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Modular co-design of controllers and transmission schedules in *Wireless*HART

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***Abstract*— We consider the joint design of transmission schedules and controllers for networked control loops that use *Wireless*HART communication for sensor and actuator data. By parameterizing the design problem in terms of the sampling rate of the control loop, the co-design problem separates into two well-defined subproblems which admit optimal solutions: transmission scheduling should be done to maximize the delay-constrained reliability while the control design should optimize closed-loop performance under packet loss. We illustrate how these problems can be solved and demonstrate our co-design framework for the case of linear-quadratic control.**

I. INTRODUCTION

Networked control has been an active area of research for more than a decade (*e.g.* [1] and the references therein) and the literature is by now rather extensive. The research has mainly focused on structured control design methods for a given (typically linear) system and a high-level abstraction of the underlying communication system in terms of its latency or loss. The state-of-the art control design techniques are very powerful when the control system is able to cope with the network deficiencies. However, when the resulting closed-loop performance is unsatisfactory, they typically do not provide any feedback on how the communication system should be modified to yield better system performance.

In wired control systems where sensor data and actuator commands are transferred over a high-speed bus, extensive latencies and losses are often due to interaction with other traffic. Hence, co-design issues do not appear until the system is very highly loaded and can often be dealt with improved priorities between traffic flows or simply by adding an additional communication bus. The situation is very different for low-power wireless multi-hop communications. Non-negligible latencies and packet losses are to be expected already in a lightly loaded network and the use of capacity expansion for dealing with high-traffic scenarios is non-trivial since the wireless medium is shared with existing equipment. Furthermore, transmission scheduling policies have a strong impact on the guaranteed latency, loss and energy consumption of end-to-end transmissions, particularly when link losses are correlated in time [2]. Hence, co-design issues arise already for single-loop controllers operating alone in a wireless network.

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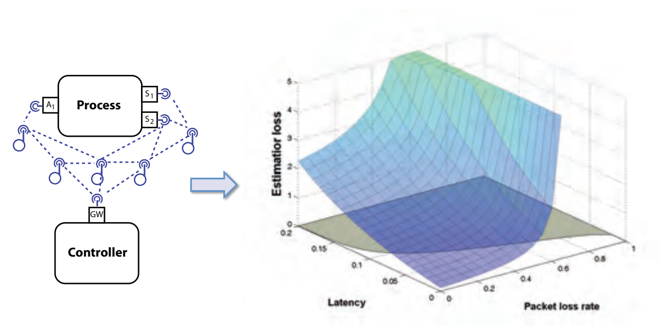


Fig. 1. Our framework separates the system design into two well-defined subproblems that admit optimal solutions: delay-constrained maximum reliability routing and the associated characterization of the achievable loss-latency pairs (the shaded region) and the design of optimal controllers under latency and loss and their associated performance evaluation (blue surface).

We would like to stress that the design-space of networked control systems is *huge*. It involves the selection of networking technology, protocol decisions from physical to application layers, the selection of sampling strategies and control laws. While one could attempt to pick certain parameters such as sampling time using classical rules-of-thumb, this would impact other parameters such as the achievable end-to-end reliability. It is not immediately clear if a shorter sampling interval is better if it also results in increased loss probability. Adding energy consumption and network life-time to the picture complicates decisions further.

This paper proposes a co-design framework for wireless control systems targeting *Wireless*HART and emerging scheduled multi-channel medium access control standards such as IEEE 802.15.4e. By parameterizing the system design in terms of the sample-time of the digital control loop, the co-design problem separates into two well-defined and solvable problems: to schedule the multi-hop network to maximize the deadline-constrained reliability, and to design a controller with optimum performance under (independent) packet losses. The deadline-constrained maximum reliability routing problem has recently been addressed [2], [3] and allows to compute optimal scheduling policies and characterize the achievable loss-latency region for multi-hop wireless networks. Similarly, for a given communication latency and loss probability, we can derive optimal controllers and estimate the achievable closed-loop performance, see Figure 1. In this paper, we detail our co-design framework for system-level design of a single-loop wireless control system with an LQG performance criterion. We derive the optimal scheduling policy and control law and illustrate the co-design process in numerical examples.

II. BACKGROUND AND MOTIVATION

A well-designed co-design framework requires insight and understanding of networking and control and how they interact. To this end, this section contains some background information on current and emerging standards on wireless control, basic insight about control and estimation under latency and loss, and related co-design efforts in the literature.

A. Wireless technologies for networked process control

Recent communication standards for real-time wireless control, such as *Wireless*HART [4], ISA-100 [5] and IEEE 802.15.4e [6], are converging toward a design that combines a multi-hop multi-path routing layer with a globally time synchronized channel hopping (TSCH) medium access control. Global scheduling removes the non-determinism and sometimes significant delays associated with random access protocols while the diversity offered by multipath routing and channel hopping can improve the end-to-end reliability.

These standards operate over low-power radios compliant with the IEEE 802.15.4-2006 physical layer which supports 16 channels in the 2.4GHz ISM band. Channel blacklisting is used to avoid channels with consistently high interference levels (*e.g.* due to coexistence with IEEE 802.11 standard) or to protect wireless services that share the ISM band.

The medium access control layer is based on a globally synchronized multi-channel TDMA that performs channel hopping at each slot boundary. One time slot is typically 10 ms long and allows for channel switching and the transmission of a single packet and the associated acknowledgement. Transmission opportunities can be dedicated or shared. To logically structure the global transmission schedule, time slots are organized into multiple frame structures (so called *superframes* in *Wireless*HART or *frame cycles* in the IEEE 802.15.4e proposals) that are repeated periodically, see *e.g.* Figure 2. Each frame can be used for scheduling one networking operation such as the collection of measurements from a subset of sensors, or the dissemination of commands to a set of actuators. Multiple superframes with different lengths can operate at the same time and unless conflicts between superframes have been eliminated in the scheduling phase, the standard prescribes how nodes should behave to resolve such conflicts. For a thorough description, we refer to the documentation of each standard [4], [5], [6].

