2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Mixed Criteria Control Design with Finite-Time Boundedness and H_∞ Property for a Class of Discrete-Time Nonlinear Systems

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Abstract— **A feedback controller design which guarantees both finite-time boundedness and H_∞ attenuation for a class of nonlinear systems with conic type nonlinearities and additive disturbances is presented. Conditions which guarantee the existence of a robust state-feedback controller for maintaining a bound on the transient response and satisfying an H_∞ bound in the steady state for this class of systems are derived. A solution for the controller gain is obtained through the application of linear matrix inequality techniques. The controller developed is robust for all nonlinearities satisfying the conic inequality and all admissible disturbances. We conclude the paper with a numerical example illustrating the applicability of the controller design.**

I. Introduction

LYAPUNOV Asymptotic Stability, LAS, allows for the analysis of the behavior of a dynamical system over an infinite time interval. However, there are several applications where it is desired to maintain the state of a system within a prescribed region in its space over a fixed finite-time interval. Moreover, for example in vehicle maneuvering applications, the sole interest is in the behavior of the given system over a specified finite-time interval. Therefore, the concept of Finite-Time Stability was introduced [1], [2].

A system is said to be Finite-Time Stable, FTS, if, for any initial condition lying within a prescribed bounded region in the state space, the state of the system does not exceed a specified threshold over a finite-time interval. It is necessary to note that LAS and FTS are two independent concepts. A system which is LAS may not be FTS and vice versa. Another concept which is an extension to that of FTS is Finite-Time Boundedness, FTB. A system with additive disturbances is said to be Finite-Time Bounded, FTB, if, given the dynamics of the disturbances and an initial bound on their state, the system remains FTS for all the admissible disturbances [3].

Various finite-time controller design results can be found in the literature related to this field. Nevertheless, most of these results apply to linear systems. For instance, in [3], the authors present the design of a robust finite-time controller of continuous linear systems with polytopic uncertainties. Furthermore, in [3] and [5]-[7], several variations of the problem of FTS and finite-time control of linear systems are considered. However, to the best of our knowledge, the study of FTS and stabilization of nonlinear systems is rarely addressed in the literature. The authors in [8] consider nonlinear systems that are hybrid and stochastic. Other works have studied the FTS and stabilization of nonlinear quadratic systems [9]. Furthermore, in [10], the stabilization of a class of uncertain nonlinear systems with time-delay is presented.

In this paper, a finite-time state-feedback controller design for a class of discrete-time nonlinear systems with conic type nonlinearities and additive disturbances is considered. In fact, the work presented here is applicable to all nonlinear systems which are locally Lipschitz [11]. Moreover, the controller developed is designed to also satisfy the H_∞ performance criterion. Thus, with such a controller, we are able to guarantee a bounded response with a prescribed bound during the transient time and also guarantee that the energy of the system of the performance output remains below a given value in the steady state, despite the presence of disturbances, due to the H_∞ property of the controller. Conditions under which the closed loop system satisfies both FTB and H_∞ performance criterion are derived. The controller gain is solved for via Linear Matrix Inequality, LMI, techniques.

The paper is divided into 5 sections. Next, we introduce the system model and control problem. In section 3, we recall the basic definitions of FTB and H_∞. In section 4, we present the main results of the mixed finite-time and H_∞ criteria control and derive the LMI conditions. In section 5, a numerical example is presented to illustrate the applicability of the results obtained.

The notation used in this paper is shown in Table I.

TABLE I
NOTATION

Notation	Definition
$x \in R^n$	An n-dimensional real vector
$\|x\| = \left(x^T x\right)^{1/2}$	Euclidean norm
$(.)^T$	Matrix transpose
$A \in R^{m\times n}$	An $m\times n$ real matrix
A^{-1}	Inverse of matrix A
$A > 0 (A < 0)$	A is a positive (negative) definite matrix
I	Identity matrix of appropriate dimensions
$\lambda_{\min(A)} (\lambda_{\max(A)})$	Minimum (Maximum) eigenvalue of the symmetric matrix A
$\mathbb{N}_0$	Set of nonnegative integers
$w_k \in L_2$	w_k is a finite energy disturbance where $\sum_{k=0}^{\infty} w_k^T w_k < \infty$

