\mathbf{G_e}^{n\mathrm{T}},\ \forall n \geq 0.$$

Since (14) or (15) implies $\lim_{n\to\infty}\mathbf{G_e}^{n} = 0$, let $n \to \infty$ to obtain $\widehat{\Pi} = \Pi$ and then the uniqueness of the solution Π. In this case and by Lemma 1, a series form of Π can be derived from (12) as

$$\begin{aligned}\Pi &= \sum_{i=0}^{\infty}\mathbf{G_e}^{i}\left[\mathbf{G_p}\mathbf{L}\mathbf{V} + \mathbf{V}\left(\mathbf{G_p}\mathbf{L}\right)^{\mathrm{T}}\right]\mathbf{G_e}^{i\mathrm{T}} \\ &= \mathbf{G_p}\mathbf{L}\sum_{i=0}^{\infty}\mathbf{G_e}^{i}\mathbf{V}\mathbf{G_e}^{i\mathrm{T}} + \sum_{i=0}^{\infty}\mathbf{G_e}^{i}\mathbf{V}\mathbf{G_e}^{i\mathrm{T}}\left(\mathbf{G_p}\mathbf{L}\right)^{\mathrm{T}}.\end{aligned}$$

Also, the above convergence properties can be pulled from the basic properties of the Lyapunov equation.

IV. Indirect Approach

Next, a convergence analysis of the ILC process is developed by using the control input error[1]. To this end, two conditions related to $\mathbf{G_u} = I - \mathbf{L}\mathbf{G_p}$ are introduced as

$$\rho\left(\mathbf{G_u}\right) < 1 \tag{16}$$

$$\left\|\mathbf{G_u}\right\|_2 < 1. \tag{17}$$

Obviously, both of them can ensure that $\mathbf{L}\mathbf{G_p}$ is nonsingular. With this fact, define $\mathbf{u}_\infty = \left(\mathbf{L}\mathbf{G_p}\right)^{-1}\mathbf{L}\mathbf{y_d}$, $\Delta\mathbf{u}_k = \mathbf{u}_\infty - \mathbf{u}_k$, $\Delta\hat{\mathbf{u}}_k = \Delta\mathbf{u}_k - \mathrm{E}\left[\Delta\mathbf{u}_k\right]$ and $\mathrm{Var}\left[\Delta\mathbf{u}_k\right] = \mathrm{E}\left[\Delta\hat{\mathbf{u}}_k\Delta\hat{\mathbf{u}}_k^{\mathrm{T}}\right]$. Then the results of Propositions 2-6 can be proved, and their proofs are given in the Appendix.

Proposition 2: Consider system (1) satisfying Assumptions A1) and A2). If the ILC law (4) is applied, then

1) $\lim_{k\to\infty}\mathrm{E}\left[\Delta\mathbf{u}_k\right] = 0$ if and only if (16) holds, and this convergence is monotonic in the sense of the 2-norm if (17) holds;
2) $\lim_{k\to\infty}\mathrm{Var}\left[\Delta\mathbf{u}_k\right] = \Pi_u$ if and only if (16) holds, and this convergence is monotonic in the sense of the F-norm if (17) holds;
3) $\lim_{k\to\infty}\mathrm{E}\left[\mathbf{e}_k\right] = \mathbf{e}_\infty$ and $\lim_{k\to\infty}\mathrm{Var}\left[\mathbf{e}_k\right] = \Pi_e$ if (16) or (17) holds;

where

$$\Pi_u = \mathbf{G_u}\Pi_u\mathbf{G_u}^{\mathrm{T}} + \mathbf{L}\mathbf{V}\mathbf{L}^{\mathrm{T}} \tag{18}$$

$$\mathbf{e}_\infty = \left[I - \mathbf{G_p}\left(\mathbf{L}\mathbf{G_p}\right)^{-1}\mathbf{L}\right]\mathbf{y_d} \tag{19}$$

$$\Pi_e = \mathbf{G_p}\Pi_u\mathbf{G_p}^{\mathrm{T}} + \mathbf{V}. \tag{20}$$

From Proposition 2, it is clear that $\lim_{k\to\infty}\mathrm{E}\left[\mathbf{u}_k\right] = \mathbf{u}_\infty$, and then $\lim_{k\to\infty}\mathrm{E}\left[\mathbf{e}_k\right] = \mathbf{e}_\infty$, hold under the condition (16) or (17). In particular, $\mathbf{e}_\infty = 0$ always holds for SISO systems since $\mathbf{L}$ and $\mathbf{G_p}$ are both nonsingular in this case. However, the zero convergence of $\mathrm{E}\left[\mathbf{e}_k\right]$ may not be achieved for MIMO systems, as stated in the following proposition.