B. Insight from estimation under latency and loss

While there is a large body of work on networked estimation and control under packet loss, there are few closed-form expressions that allows to gain analytical insight into how various networking parameters influence the overall system performance. To develop such an insight, we will consider a simple scenario where the state of a stochastic linear system

$$dx(t) = ax(t)dt + dw(t)$$

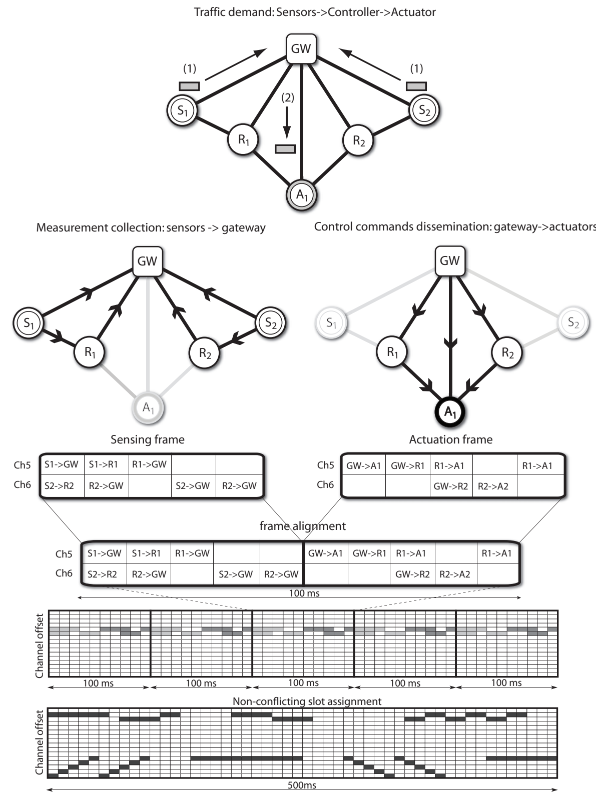


Fig. 2. This example describes the data collection from the two sensors, and then control command dissemination to an actuator. These basic operations are scheduled separately and aligned to form a 100ms slot frame. The slot frame is repeatedly scheduled to form a superframe. The global transmission schedule consists of multiple (non-conflicting) superframes.

should be estimated based on periodic noise-free samples of the state $x(kh)$. The aim is maintain a state estimate $\hat{x}(t)$ that minimizes the expected mean-square error

$$J_e = \lim \sup_{T\to\infty} \frac{1}{T} \int_0^T \mathbb{E}\left\{(x(t) - \hat{x}(t))^2\right\} dt$$

Samples transmitted from the sensor to the estimator over an unreliable channel where packet experience a stochastic delay D. If the delay exceeds one sampling interval, the transmission is aborted and the packet is considered lost. This gives rise to a networked control system with both a time-varying delay and a loss probability. As shown in [7], it is then possible to explicitly characterize the optimal loss:

$$J_e = \mathbb{E}\left\{e^{2aD}\right\} \frac{(1-p)(e^{2ah}-1)}{4a^2h(1-pe^{2ah})} - \frac{1}{2a} \qquad (1)$$

provided that $pe^{2ah} < 1$. This expression reveals that the achievable loss depends both the sampling interval h, the loss probability p and the complete delay distribution. Specifically, J_e is monotone increasing in h and p, and the smaller the average delay and its variance, the lower the loss.

However, latency and loss can typically not both be made small. As an illustration, consider communication over a single link and assume that the medium is divided into time slots of length t_s. A single time slot allows to transmit one packet, and a transmission attempt fails with probability p_s.

Assuming that $h = kt_s$ and that unacknowledged packets are retransmitted in the next time slot, the probability that the packet arrives with delay nt_s equals $(1-p_s)p_s^{n-1}$ and the probability that the packet is not transmitted until the end of the sampling interval is $p = p_s^k$. Note that to guarantee a small loss probability we must allow for more retransmission attempts and hence a longer sampling interval. Unfortunately, for this specific scenario there is no interesting co-design: J_e is bounded if $p_s^k e^{2akt_s} = \left(p_s e^{2at_s}\right)^k < 1$, so if we cannot stabilize the estimation error variance for $h = t_s$, then we cannot do so no matter how many retransmissions we allow for; moreover, $J_e = c\left(e^{2at_sk} - 1\right)/k$ which is monotone increasing in k, so the optimal performance is obtained for $h = t_s$. As we will see in Section V, however, these observations do not hold when we move from this restricted setting to closed-loop control over multi-hop wireless networks. In such cases, there is a non-trivial co-design decision to trade off latency and loss.

C. Related work on co-design of wireless control systems

This paper is by no means the first to consider controller-communication co-design. Early attempts, *e.g.* [8] focus on resource-constrained scenarios where the amount of bits that can be communicated over a wireless channel during a sampling interval is limited and needs to be allocated to different control loops, or assumes that only a single controller can access the communication medium at each sampling instant [9]. The paper [7] focuses on co-design of contention-based medium access and networked estimation and studies the dependencies between the number of contenders, the sampling interval, and the latency and loss distributions of sensor-estimator communication. For *WirelessHART* networks, our earlier paper [10] argues for structuring the communication schedule into network primitives such as unicast and convergecast and develops latency-optimal schedules under the assumption that communication links are reliable. The co-design aspect investigates how the additional latency introduced by heuristic retransmission policies (which improve end-to-end reliability) impact the closed-loop performance. Related is also the work by Hou [11] which extends the schedulability analysis tools for embedded systems to unreliable wireless systems.

III. A MODULAR CO-DESIGN FRAMEWORK

We propose a co-design framework that separates the co-design problem into well-defined networking and control design tasks. These tasks are parameterized by a single parameter: the sampling interval of sensors. The optimal co-designed system is found by evaluating the optimal performance of the networked control loop for each fixed sampling interval and select the one that yields the best overall performance.