II. System Model and Control

Consider the following discrete-time nonlinear system:

$$x_{k+1} = f\left(x_k, u_k, w_k\right) = f_k \quad (1)$$

where $x_k \in W_n \subset R^n$ is the system state vector, $u_k \in W_m \subset R^m$ is the input vector, $w_k \in W_r \subset R^r$ is the disturbance input, $A \in R^{n\times n}$, $B \in R^{n\times m}$, and $F \in R^{n\times r}$ such that the domains W_n, W_m, and W_r are open and connected sets. In regard to the disturbance w_k, the dynamical model of the disturbance is assumed to be known. But since the H_∞ condition, or finite energy condition, needs to be satisfied simultaneously, asymptotically stable dynamics for the disturbance are assumed.

Moreover, f_k is assumed to be an unknown nonlinearity whose dynamics have the following conic sector description:

$$\left\| f_k - \left(Ax_k + Bu_k + Fw_k\right)\right\| \le \left\| C_f x_k + D_f u_k + F_f w_k \right\| \quad (2)$$

for all time $k \in \mathbb{N}_0$, $x_k \in W_n$, $u_k \in W_m$, and, $w_k \in W_r$.

Even though the nonlinearity f is assumed to be unknown, we assume that it is possible for the matrices A, B, F, C_f, D_f, and F_f to be known. The inequality shown in (2) implies that the unknown nonlinearity lies in an n-dimensional hypersphere whose center is the linear system $Ax_k + Bu_k + Fw_k$ and whose radius is bounded by the right hand side term of (2). So, all nonlinearities which are locally continuously differentiable satisfy (2)

Also, given system (1), a linear state-feedback controller

$$u_k = Kx_k \quad (3)$$

is considered where $K \in R^{m\times n}$ is the controller gain. In the following section, we recall the basic definition of performance indices to be used.

III. Definitions

Generally, a system is said to be FTB, if the states of the system do not exceed a given bound over a fixed time interval and for all admissible disturbances. In this work, the definitions stated in [3] are adopted here and are generalized to include nonlinear systems.

Definition 1: (Finite-Time Boundedness)

System (1) is said to be finite-time bounded with respect to $\left(\alpha_x, \alpha_w, \beta, R, N\right)$ where $R > 0$, $\alpha_w \ge 0$, $0 \le \alpha_x \le \beta$, and $N \in \mathbb{N}_0$ if

$$\begin{cases} x_0^T R x_0 \le \alpha_x^2 \\ w_0^T w_0 \le \alpha_w^2 \end{cases} \Rightarrow x_k^T R x_k \le \beta^2 \quad \forall k = 1, \ldots, N$$

Definition 2: (H_∞ Property)

Consider system (1) with $w_k \in L_2$ and assume a performance output z_k such that

$$z_k = C_z x_k + D_z w_k \quad (4)$$

where $C_z \in R^{1\times n}$ and $D_z \in R^{1\times r}$

The system is said to have H_∞ property with degree α if

$$\frac{\sum_{k=0}^{\infty} \left\| z_k \right\|^2}{\sum_{k=0}^{\infty} \left\| w_k \right\|^2} < \alpha \quad (5)$$

where α is called the H_∞ bound.

Now, we proceed to present the main results of this paper.

IV. Main Results

The objective of this work is to find a robust state feedback controller that will guarantee the FTB during the transient period of the closed-loop system obtained from (1) and (3) as long as the nonlinearity is within the hypersphere defined by (2). Furthermore, it is desired that the obtained controller satisfies the H_∞ criterion. Therefore, first, the set of conditions corresponding to the FTB property of the closed-loop system are derived and, then, we proceed to derive those relevant to the H_∞ criterion.

Consider the closed-loop system resulting from applying controller (3) to system (1) and let $\Im_k = f - \left(A + BK\right) x_k - Fw_k$:

$$x_{k+1} = \left(A + BK\right) x_k + Fw_k + \Im_k \quad (6)$$

where the disturbance input is described by

$$w_{k+1} = \Phi w_k \quad (7)$$

where $\left|\lambda_i(\Phi)\right| < 1$ for $w_k \in L_2$.