Proposition 3: Consider system (1) satisfying Assumptions A1) and A2). If the ILC law (4) is applied, then under the condition (16) or (17), $\lim_{k\to\infty}\mathrm{E}\left[\mathbf{e}_k\right] = 0$ if and only if $m = p$ holds, i.e., the number of inputs and that of outputs are equal.

Clearly, $m = p$ is required by MIMO systems to ensure that the expected tracing error converges to zero. Otherwise, if $m < p$, then there exists a residual error $\mathbf{e}_\infty$. Actually, in this case, $\rho\left(\mathbf{G_u}\right) < 1$ implies $\rho\left(\mathbf{G_e}\right) = 1$ and, hence, it can also be proved directly from (9) that the expected tracking error still converges, but it does not converge to zero any longer.

For $\mathrm{Var}\left[\Delta\mathbf{u}_k\right]$, it can be easily proved that under the condition (16) or (17), its stationary value Π_u is the unique solution to (18) (see Remark 4 for the same sketch of proof). In this case, the series form of Π_u can be obtained from (18) as follows

$$\Pi_u = \sum_{i=0}^{\infty}\mathbf{G_u}^{i}\mathbf{L}\mathbf{V}\mathbf{L}^{\mathrm{T}}\mathbf{G_u}^{i\mathrm{T}}.$$

[1] Although convergence of the tracking error can still be achieved, it is worth pointing out that the convergence results of the tracking error derived in the previous subsection can not be always derived any longer using the control input error for analysis in this section.

But, for the induced Π_e, such development about uniqueness may not work.

Proposition 4: Let Π_u be the solution to (18) and Π_e be given by (20). Then Π_e still satisfies the equation (12), i.e.,

$$\Pi_e = \mathbf{G_e}\Pi_e\mathbf{G_e}^{\mathrm{T}} + \mathbf{G_p}\mathbf{L}\mathbf{V} + \mathbf{V}\left(\mathbf{G_p}\mathbf{L}\right)^{\mathrm{T}}. \tag{21}$$

Particularly, if $m = p$ is satisfied, then under the condition (16) or (17), $\Pi_e = \Pi$ holds and is the unique solution to (12).

As a matter of fact, the expression of (11) can still work for the variance matrix of the tracking error. Hence, it can be obtained from (11) that $\mathrm{Var}[\mathbf{e}_k]$ converges when $k \to \infty$, but it can not be derived from (13) any more that $\lim_{k\to\infty}\mathrm{Var}[\mathbf{e}_k] = \Pi$, since now $\rho(\mathbf{G_e}) \le 1$ can only be ensured. In this case, it is obvious from Proposition 4 that the stationary variance matrix of the tracking error should satisfy not only (12) but also (20). Furthermore, this proposition implies that only when $m = p$ holds, the matrix expressions for the variance of both input error and tracking error have unique stationary solutions simultaneously.

In the literature that uses the input for the ILC convergence analysis, a general assumption is the realizability of the desired output trajectory (see, e.g., [16]-[18]). An advantage of this assumption is that the errors for both control input and output tracking can be guaranteed to converge to zero. Here such an assumption is made as follows.

A3) It is assumed that $y_d(t)$ is a realizable desired output trajectory. That is, there exists a unique control input $u_d(t)$ generating this trajectory for the nominal plant, i.e., $y_d(t) = G_p(q)u_d(t)$.

Based on Assumption A3), let $\mathbf{u_d}$ be defined in the same way with $\mathbf{u}_k$ and then let $\delta\mathbf{u}_k = \mathbf{u_d} - \mathbf{u}_k$, $\delta\hat{\mathbf{u}}_k = \delta\mathbf{u}_k - \mathrm{E}[\delta\mathbf{u}_k]$ and $\mathrm{Var}[\delta\mathbf{u}_k] = \mathrm{E}\left[\delta\hat{\mathbf{u}}_k\delta\hat{\mathbf{u}}_k^{\mathrm{T}}\right]$. In this case, the following result can be proved.