Our aim is to solve the following problem

$$\begin{array}{ll} \text{minimize} & J(\mathcal{D}, p, h) \\ \text{subject to} & (\mathcal{D}, p, h) \in \mathcal{S} \end{array}$$

Here, J is the control loss function which depends on the communication latency distribution $\mathcal{D}$, the loss probability p and the sampling interval h, while the notation $(\mathcal{D}, p, h) \in \mathcal{S}$ denotes that the triple $(\mathcal{D}, p, h)$ should be schedulable, *i.e.* there should exist a transmission schedule that realizes latency distribution $\mathcal{D}$ and end-to-end loss probability p when packets are generated every h seconds.

In interest of simplicity, our co-design framework has some limitations. The main restriction is that we use a time-triggered control architecture that operates on the data that is available a fixed time-interval after it has been measured at the sensors. Nilsson [1] demonstrated that this control structure is suboptimal and that better performance can be obtained by an event-triggered architecture that acts immediately as data arrives. However, our assumption simplifies both the control design and the network scheduling. In particular, the time-triggered architecture guarantees that the controller-to-actuator communication occurs at well-defined points in time, which is critical for efficient reservation of network resources. For example, in the scheduling exercise in Figure 2, the time-triggered architecture allows to separate sensor and command scheduling into two non-conflicting operations. If we would use an event-triggered controller there is a chance that the sensor readings will be already delivered after the second time slot, but to be able to act at this point in time, we would need to reserve network resources for the controller-actuator communication over time slots $3-10$. Not only would such a resource reservation demand more energy (all devices that are scheduled to possibly receive in a time slot must have their radios on) but it would also complicate the superframe alignment process, especially when there are multiple control loops and many sensor and actuator communication flows. For simplicity, we set the communication interval equal to the sampling interval (*i.e.* we allow the communication to take a full sampling interval). The co-design problem now becomes

$$\begin{array}{ll} \text{minimize} & J(\mathbb{I}_h, p(h), h) \\ \text{subject to} & (\mathbb{I}_h, p(h), h) \in \mathcal{S} \end{array}$$

where $\mathcal{D} = \mathbb{I}_h$ denotes that the communication delay is one sampling interval, and $p(h)$ is now the probability that the packet is not delivered within this time.

Now, since $J(\mathbb{I}_h, p(h), h)$ is monotone increasing in $p(h)$ (see [12] for the details), the optimal solution to the co-design problem for any fixed h separates into two well-defined problems which we can solve to optimality when link losses are independent in space and time. Network resources should be allocated to minimize $p(h)$. This *deadline-constrained maximum-reliability forwarding* problem is defined and solved in Section IV-A. The *optimal control under independent packet losses* has been addressed for linear-quadratic loss in *e.g.* [13], but we present some of the extensions necessary for the co-design setting in Section IV-B. This includes sampling of the continuous-time loss functions to allow the comparison of the optimal losses for different values of h and the treatment of cross-terms in the loss function which now becomes instrumental.

IV. CO-DESIGN FOR LINEAR-QUADRATIC CONTROL

In this section we describe the networking and control design tasks. Firstly we describe the optimal networking design for a specified packet deadline (the next sampling time for sensor recallings). This allows to characterize the achievable latency-reliability pairs for a given network. This input is then used in the control task to find the optimal sampling period.

A. Deadline-constrained maximum reliability forwarding

We consider a multi-hop wireless mesh network whose routing topology can be represented by a directed acyclic graph (DAG) $\mathcal{G} = (\mathcal{N}, \mathcal{L})$ with nodes $\mathcal{N} = \{1, \ldots, N\}$ and links $\mathcal{L} = \{1, \ldots, L\}$. The presence of a directed link $(n, m) \in \mathcal{L}$ means that node n is able to successfully transmit a packet to node m. Each device is equipped with a half-duplex radio transceiver and cannot transmit and receive simultaneously. Communication is slotted, and each time slots (of length t_s ms) allows the transmission of a single packet and its associated acknowledgement. Links are unreliable with independent erasure events following a Bernoulli process with average link-loss probabilities $\mathbf{p} = [p_l]$ (*i.e.* communication on link l in time slot t fails with probability p_l, independently on other links). We assume that each node n knows its shortest hop-count $d_{n,\min}$ from the destination and the set $\mathcal{O}(n)$ of its outgoing links, which can be easily obtained during DAG construction [14].

On this network, we consider the *deadline-constrained maximum reliability unicast problem*. A sensor device periodically produces data with period h ms to be delivered to a controller node, the gateway, within a strict latency bound of D time slots. We consider a co-design problem where $h = D \cdot t_s$, that is, sensors inject a single packet into the network each sampling interval, and declare the packet lost if it has not arrived at the controller node within one sampling interval. For this problem, we study jointly optimal transmission scheduling and routing policy that maximize the probability of delivering the packet on time.

At time slot $t \geq 1$, let $d \triangleq D - t + 1$ be the time left to deliver the packet from any node n to the gateway, and let $\overline{\rho}_n(d)$ denote the maximum end-to-end reliability for packet delivery from node n to the destination with deadline d. Clearly,

$$\overline{\rho}_n(d) = 0 \quad \forall\, d < d_{n,\min}, \qquad \text{and} \quad \overline{\rho}_{gw}(d) = 1 \quad \forall\, d. \quad (2)$$

A node holding a packet at time t with deadline $d \in [1, D]$ has two choices: it can hold its transmission, thus yielding a reliability $\overline{\rho}_n(d) = \overline{\rho}_n(d-1)$; Or it can forward the packet to a neighbor, yielding a forward reliability $\overline{\rho}_n^f(d)$ defined as

$$\overline{\rho}_n^f(d) \triangleq \max_{l=(n,j)\in\mathcal{O}(n)} \{(1-p_l)\overline{\rho}_j(d) + p_l\overline{\rho}_n(d-1)\}. \quad (3)$$

The next theorem characterizes the optimal forwarding policy.