Theorem 1: *System (6)-(7) is FTB with respect to $\left(\alpha_x, \alpha_w, \beta, R, N\right)$ and satisfies the H_∞ property if there exist positive-definite matrices $Q_1 \in R^{n\times n}$ and $Q_2 \in R^{r\times r}$, a matrix $Y \in R^{m\times n}$, and positive scalars $\gamma \ge 1$, b_1, α, and δ such that*

$$\begin{bmatrix} \gamma Q_1 & 0 & Q_1 A^T + Y^T B^T & Q_1 C_f^T + Y^T D_f^T & 0 \\ * & \gamma Q_2 & Q_2 F^T & Q_2 F_f^T & Q_2 \Phi^T \\ * & * & Q_1 - b_1 I & 0 & 0 \\ * & * & * & b_1 I & 0 \\ * & * & * & * & Q_2 \end{bmatrix} > 0 \quad (8)$$

$$\begin{bmatrix} Q_1 - \delta R^{-1} & 0 \\ 0 & Q_2 - I \end{bmatrix} > 0 \quad (9)$$

$$\delta R^{-1} \frac{\beta^2 \gamma^{-N}}{\alpha_x^2 + \alpha_w^2} - Q_1 > 0 \quad (10)$$

$$\begin{bmatrix} Q_1 & 0 & Q_1 A^T + Y^T B^T & Q_1 C_f^T + Y^T D_f^T & Q_1 C_z^T \\ * & \alpha I & F^T & F_f^T & D_z^T \\ * & * & Q_1 - b_1 I & 0 & 0 \\ * & * & * & b_1 I & 0 \\ * & * & * & * & I \end{bmatrix} > 0 \quad (11)$$

where * denotes the elements of the matrix that need to be added to make the matrix symmetric. The controller gain is given by $K = YQ_1^{-1}$.

Proof of Theorem 1:

We start the proof with that of the conditions under which the system is FTB.

Assume that $x_0^T R x_0 \le \alpha_x^2$, $w_0^T w_0 \le \alpha_w^2$, and that $x_k^T R x_k \le \beta^2$ $\forall k = 1,...,N$. Consider the energy function,

$$V_k = x_k^T P_1 x_k + w_k^T P_2 w_k \tag{12}$$

such that

$$V_{k+1} < \gamma V_k \tag{13}$$

where $P_1 > 0$, $P_2 > 0$ and $\gamma \ge 1$

Moreover, consider the inequality shown in (2) which can be rewritten as follows:

$$\Im_k^T \Im_k \le \left(A_f x_k + F_f w_k\right)^T \left(A_f x_k + F_f w_k\right) \tag{14}$$

where $A_f = C_f + D_f K$.

Substituting (12) into (13), then replacing x_{k+1} and w_{k+1} with the equations of system (6)-(7), and applying Schur's complement [12] , the following matrix inequality is obtained.

$$\begin{bmatrix} h_{11} & h_{12} \\ h_{12}^T & h_{22} \end{bmatrix} > \begin{bmatrix} 0 & -\Im_k^T P_1 \\ -P_1 \Im_k & 0 \end{bmatrix} \tag{15}$$

where

$h_{11} = \gamma\left(x_k^T P_1 x_k + w_k^T P_2 w_k\right) - w_k^T \Phi^T P_2 \Phi w_k$, $h_{22} = P_1$,

and $h_{12} = \left((A+BK)x_k + F w_k\right)^T P_1$

For any $b_1 > 0$, it is true that

$$\begin{bmatrix} b_1^{-1/2} \Im_k^T \\ b_1^{1/2} P_1 \end{bmatrix} \begin{bmatrix} b_1^{-1/2} \Im_k & b_1^{1/2} P_1 \end{bmatrix} \ge 0 \tag{16}$$

which can be rewritten as follows:

$$\begin{bmatrix} b_1^{-1} \Im_k^T \Im_k & 0 \\ 0 & b_1 P_1^2 \end{bmatrix} \ge \begin{bmatrix} 0 & -\Im_k^T P_1 \\ -P_1 \Im_k & 0 \end{bmatrix} \tag{17}$$

Using (17), the following is a sufficient condition for (15):

$$\begin{bmatrix} h_{11} & h_{12} \\ h_{12}^T & h_{22} \end{bmatrix} > \begin{bmatrix} b_1^{-1} \Im_k^T \Im_k & 0 \\ 0 & b_1 P_1^2 \end{bmatrix} \tag{18}$$