Proposition 5: Consider system (1) satisfying Assumptions A1)-A3). If the ILC law (4) is applied, then the following convergence results hold.

a) $\lim_{k\to\infty}\mathrm{E}[\delta\mathbf{u}_k] = 0$ if and only if (16) holds, and this convergence is monotonic in the sense of the 2-norm if (17) holds;
b) $\lim_{k\to\infty}\mathrm{Var}[\delta\mathbf{u}_k] = \Pi_u$ if and only if (16) holds, and this convergence is monotonic in the sense of the F-norm if (17) holds;
c) $\lim_{k\to\infty}\mathrm{E}[\mathbf{e}_k] = 0$ and $\lim_{k\to\infty}\mathrm{Var}[\mathbf{e}_k] = \Pi_e$ if (16) or (17) holds;

where Π_u and Π_e are given by (18) and (20), respectively.

Clearly, the condition (16) or (17) enables both $\mathrm{E}[\delta\mathbf{u}_k]$ and $\mathrm{E}[\mathbf{e}_k]$ to possess zero convergence. However, the results of Proposition 5 work in limited cases because Assumption A3) requires that the number of inputs should be equal to that of outputs, as claimed in the following proposition.

Proposition 6: The Assumption A3) is valid only when $m = p$ holds.

It is worth noting that if $m = p$, then (16) or (17) implies that $\mathbf{G_p}$ is nonsingular. Consequently, it follows that $\mathbf{u_d} = \mathbf{G_p}^{-1}\mathbf{y_d}$. This, together with Proposition 6, further implies that the Assumption A3) requires the system inverse in essence. Hence, with this assumption, the model inverse-based ILC can be discussed, in particular, for the SISO systems (see, e.g., [20]). On the contrary, when $m \ne p$, Proposition 6 implies that Proposition 5 does not hold and, in this case, only the general results of Proposition 2 hold. For more details of the invertibility of discrete-time systems, see [25].

Remark 5: For the SISO systems, some statistical expressions have been provided for disturbed ILC in time [19] and frequency [20] domains, but convergence conditions have not been studied. Here for the MIMO systems, error expressions have been developed for evaluation of stochastic ILC, and two general methods in the ILC analysis have been both considered and differences between them have been disclosed. Moreover, conditions have been presented for both asymptotic stability and monotonic convergence analyses of the stochastic ILC process.

V. Conclusions

In this paper, the robust convergence of ILC for discrete-time systems subject to random disturbances have been discussed, and two alternative approaches have been considered. Using the tracking error for analysis directly, the zero output tracking of the desired trajectory can be guaranteed in the sense of both expectation and variance, which however may not hold if the input error is used for analysis. It has been demonstrated that only when the input number is equal to the output number, the two approaches to ILC can obtain the equal convergence results.

Appendix

Proofs of Propositions 2-6

Proof: [Proof of Proposition 2]: Either (16) or (17) implies that $\mathbf{u}_\infty$ is well defined and $\mathbf{L}\mathbf{y_d} = \mathbf{L}\mathbf{G_p}\mathbf{u}_\infty$ is satisfied. Pre-multiply (7) by $\mathbf{L}$ and then subtract it from $\mathbf{L}\mathbf{y_d} = \mathbf{L}\mathbf{G_p}\mathbf{u}_\infty$ to obtain

$$\mathbf{L}\mathbf{e}_k = \mathbf{L}\mathbf{G_p}\Delta\mathbf{u}_k - \mathbf{L}\mathbf{v}_k. \tag{22}$$