Theorem 4.1: Given a DAG routing topology with independent link loss probabilities $p_l \in [0, 1)$, the optimal policy for node n holding a packet at time t with deadline $d \in [d_{n,\min}, D]$ is to maximize the end-to-end reliability

$$\overline{\rho}_n(d) = \max\{\overline{\rho}_n^f(d), \overline{\rho}_n(d-1)\}, \quad (4)$$

scheduling a transmission on the link $l_t^\star$ defined as

$$l_t^\star = \underset{l=(n,j)\in\mathcal{O}(n)}{\arg\max} \{(1-p_l)\overline{\rho}_j(d) + p_l\overline{\rho}_n(d-1)\}. \quad (5)$$

Proof: Please refer to [3] for the proof details. ■

Algorithm 1 Optimal multi-path routing and scheduling.

Initialize: $\overline{\rho}_{\mathrm{gw}}(d) = 1, \forall d \in [0, D]$; $\overline{\rho}_n(d) = 0$, $\forall n \in \mathcal{N}, \forall d \in [0, D]$; $S(t) = \emptyset, \forall n \in \mathcal{N}, \forall t \in [1, D]$;
for $d_{\min} = 1$ to $d_{\max}$ **do**
  for each n with $d_{n,\min} = d_{\min}$ **do**
    for $d = d_{\min}$ to $D - (d_{\max} - d_{\min})$ **do**
      Time slot: $t = D - d + 1$.
      Update $S(t) = S(t) \bigcup l_t^\star$ with $l_t^\star$ from Eq. (5).
      Compute $\overline{\rho}_n(d)$ with Eq. (4).
    end for
  end for
end for

Thus, $\overline{\rho}_n(d)$ can be efficiently computed through (4)-(5) using dynamic programming as in Algorithm 1, where $S(t)$ is the set of links that can be potentially used at time t and $\Theta = \max_n \mid \mathcal{O}(n) \mid$. Since the time complexity to compute $l_t^\star$ is $O(\Theta)$, the time complexity of the algorithm is $O(\Theta N(D - d_{\max}))$ where $d_{\max} = \max_{n\in\mathcal{N}} d_{n,\min}$. Theorem 4.1 characterizes the achievable latency-reliability pairs $(h, \overline{\rho}(h))$ for a given DAG, link loss probability $\mathbf{p} = [p_l]$, and sampling time h. The solution to this problem is a jointly optimal scheduling and routing policy, where higher reliability is achieved with more retransmissions. Figure 3 illustrates this tradeoff for sampling times $h \in [0, 450]$ms with $t_s = 10$ms and three scenarios in which the network becomes increasingly unreliable by randomly picking the average links loss probability p_l in various intervals. Several extensions to this basic deadline-constrained scheduling can be found in [2], [3].

The control design task exploits the latency-reliability pairs $(h, \overline{\rho}(h))$ to find the optimal sampling time $h^\star$ that minimizes a control cost. Characterizing the optimal linear-quadratic loss function of a linear system under latency and loss is done next.

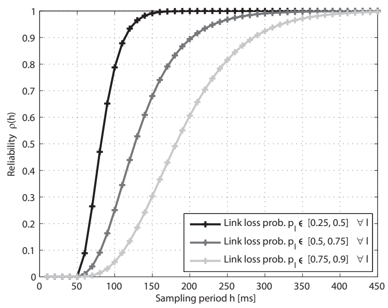
B. Linear-quadratic control under independent packet losses

We consider a stochastic linear continuous-time system

$$dx = Axdt + Budt + dv_c, \quad (6)$$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are the system matrices and v_c is a Wiener process with the incremental covariance $R_v^c dt$. We assume that a noisy measurement of the system is taken every sample period h and is sent to the controller over an unreliable network. The corresponding sampled system is

$$x(kh + h) = \Phi x(kh) + \Gamma u(kh) + v(kh),$$
$$y(kh) = \rho C x(kh) + w(kh),$$



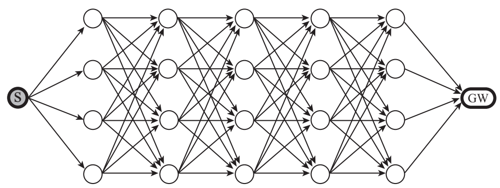


Fig. 3. DAG routing topology with a sensor at 6-hop from the gateway.

where $\Phi = e^{Ah}$, $\Gamma = \int_0^h e^{As} ds B$ and $\rho \in \{0, 1\}$ is a stochastic independent and identically distributed (i.i.d.) binary variable with

$$\text{Prob}\{\rho = 1\} = \mathbb{E}[\rho] := \bar{\rho}\,,$$
$$\text{Prob}\{\rho = 0\} = 1 - \mathbb{E}[\rho] := 1 - \bar{\rho}\,.$$

Additionally, $v(kh)$ and $w(kh)$ are discrete-time Gaussian white-noise processes with zero-mean and covariances:

$$\mathbb{E}\{v(kh)v^T(kh)\} = R_v = \int_0^h e^{A\tau} R_v^c e^{A^T\tau} d\tau\,,$$
$$\mathbb{E}\{w(kh)w^T(kh)\} = R_w\,,$$
$$\mathbb{E}\{v(kh)w^T(kh)\} = 0\,.$$

We define the following information set

$$\mathcal{I}_k \triangleq \{\mathcal{Y}_k\,, \mathcal{U}_{k-1}\,, \mathcal{R}_k\}\,,$$

where $\mathcal{Y}_k = (y_k, \ldots, y_1)$, and $\mathcal{U}_{k-1} = (u_{k-1}, \ldots, u_1)$ are the output and input sets up to k and $k-1$, while $\mathcal{R}_k = (\rho_k, \ldots, \rho_1)$ is the set of realizations of ρ until time k.