Moreover, based on (14), (18) will still be satisfied if the following inequality holds.

$$\begin{bmatrix} h_{11} - b_1^{-1}\left(A_f x_k + F_f w_k\right)^T \left(A_f x_k + F_f w_k\right) & h_{12} \\ h_{12}^T & h_{22} - b_1 P_1^2 \end{bmatrix} > 0 \tag{19}$$

Now, apply Schur's complement to (19) to obtain

$$\begin{aligned} & h_{11} - b_1^{-1}\left(A_f x_k + F_f w_k\right)^T \left(A_f x_k + F_f w_k\right) \\ & \qquad - h_{12}\left(h_{22} - b_1 P_1^2\right)^{-1} h_{12}^T > 0 \end{aligned} \tag{20}$$

and

$$h_{22} - b_1 P_1^2 > 0 \tag{21}$$

Substitute the expressions of h_{11}, h_{12}, and h_{22} in (20) and then rearrange the obtained expression in a quadratic format as shown in (22).

$$\begin{bmatrix} x_k^T & w_k^T \end{bmatrix} \begin{bmatrix} d_{11} & d_{12} \\ d_{12}^T & d_{22} \end{bmatrix} \begin{bmatrix} x_k \\ w_k \end{bmatrix} > 0 \tag{22}$$

where $d_{11} = \gamma P_1 - b_1^{-1} A_f^T A_f - A_c^T P_1 \left(P_1 - b_1 P_1^2\right)^{-1} P_1 A_c$,

$d_{12} = -A_c^T P_1 \left(P_1 - b_1 P_1^2\right)^{-1} P_1 F - b_1^{-1} A_f^T F_f$, and

$d_{22} = \gamma P_2 - \Phi^T P_2 \Phi - b_1^{-1} F_f^T F_f - F^T P_1 \left(P_1 - b_1 P_1^2\right)^{-1} P_1 F$

Inequality (22) implies that matrix $\begin{bmatrix} d_{11} & d_{12} \\ d_{12}^T & d_{22} \end{bmatrix} > 0$, which can be rewritten as

$$\begin{aligned} & \begin{bmatrix} \gamma P_1 - b_1^{-1} A_f^T A_f & -b_1^{-1} A_f^T F_f \\ -b_1^{-1} F_f^T A_f & \gamma P_2 - \Phi^T P_2 \Phi - b_1^{-1} F_f^T F_f \end{bmatrix} \\ & \qquad - \begin{bmatrix} A_c^T P_1 \\ F^T P_1 \end{bmatrix} \left(P_1 - b_1 P_1^2\right)^{-1} \begin{bmatrix} P_1 A_c & P_1 F \end{bmatrix} > 0 \end{aligned} \tag{23}$$

By applying Schur's complement to (23), we obtain

$$\begin{bmatrix} \gamma P_1 - b_1^{-1} A_f^T A_f & -b_1^{-1} A_f^T F_f & A_c^T P_1 \\ -b_1^{-1} F_f^T A_f & \gamma P_2 - \Phi^T P_2 \Phi - b_1^{-1} F_f^T F_f & F^T P_1 \\ P_1 A_c & P_1 F & P_1 - b_1 P_1^2 \end{bmatrix} > 0 \tag{24}$$

where $A_c = A + BK$.

Note that if inequality (24) is satisfied, condition (21) implicitly holds too since it appears as one of the diagonal elements of (24). Therefore, it is redundant to include it as one of the conditions for the existence of the controller designed.

Now, pre and post multiply (24) by

$$\begin{bmatrix} P_1^{-1} & 0 & 0 \\ 0 & P_2^{-1} & 0 \\ 0 & 0 & P_1^{-1} \end{bmatrix} \tag{25}$$

and, again, apply Schur's complement to the resulting matrix after rearranging it in an appropriate form. We, then, obtain the following inequality:

$$\begin{bmatrix} \gamma P_1^{-1} & 0 & P_1^{-1} A_c^T & P_1^{-1} A_f^T \\ 0 & \gamma P_2^{-1} - P_2^{-1} \Phi^T P_2 \Phi P_2^{-1} & P_2^{-1} F^T & P_2^{-1} F_f^T \\ A_c P_1^{-1} & F P_2^{-1} & P_1^{-1} - b_1 I & 0 \\ A_f P_1^{-1} & F_f P_2^{-1} & 0 & b_1 I \end{bmatrix} > 0 \tag{26}$$

Apply similar manipulations as before to (26) , let $Q_1 = P_1^{-1}$ and $Q_2 = P_2^{-1}$, substitute the expressions of A_f and A_c, let $Y = KQ_1$, and condition (8) is obtained.