Using the fact that $\Delta\mathbf{u}_k = \mathbf{u}_\infty - \mathbf{u}_k$ and inserting (22), it follows immediately from (8) that

$$\begin{aligned}\Delta\mathbf{u}_k &= \Delta\mathbf{u}_{k-1} - \mathbf{L}\mathbf{e}_{k-1}\\ &= \mathbf{G_u}\Delta\mathbf{u}_{k-1} + \mathbf{L}\mathbf{v}_{k-1}\end{aligned} \tag{23}$$

and hence

$$\begin{aligned}\mathrm{E}[\Delta\mathbf{u}_k] &= \mathbf{G_u}\mathrm{E}[\Delta\mathbf{u}_{k-1}] + \mathbf{L}\mathrm{E}[\mathbf{v}_{k-1}]\\ &= \mathbf{G_u}\mathrm{E}[\Delta\mathbf{u}_{k-1}]\\ &= \mathbf{G_u}^k\mathrm{E}[\Delta\mathbf{u}_0]\end{aligned} \tag{24}$$

which implies that $\lim_{k\to\infty}\mathrm{E}[\Delta\mathbf{u}_k] = 0$ if and only if (16) holds. Moreover, using (24) leads to

$$\begin{aligned}\|\mathrm{E}[\Delta\mathbf{u}_k]\|_2 &\le \|\mathbf{G_u}\|_2\|\mathrm{E}[\Delta\mathbf{u}_{k-1}]\|_2\\ &\le \|\mathbf{G_u}\|_2^k\|\mathrm{E}[\Delta\mathbf{u}_0]\|_2.\end{aligned}$$

In view of (17), one can conclude that the expected input error $\mathrm{E}[\Delta\mathbf{u}_k]$ converges monotonically to zero as $k \to \infty$, in the sense of the 2-norm.

Now let us consider the variance of the input error. Subtract (24) from (23) to derive

$$\Delta\hat{\mathbf{u}}_k = \mathbf{G_u}\Delta\hat{\mathbf{u}}_{k-1} + \mathbf{L}\mathbf{v}_{k-1} \tag{25}$$

which yields

$$\begin{aligned}\Delta\hat{\mathbf{u}}_k\Delta\hat{\mathbf{u}}_k^{\mathrm{T}} = &\mathbf{G_u}\Delta\hat{\mathbf{u}}_{k-1}\Delta\hat{\mathbf{u}}_{k-1}^{\mathrm{T}}\mathbf{G_u}^{\mathrm{T}} + \mathbf{G_u}\Delta\hat{\mathbf{u}}_{k-1}\mathbf{v}_{k-1}^{\mathrm{T}}\mathbf{L}^{\mathrm{T}} \\ &+ \mathbf{L}\mathbf{v}_{k-1}\Delta\hat{\mathbf{u}}_{k-1}^{\mathrm{T}}\mathbf{G_u}^{\mathrm{T}} + \mathbf{L}\mathbf{v}_{k-1}\mathbf{v}_{k-1}^{\mathrm{T}}\mathbf{L}^{\mathrm{T}}.\end{aligned} \tag{26}$$

It is easy to show that $\mathrm{E}\left[\Delta\hat{\mathbf{u}}_k\mathbf{v}_k^{\mathrm{T}}\right] = 0$. Thus, it follows from (26) that an expression of $\mathrm{Var}\left[\Delta\mathbf{u}_k\right]$ is given as

$$\begin{aligned}\mathrm{Var}\left[\Delta\mathbf{u}_k\right] &= \mathrm{E}\left[\Delta\hat{\mathbf{u}}_k\Delta\hat{\mathbf{u}}_k^{\mathrm{T}}\right] \\ &= \mathbf{G_u}\mathrm{E}\left[\Delta\hat{\mathbf{u}}_{k-1}\Delta\hat{\mathbf{u}}_{k-1}^{\mathrm{T}}\right]\mathbf{G_u}^{\mathrm{T}} + \mathbf{L}\mathrm{E}\left[\mathbf{v}_{k-1}\mathbf{v}_{k-1}^{\mathrm{T}}\right]\mathbf{L}^{\mathrm{T}} \\ &= \mathbf{G_u}\mathrm{Var}\left[\Delta\mathbf{u}_{k-1}\right]\mathbf{G_u}^{\mathrm{T}} + \mathbf{L}\mathbf{V}\mathbf{L}^{\mathrm{T}}.\end{aligned} \tag{27}$$