The loss function for the networked control system is

$$J = \mathbb{E}\bigg\{ \int_0^{Nh} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} Q_{xx}^c & Q_{xu}^c \\ Q_{xu}^{cT} & Q_{uu}^c \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt + x^T(Nh) Q_0^c x(Nh) \bigg\} \tag{7}$$

where the matrix Q_{xx}^c is symmetric and positive semi-definite while Q_{uu}^c is symmetric and positive definite. When we use piecewise constant control signal, (7) can be transformed into an equivalent discrete-time loss

$$J = \mathbb{E}\bigg\{ \sum_{k=0}^{N-1} \begin{bmatrix} x_k \\ u_k \end{bmatrix}^T \begin{bmatrix} Q_{xx} & Q_{xu} \\ Q_{xu}^T & Q_{uu} \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix} + x_N^T Q_0^c x_N \bigg\}$$

where

$$Q_{xx} = \int_{kh}^{kh+h} \Phi^T(s) Q_{xx}^c \Phi(s) ds\,,$$
$$Q_{xu} = \int_{kh}^{kh+h} \Phi^T(s) \big(Q_{xx}^c \Gamma(s) + Q_{xu}^c\big) ds\,,$$
$$Q_{uu} = \int_{kh}^{kh+h} \big(\Gamma^T(s) Q_{xx}^c \Gamma(s) + 2\Gamma^T(s) Q_{xu}^c + Q_{uu}^c\big) ds\,.$$

Minimizing the loss function of (7) when $u(t)$ is constant over the sampling period is thus the same as minimizing the discrete-time loss function. However, it is important to note that the discrete-time loss has cross-terms also when the continuous-time loss function has not. The authors in [15] studied a similar problem omitting the cross-terms in the loss function. In what follow, we extend the framework of [15] to include the cross-coupling terms in the loss function and derive the optimal controller and bound its performance for the case of reliable controller-actuator communication.

1) Estimator Design: Similar to [15], we use a Kalman filter to design the optimal estimator. The minimum mean square error (MMSE) estimate $\hat{x}_{k|k}$ of x_k given by $\hat{x}_{k|k} = \mathbb{E}[x_k | \mathcal{I}_k]$ can be computed recursively in two steps starting from the initial conditions $\hat{x}_{0|-1} = 0$ and $P_{0|-1} = P_0$. The innovation step is

$$\hat{x}_{k+1|k} \triangleq \mathbb{E}[x_{k+1} | \mathcal{I}_k] = \Phi \hat{x}_{k|k} + \Gamma u_k \tag{8}$$
$$e_{k+1|k} \triangleq x_{k+1} - \hat{x}_{k+1|k} = \Phi e_{k|k} + v_k \tag{9}$$
$$P_{k+1|k} \triangleq \mathbb{E}[e_{k+1|k} e_{k+1|k}^T | \mathcal{I}_k] = \Phi P_{k|k} \Phi^T + R_v \tag{10}$$

with independent v_k and $\mathcal{I}_k$. The subsequent correction step is

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + \rho_{k+1} K_{k+1} (y_{k+1} - C\hat{x}_{k+1|k}) \tag{11}$$
$$\begin{aligned} e_{k+1|k+1} &= x_{k+1} - \hat{x}_{k+1|k+1} \\ &= e_{k+1|k} - \rho_{k+1} K_{k+1} (C x_{k+1} + w_{k+1} - C\hat{x}_{k+1|k}) \end{aligned}$$
$$P_{k+1|k+1} = P_{k+1|k} - \rho_{k+1} K_{k+1} C P_{k+1|k} \tag{12}$$
$$K_{k+1} \triangleq P_{k+1|k} C^T (C P_{k+1|k} C^T + R_w)^{-1}\,. \tag{13}$$

2) Controller Design: We next develop the optimal feedback control law and the corresponding value of the loss function for both finite and infinite horizon cases.

Theorem 4.2: Consider the aforementioned finite horizon LQG control problem. The optimal control law

$$u_k = -\underbrace{(\Gamma^T S_{k+1} \Gamma + Q_{uu})^{-1} (\Gamma^T S_{k+1} \Phi + Q_{xu}^T)}_{L_k} \hat{x}_{k|k}$$

is a linear function of the estimated state. The matrix S_k evolves according to the backward Riccati recursion

$$\begin{aligned} S_k = Q_{xx} + \Phi^T S_{k+1} \Phi - (\Phi^T S_{k+1} \Gamma + Q_{xu})(\Gamma^T S_{k+1} \Gamma \\ + Q_{uu})^{-1} (\Gamma^T S_{k+1} \Phi + Q_{xu}^T) \end{aligned}$$

where $\hat{x}_{k|k}$ is the MMSE estimate of the state x_k based on the information set $\mathcal{I}_k$ computed with the Kalman filter above.

Proof: To derive the optimal feedback control law and the corresponding value for the objective function, we apply dynamic programming. Define the optimal value function $V_k(x_k)$ as follows:

$$V_k(x_k) \triangleq \min_{u_k} \mathbb{E}\{x_k^T Q_{xx} x_k + 2x_k^T Q_{xu} u_k + u_k^T Q_{uu} u_k + V_{k+1}|\mathcal{I}_k\} \quad (14)$$

$$V_N(x_N) \triangleq \mathbb{E}\{x_N^T Q_0 x_N|\mathcal{I}_N\} \quad (15)$$

where $k = \{N-1, \ldots, 1\}$. We show that $J_N^\star = V_0(x_0)$. The solution of the Bellman equation (14) with the initial condition (15) is given by

$$V_k(x_k) = \mathbb{E}\{x_k^T S_k x_k|\, \mathcal{I}_k\} + c_k\,, \quad (16)$$

where the nonnegative matrix S_k and the scaler c_k are independent of the information set $\mathcal{I}_k$. The aforementioned solution is apparently true for $k = N$. Proceeding by induction, we assume that (16) holds for $k+1$ and then we show it holds for k, as well. Hence, we have