As a consequence of (18) and (27), the variance of the input error satisfies

$$\begin{aligned}\mathrm{Var}\left[\Delta\mathbf{u}_k\right] - \Pi_u &= \mathbf{G_u}\left(\mathrm{Var}\left[\Delta\mathbf{u}_{k-1}\right] - \Pi_u\right)\mathbf{G_u}^{\mathrm{T}} \\ &= \mathbf{G_u}^k\left(\mathrm{Var}\left[\Delta\mathbf{u}_0\right] - \Pi_u\right)\mathbf{G_u}^{k\mathrm{T}}\end{aligned} \tag{28}$$

which implies that $\lim_{k\to\infty}\mathrm{Var}\left[\Delta\mathbf{u}_k\right] = \Pi_u$ if and only if (17) holds. In view of Lemma 1, using (28) also results in

$$\begin{aligned}\left\|\mathrm{Var}\left[\Delta\mathbf{u}_k\right] - \Pi_u\right\|_{\mathrm{F}} &\le \left\|\mathbf{G_u}\right\|_2^2\left\|\mathrm{Var}\left[\Delta\mathbf{u}_{k-1}\right] - \Pi_u\right\|_{\mathrm{F}} \\ &\le \left\|\mathbf{G_u}\right\|_2^{2k}\left\|\mathrm{Var}\left[\Delta\mathbf{u}_0\right] - \Pi_u\right\|_{\mathrm{F}}.\end{aligned}$$

Clearly, (17) ensures that the variance matrix of the input error $\mathrm{Var}\left[\Delta\mathbf{u}_k\right]$ converges monotonically to Π_u as $k\to\infty$, in the sense of the F-norm.

Again using (7), it yields

$$\mathrm{E}\left[\mathbf{e}_k\right] = \mathbf{y_d} - \mathbf{G_p}\mathrm{E}\left[\mathbf{u}_k\right]. \tag{29}$$

This, together with $\lim_{k\to\infty}\mathrm{E}\left[\Delta\mathbf{u}_k\right] = 0$, ensures that $\lim_{k\to\infty}\mathrm{E}\left[\mathbf{e}_k\right] = \mathbf{e}_\infty$, where

$$\mathbf{e}_\infty = \mathbf{y_d} - \mathbf{G_p}\mathbf{u}_\infty = \left[I - \mathbf{G_p}\left(\mathbf{L}\mathbf{G_p}\right)^{-1}\mathbf{L}\right]\mathbf{y_d}.$$

That is, $\mathbf{e}_\infty$ is given by (19). Next using (7), (29) and the fact that $\mathbf{u}_k - \mathrm{E}\left[\mathbf{u}_k\right] = -\left(\Delta\mathbf{u}_k - \mathrm{E}\left[\Delta\mathbf{u}_k\right]\right) = -\Delta\hat{\mathbf{u}}_k$, it results in

$$\hat{\mathbf{e}}_k = \mathbf{e}_k - \mathrm{E}\left[\mathbf{e}_k\right] = -\mathbf{G_p}\left(\mathbf{u}_k - \mathrm{E}\left[\mathbf{u}_k\right]\right) - \mathbf{v}_k = \mathbf{G_p}\Delta\hat{\mathbf{u}}_k - \mathbf{v}_k$$

and thus

$$\hat{\mathbf{e}}_k\hat{\mathbf{e}}_k^{\mathrm{T}} = \mathbf{G_p}\Delta\hat{\mathbf{u}}_k\Delta\hat{\mathbf{u}}_k^{\mathrm{T}}\mathbf{G_p}^{\mathrm{T}} - \mathbf{G_p}\Delta\hat{\mathbf{u}}_k\mathbf{v}_k^{\mathrm{T}} - \mathbf{v}_k\Delta\hat{\mathbf{u}}_k^{\mathrm{T}}\mathbf{G_p}^{\mathrm{T}} + \mathbf{v}_k\mathbf{v}_k^{\mathrm{T}}$$

which leads to

$$\mathrm{Var}\left[\mathbf{e}_k\right] = \mathbf{G_p}\mathrm{Var}\left[\Delta\mathbf{u}_k\right]\mathbf{G_p}^{\mathrm{T}} + \mathbf{V}. \tag{30}$$

As a consequence, $\lim_{k\to\infty}\mathrm{Var}\left[\mathbf{e}_k\right] = \Pi_e$ can be obtained, where Π_e is defined by (20). The proof is complete. ■

Proof: [Proof of Proposition 3]: Note that a necessary condition for (16) or (17) is that the matrix $\mathbf{G_p}$ has full column rank. This implies that $m \le p$ holds. From Proposition 2, it follows immediately that $\mathbf{e}_\infty = 0$ holds for $\forall\mathbf{y_d}$ if and only if

$$\mathbf{G_p}\left(\mathbf{L}\mathbf{G_p}\right)^{-1}\mathbf{L} = I \in \mathbb{R}^{pT\times pT}. \tag{31}$$

With these facts, the proofs of necessity and sufficiency can be given.

If $\mathbf{e}_\infty = 0$ holds, then (31) implies

$$pT = \mathrm{rank}\left[\mathbf{G_p}\left(\mathbf{L}\mathbf{G_p}\right)^{-1}\mathbf{L}\right] \le \mathrm{rank}\left[\mathbf{L}\mathbf{G_p}\right] = mT$$

and consequently, $m = p$ is immediate. That is, the necessity is derived. On the contrary, if $m = p$ holds, then matrices $\mathbf{L}$ and $\mathbf{G_p}$ are square. Consequently, the nonsingularity of $\mathbf{L}\mathbf{G_p}$ implies that both $\mathbf{L}$ and $\mathbf{G_p}$ are nonsingular. Hence, (31) can be derived which leads to $\mathbf{e}_\infty = 0$. The sufficiency is satisfied. The proof is complete. ■

Remark 6: To achieve $\mathbf{e}_\infty = 0$, the above proof only shows the case where $m \le p$ due to the requirement of (16) and (17). While for the case where $m \ge p$, $\mathbf{e}_\infty = 0$ is immediate only if (14) or (15) holds, as shown in [22], [23].

Proof: [Proof of Proposition 4]: Using the fact that $\mathbf{G_p}\mathbf{G_u} = \mathbf{G_e}\mathbf{G_p}$, it immediately follows from (18) that

$$\begin{aligned}\mathbf{G_p}\Pi_u\mathbf{G_p}^{\mathrm{T}} &= \mathbf{G_p}\mathbf{G_u}\Pi_u\mathbf{G_u}^{\mathrm{T}}\mathbf{G_p}^{\mathrm{T}} + \mathbf{G_p}\mathbf{L}\mathbf{V}\left(\mathbf{G_p}\mathbf{L}\right)^{\mathrm{T}} \\ &= \mathbf{G_e}\mathbf{G_p}\Pi_u\mathbf{G_p}^{\mathrm{T}}\mathbf{G_e}^{\mathrm{T}} + \mathbf{G_p}\mathbf{L}\mathbf{V}\left(\mathbf{G_p}\mathbf{L}\right)^{\mathrm{T}}\end{aligned}$$

which, in view of (20), becomes

$$\mathbf{G_p}\Pi_u\mathbf{G_p}^{\mathrm{T}} = \mathbf{G_e}\Pi_e\mathbf{G_e}^{\mathrm{T}} - \mathbf{G_e}\mathbf{V}\mathbf{G_e}^{\mathrm{T}} + \mathbf{G_p}\mathbf{L}\mathbf{V}\left(\mathbf{G_p}\mathbf{L}\right)^{\mathrm{T}}. \tag{32}$$

By using the fact that $\mathbf{G_e} = I - \mathbf{G_p}\mathbf{L}$, it leads to

$$\mathbf{G_e}\mathbf{V}\mathbf{G_e}^{\mathrm{T}} = \mathbf{V} - \mathbf{G_p}\mathbf{L}\mathbf{V} - \mathbf{V}\left(\mathbf{G_p}\mathbf{L}\right)^{\mathrm{T}} + \mathbf{G_p}\mathbf{L}\mathbf{V}\left(\mathbf{G_p}\mathbf{L}\right)^{\mathrm{T}}.$$

Inserting this into the previous equation (32) yields

$$\mathbf{G_p}\Pi_u\mathbf{G_p}^{\mathrm{T}} = \mathbf{G_e}\Pi_e\mathbf{G_e}^{\mathrm{T}} + \mathbf{G_p}\mathbf{L}\mathbf{V} + \mathbf{V}\left(\mathbf{G_p}\mathbf{L}\right)^{\mathrm{T}} - \mathbf{V}.$$