$$\begin{aligned} V_k(x_k) =& \min_{u_k} \mathbb{E}\big[x_k^T Q_{xx} x_k + 2x_k^T Q_{xu} u_k + u_k^T Q_{uu} u_k \\ &+ V_{k+1}(x_{k+1})|\mathcal{I}_k\big] \\ =& \min_{u_k} \mathbb{E}\big[x_k^T Q_{xx} x_k + 2x_k^T Q_{xu} u_k + u_k^T Q_{uu} u_k \\ &+ \mathbb{E}\big[x_{k+1}^T S_{k+1} x_{k+1} + c_{k+1}|\mathcal{I}_{k+1}\big]\ |\mathcal{I}_k\big] \\ =& \min_{u_k} \mathbb{E}\big[x_k^T Q_{xx} x_k + 2x_k^T Q_{xu} u_k + u_k^T Q_{uu} u_k \\ &+ x_{k+1}^T S_{k+1} x_{k+1} + c_{k+1}|\mathcal{I}_k\big] \\ =& \mathbb{E}\big[x_k^T Q_{xx} x_k + x_k^T \Phi^T S_{k+1} \Phi x_k|\mathcal{I}_k\big] + \mathrm{tr}(S_{k+1} R_v) \\ &+ \min_{u_k} \big(u_k^T (Q_{uu} + \Gamma^T S_{k+1} \Gamma) u_k + 2u_k^T (\Gamma^T S_{k+1} \Phi \\ &+ Q_{xu}^T)\hat{x}_{k|k}\big) + \mathbb{E}\big[c_{k+1}|\mathcal{I}_k\big] \end{aligned} \quad (17)$$

Hence, $V_k(x_k)$ is a quadratic function with respect to u_k and its minimizer can be computed by solving $\frac{\partial V_k}{\partial u_k} = 0$ as

$$u_k = -(\Gamma^T S_{k+1} \Gamma + Q_{uu})^{-1} (\Gamma^T S_{k+1} \Phi + Q_{xu}^T) \hat{x}_{k|k} \quad (18)$$

The optimal control law is a linear function of the state estimate. Substituting the minimizer into (17) we find

$$\begin{aligned} V_k(x_k) =& \mathbb{E}\big[x_k^T Q_{xx} x_k + x_k^T \Phi^T S_{k+1} \Phi x_k|\mathcal{I}_k\big] + \mathrm{tr}(S_{k+1} R_v) \\ &- \hat{x}_{k|k}^T (\Phi^T S_{k+1} \Gamma + Q_{xu})(\Gamma^T S_{k+1} \Gamma + Q_{uu})^{-1} \\ &\times (\Gamma^T S_{k+1} \Phi + Q_{xu}^T)\hat{x}_{k|k} + \mathbb{E}\big[c_{k+1}|\mathcal{I}_k\big] \end{aligned}$$

Hence, (16) yields

$$\begin{aligned} &\mathbb{E}\big[x_k^T S_k x_k|\mathcal{I}_k\big] + c_k = \mathbb{E}\big[x_k^T Q_{xx} x_k + x_k^T \Phi^T S_{k+1} \Phi x_k \\ &- x_{k|k}^T (\Phi^T S_{k+1} \Gamma + Q_{xu})(Q_{uu} + \Gamma^T S_{k+1} \Gamma)^{-1} \\ &\times (\Gamma^T S_{k+1} \Phi + Q_{xu}^T) x_{k|k}|\mathcal{I}_k\big] + \mathrm{tr}(S_{k+1} R_v) \\ &+ \mathbb{E}\big[c_{k+1}|\mathcal{I}_k\big] + \mathrm{tr}\big((\Phi^T S_{k+1} \Gamma + Q_{xu})(Q_{uu} \\ &+ \Gamma^T S_{k+1} \Gamma)^{-1} (\Gamma^T S_{k+1} \Phi + Q_{xu}^T) P_{k|k}\big) \end{aligned}$$

Since this equation holds for all x_k, we have

$$\begin{aligned} S_k =& \Phi^T S_{k+1} \Phi + Q_{xx} - (\Phi^T S_{k+1} \Gamma + Q_{xu})(Q_{uu} \\ &+ \Gamma^T S_{k+1} \Gamma)^{-1} (\Gamma^T S_{k+1} \Phi + Q_{xu}^T) \\ c_k =& \mathrm{tr}\big((\Phi^T S_{k+1} \Gamma + Q_{xu})(Q_{uu} + \Gamma^T S_{k+1} \Gamma)^{-1} \\ &\times (\Gamma^T S_{k+1} \Phi + Q_{xu}^T) P_{k|k}\big) + \mathrm{tr}\big(S_{k+1} R_v\big) + \mathbb{E}\big[c_{k+1}|\mathcal{I}_k\big] \end{aligned}$$

■

The loss function of the finite horizon LQG for the networked control system can be written as

$$\begin{aligned} J_N^\star =& V_0(x_0) = \bar{x}_0^T S_0 \bar{x}_0 + \mathrm{tr}\big(S_0 P_0\big) + \sum_{k=0}^{N-1} \mathrm{tr}\big(S_{k+1} R_v\big) \\ &+ \sum_{k=0}^{N-1} \mathrm{tr}\big((\Phi^T S_{k+1} \Phi + Q_{xx} - S_k)\mathbb{E}_\rho\big[P_{k|k}\big]\big) \end{aligned} \quad (19)$$

Lemma 4.3: The expected error covariance matrix $\mathbb{E}_\rho[P_{k|k}]$ satisfies the following bounds

$$\underline{P}_{k|k} \le \mathbb{E}_\rho[P_{k|k}] \le \overline{P}_{k|k}, \quad \forall k > 0\,.$$

where the matrices $\overline{P}_{k|k}$ and $\underline{P}_{k|k}$ can be computed as

$$\begin{aligned} \overline{P}_{k+1|k} =& \Phi \overline{P}_{k|k-1} \Phi^T + R_v \\ &- \bar{\rho} \Phi \overline{P}_{k|k-1} C^T (C \overline{P}_{k|k-1} C^T + R_w)^{-1} C \overline{P}_{k|k-1} \Phi^T \\ \overline{P}_{k|k} =& \overline{P}_{k|k-1} \\ &- \bar{\rho} \overline{P}_{k|k-1} C^T (C \overline{P}_{k|k-1} C^T + R_w)^{-1} C \overline{P}_{k|k-1} \\ \underline{P}_{k+1|k} =& (1-\bar{\rho}) \Phi \underline{P}_{k|k-1} \Phi^T + R_v \\ \underline{P}_{k|k} =& (1-\bar{\rho}) \underline{P}_{k|k-1} \end{aligned}$$

starting from the initial conditions $\underline{P}_{0|-1} = \overline{P}_{0|-1} = P_0$.