Hence, it is clear from (20) that $\Pi_e = \mathbf{G_p}\Pi_u\mathbf{G_p}^{\mathrm{T}} + \mathbf{V}$ satisfies (21). Furthermore, if $m = p$ holds, $\rho\left(\mathbf{G_e}\right) = \rho\left(\mathbf{G_u}\right) < 1$ can be obtained under the condition (16) or (17). As a consequence, $\Pi_e = \Pi$ holds and is the unique solution to (12) (see also Remark 4). The proof is complete. ■

Proof: [Proof of Proposition 5]: It is clear that $\mathbf{y_d} = \mathbf{G_p}\mathbf{u_d}$. Subtract (7) from this to obtain

$$\mathbf{e}_k = \mathbf{G_p}\delta\mathbf{u}_k - \mathbf{v}_k. \tag{33}$$

With (33) and the fact that $\delta\mathbf{u}_k = \mathbf{u_d} - \mathbf{u}_k$, it follows immediately from (8) that

$$\begin{aligned}\delta\mathbf{u}_k &= \delta\mathbf{u}_{k-1} - \mathbf{L}\mathbf{e}_{k-1} \\ &= \mathbf{G_u}\delta\mathbf{u}_{k-1} + \mathbf{L}\mathbf{v}_{k-1}\end{aligned} \tag{34}$$

which yields

$$\begin{aligned}\mathrm{E}\left[\delta\mathbf{u}_k\right] &= \mathbf{G_u}\mathrm{E}\left[\delta\mathbf{u}_{k-1}\right] + \mathbf{L}\mathrm{E}\left[\mathbf{v}_{k-1}\right] \\ &= \mathbf{G_u}\mathrm{E}\left[\delta\mathbf{u}_{k-1}\right] \\ &= \mathbf{G_u}^k\mathrm{E}\left[\delta\mathbf{u}_0\right].\end{aligned} \tag{35}$$

Consequently, the result of the first item a) can be proved by using (35).

Notice that $\delta\hat{\mathbf{u}}_k = -\left(\mathbf{u}_k - \mathrm{E}\left[\mathbf{u}_k\right]\right) = \Delta\hat{\mathbf{u}}_k$ and, hence, $\mathrm{Var}\left[\delta\mathbf{u}_k\right] = \mathrm{Var}\left[\Delta\mathbf{u}_k\right]$. Subtract (35) from (34) to derive

$$\delta\hat{\mathbf{u}}_k = \mathbf{G_u}\delta\hat{\mathbf{u}}_{k-1} + \mathbf{L}\mathbf{v}_{k-1}$$

which is equivalent to (25). Then an immediate consequence is that (28) follows (see the proof of Proposition 2), based on which the result of the second item b) can be derived.

Again using (33), it yields

$$\mathrm{E}[\mathbf{e}_k] = \mathbf{G_p}\mathrm{E}[\delta\mathbf{u}_k]. \tag{36}$$

Since $\lim_{k\to\infty}\mathrm{E}[\delta\mathbf{u}_k] = 0$ is shown, $\lim_{k\to\infty}\mathrm{E}[\mathbf{e}_k] = 0$ can be ensured. Subtracting (36) from (33), it results in

$$\begin{aligned}\hat{\mathbf{e}}_k &= \mathbf{G_p}\delta\hat{\mathbf{u}}_k - \mathbf{v}_k \\ &= \mathbf{G_p}\Delta\hat{\mathbf{u}}_k - \mathbf{v}_k.\end{aligned}$$

Consequently, (30) follows and then $\lim_{k\to\infty}\mathrm{Var}[\mathbf{e}_k] = \Pi_e$ is immediate (see the proof of Proposition 2). That is, the result of the third item c) holds. The proof is complete. ■

Proof: [Proof of Proposition 6]: Assume that Assumption A3) also works for $m \neq p$ and it will see that the hypothesis is not true. It is clear from Proposition 5 that if Assumption A3) is assumed, then $\lim_{k\to\infty}\mathrm{E}[\mathbf{e}_k] = 0$ holds under the condition (16) or (17). Using Proposition 3, it follows that this zero convergence of the expected tracking error happens if and only if $m = p$ holds. This contradicts the hypothesis. Hence, the result claimed in this proposition holds. The proof is complete. ■

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