Using Lemma 4.3, the LQG loss function for finite horizon $J_N^\star$ in (19) can be bounded as follows

$$J_N^{\min} \le J_N^\star \le J_N^{\max} \quad (20)$$

where the lower and upper bound are

$$\begin{aligned} J_N^{\min} =& \bar{x}_0^T S_0 \bar{x}_0 + \mathrm{tr}\big(S_0 P_0\big) + \sum_{k=0}^{N-1} \mathrm{tr}\big(S_{k+1} R_v\big) \\ &+ \sum_{k=0}^{N-1} \mathrm{tr}\big((\Phi^T S_{k+1} \Phi + Q_{xx} - S_k) \underline{P}_{k|k}\big) \end{aligned} \quad (21)$$

$$\begin{aligned} J_N^{\max} =& \bar{x}_0^T S_0 \bar{x}_0 + \mathrm{tr}\big(S_0 P_0\big) + \sum_{k=0}^{N-1} \mathrm{tr}\big(S_{k+1} R_v\big) \\ &+ \sum_{k=0}^{N-1} \mathrm{tr}\big((\Phi^T S_{k+1} \Phi + Q_{xx} - S_k) \overline{P}_{k|k}\big) \end{aligned} \quad (22)$$

For infinite horizon, the bounds in (20) become

$$\begin{aligned} J_\infty^{\min} \triangleq& \lim_{N\to\infty} \frac{1}{N} J_N^{\min} = \mathrm{tr}(S_\infty R_v) \\ &+ (1-\bar{\rho}) \mathrm{tr}\big((\Phi^T S_\infty \Phi + Q_{xx} - S_\infty) \underline{P}_\infty\big) \\ J_\infty^{\max} \triangleq& \lim_{N\to\infty} \frac{1}{N} J_N^{\max} = \mathrm{tr}\big((\Phi^T S_\infty \Phi + Q_{xx} - S_\infty) \\ &\times (\overline{P}_\infty - \bar{\rho} \overline{P}_\infty C^T (C \overline{P}_\infty C^T + R_w)^{-1} C \overline{P}_\infty)\big) \\ &+ \mathrm{tr}(S_\infty R_v) \end{aligned}$$

where the matrices S_∞, $\overline{P}_\infty$ and $\underline{P}_\infty$ satisfy

$$\begin{aligned} S_\infty =& \Phi^T S_\infty \Phi + Q_{xx} - (\Phi^T S_\infty \Gamma + Q_{xu})(\Gamma^T S_\infty \Gamma \\ &+ Q_{uu})^{-1} (\Gamma^T S_\infty \Phi + Q_{xu}^T) \\ \overline{P}_\infty =& \Phi \overline{P}_\infty \Phi^T + R_v - \bar{\rho} \Phi \overline{P}_\infty C^T (C \overline{P}_\infty C^T \\ &+ R_w)^{-1} C \overline{P}_\infty \Phi^T \\ \underline{P}_\infty =& (1-\bar{\rho}) \Phi \underline{P}_\infty \Phi^T + R_v \end{aligned}$$

Additionally, the infinite horizon optimal controller gain converges to a constant value that is calculated as

$$L_\infty = \lim_{k\to\infty} L_k = -(\Gamma^T S_\infty \Gamma + Q_{uu})^{-1}(\Gamma^T S_\infty \Phi + Q_{xu}^T)\,.$$

However, the estimator cannot converge to any steady state value as distinct from the standard LQG control design.

The optimal control laws developed in this section are valid for any linear system with quadratic loss and piece-wise constant controls. The one-step delay present in our control architecture can be dealt with by first sampling the continuous time system and its loss function disregarding the delay (to obtain Φ, Γ, C, etc.) and then introducing an augmented state vector (x_k, x_{k-1}) and an augmented system description and solving the optimal control problem for this augmented system. The procedure is standard, see *e.g.* Åström and Wittenmark [16] for details.

To sum up, the optimal estimator is a time-varying Kalman filter given by (10), (12) and (13), while the control law is a static linear feedback (18). The combined performance, in the sense of the continuous-time loss function (7), can be bounded as in (21) and (22). It is this controller and these performance bounds that we will use in the optimal control design part of the co-design procedure.

V. CASE STUDY

We are now ready to demonstrate our co-design procedure on a numerical example. Consider an inverted pendulum described by the following model

$$dx = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} x dt + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u dt + dv_c$$
$$y(kh) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(kh) + w(kh),$$

where v_c and w have incremental covariances $R_v^c = 0.5 I_2$ and $R_w = 10^{-4}$, respectively. Our joint design should minimize (7) for

$$Q_{xx}^c = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad Q_{uu}^c = 1\,.$$

Periodic samples of the output $y(kh)$ are transmitted over the multi-hop wireless network in Figure 3. We assume that communication links are unreliable with erasure events following a Bernoulli process. We consider the three loss probability settings described in Section IV-A in which the network becomes increasingly unreliable. For a fixed h, Algorithm 1 yields the optimal deadline-constrained schedule and the associated probability $\overline{\rho}(h)$ of on-time delivery within $D = \lceil h/10ms \rceil$ time slots. After discretizing the loss function J, we bound the performance of the optimal controller using the bounds for the associated discrete-time loss and the computed $\overline{\rho}(h)$. Repeating the procedure for a range of sampling intervals yields the performance curves in Figure 4. Since the open-loop system is unstable and the minimum hop-count from sensor to gateway is six hops, the minimum feasible sampling interval is $h = 6 \cdot 10$ ms. The optimal sampling interval $h^\star$ that minimizes the control cost ranges from 90ms for the most reliable network scenario to 250ms for the least reliable case, corresponding to a required reliability of 65% and 82%, respectively. As the network becomes less reliable, more retransmissions (a longer $h^\star$) are required to guarantee a sufficiently high reliability with an associated significant increase of control cost. Specifically, the optimal control cost (marked with a square in Figure 4) increases of about a factor ten from the most to the least reliable network scenarios.

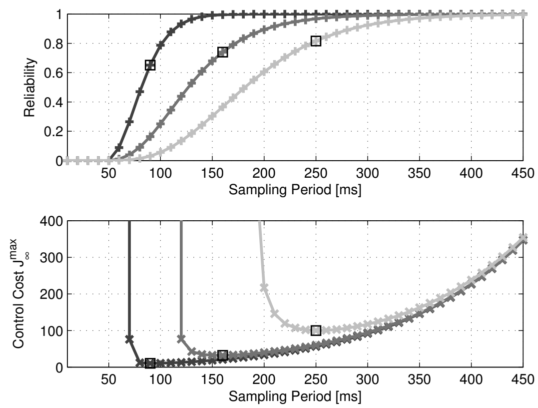


Fig. 4. Comparison of the upper bounds $J_\infty^{\max}$ for three different cases.

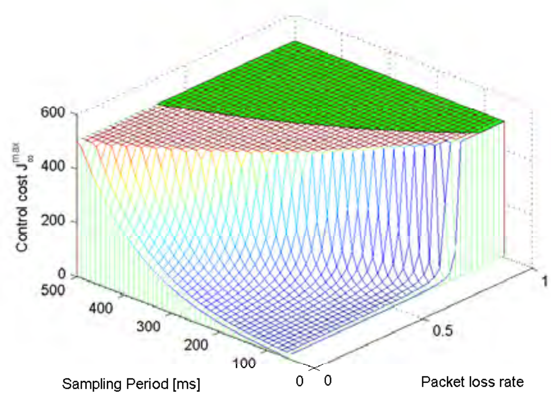


Fig. 5. The upper bounds $J_\infty^{\max}$ for the minimum control cost with respect to the different sampling periods and packet loss rates.

Figure 4 also shows that the network must provide a minimum reliability to stabilize the open-loop unstable process. To gain more insight into this aspect, Figure 5 exhibits the upper bound of the control cost $J_\infty^{\max}$ for different sampling periods and packet loss probabilities $p_{loss} = 1 - \overline{\rho}$, and the instability region with respect to the necessary stability condition $p_{loss}\lambda_{\max}(\Phi) < 1$ (green area). One can observe that $J_\infty^{\max}$ increases rapidly with respect to both sampling period and packet loss rate.

Figure 6 validates the bounds of $J_N^\star$ in (20) by computing the average value of $J_N^\star$ in (19) for the finite horizon LQG control through Monte Carlo simulations. Although the results are similar for the other scenarios, we only show the results for the least reliable networking scenario. For each pair $(h, \overline{\rho}(h))$, we compute $J_N^\star$ from (19) generating

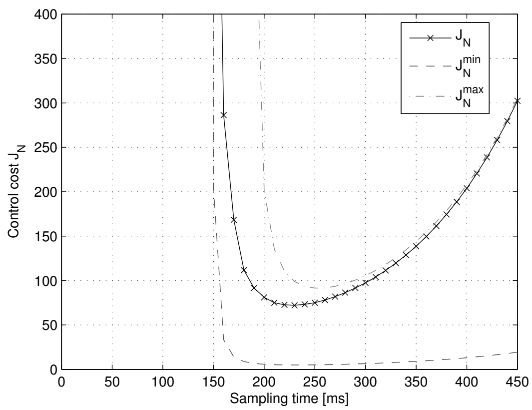


Fig. 6. The experimental $J_N^\star$, upper bound $J_N^{\max}$ and lower bound $J_N^{\min}$ for the minimum control cost with respect to sampling period. The curves are obtained by averaging 1000 Monte Carlo simulations for $N = 2500$, with the arrival sequence $\rho(k)$ generated randomly.

a synthetic data trace with average success probability $\overline{\rho}(h)$. We repeat the computation for 10^3 trials and average the cost value for the given $(h, \overline{\rho}(h))$. Figure 6 shows that the upper bound $J_N^{\max}$ becomes quite accurate for sampling intervals $h \geq 250$ms. Moreover, one can notice a good match between the minimum sampling interval and the associated control cost.

VI. Conclusions

We considered the joint design of transmission schedules and controllers for networked control loops that use *Wireless*HART communication for sensor and actuator data. By parameterizing the design problem in terms of the sampling rate of the control loop, the co-design problem separates into two well-defined networking and control design subproblems, both of which admit optimal solutions. Transmission scheduling should be done to maximize the delay-constrained reliability, and the control design should optimize closed-loop performance under packet loss. We illustrate how these problems can be solved, and demonstrate our co-design framework for the case of linear-quadratic control.

Although our framework significantly improves upon the state-of-art solutions for co-design of communication and control, several natural extensions can be considered. One is the multi-loop control problem, where multiple sensors take measurements with different sampling time and deadline, eventually different plants for different (parallel) control loops. The corresponding networking task, however, becomes a real-time deadline-constrained multi-flow scheduling, which was recently proved to be NP-complete [17]. Another extension is the optimal co-design of a system when the actuation loop is also closed through a wireless network.

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