Now, we proceed to show the derivation of conditions (9) and (10).

Applying (13) iteratively and knowing that $\gamma \ge 1$, we obtain the following:

$$V_k < \gamma^N V_0 \quad (27)$$

Replace V_k and V_0 with their corresponding expressions based on (12) and since $x_k^T P_1 x_k < x_k^T P_1 x_k + w_k^T P_2 w_k$, then

$$x_k^T P_1 x_k < \gamma^N \left(x_0^T P_1 x_0 + w_0^T P_2 w_0 \right) \quad (28)$$

After introducing $R^{1/2} R^{-1/2}$ to the left and right hand side of P_1 and expressing the right hand side of the inequality in a quadratic form, (28) can be rewritten as

$$x_k^T R^{1/2} R^{-1/2} P_1 R^{-1/2} R^{1/2} x_k < \gamma^N \left(\begin{bmatrix} x_0^T R^{1/2} & w_0^T \end{bmatrix} \begin{bmatrix} R^{-1/2} P_1 R^{-1/2} & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} R^{1/2} x_0 \\ w_0 \end{bmatrix} \right) \quad (29)$$

Recall Rayleigh's inequality which states that given $Q > 0$ then $\lambda_{\min}(Q) x_k^T x_k < x_k^T Q x_k < \lambda_{\max}(Q) x_k^T x_k$ is true. Now, applying Rayleigh's inequality and the bounds on the initial state of the system and the disturbance input to (29), we obtain the following inequality:

$$\lambda_{\min}\left(R^{-1/2} P_1 R^{-1/2} \right) x_k^T R x_k < \gamma^N \lambda_{\max}\left(\begin{bmatrix} R^{-1/2} P_1 R^{-1/2} & 0 \\ 0 & P_2 \end{bmatrix} \right) \left(\alpha_x^2 + \alpha_w^2 \right) \quad (30)$$

In order for $x_k^T Q x_k < \beta^2$ to be satisfied then

$$\lambda_{\max}\left(\begin{bmatrix} R^{-1/2} P_1 R^{-1/2} & 0 \\ 0 & P_2 \end{bmatrix} \right) < \frac{\beta^2 \gamma^{-N}}{\left(\alpha_x^2 + \alpha_w^2 \right)} \lambda_{\min}\left(R^{-1/2} P_1 R^{-1/2} \right) \quad (31)$$

must hold.

Let $\delta^{-1} > 0$ such that

$$\lambda_{\max}\left(\begin{bmatrix} R^{-1/2} P_1 R^{-1/2} & 0 \\ 0 & P_2 \end{bmatrix} \right) < \delta^{-1} \quad (32)$$

and

$$\delta^{-1} < \frac{\beta^2 \gamma^{-N}}{\left(\alpha_x^2 + \alpha_w^2 \right)} \lambda_{\min}\left(R^{-1/2} P_1 R^{-1/2} \right) \quad (33)$$

Then, conditions (9) and (10) can be derived from (32) and (33) respectively through basic algebraic manipulations.

Now, we go on to prove the condition under which the state-feedback linear controller satisfies the H_∞ criterion.

Consider the closed loop system (6)-(7). Moreover, consider the performance index:

$$V'_{k+1} - V'_k + z_k^T z_k - \alpha w_k^T w_k < 0 \quad (34)$$

where

$$V'_k = x_k^T P_1 x_k \quad (35)$$

where $P_1 > 0$ and z_k is given by (4).

Substituting (35) in (34), then replacing x_{k+1} with the equation of system (6), and applying Schur's complement, we obtain the following inequality.

$$\begin{bmatrix} g_{11} & g_{12} \\ g_{12}^T & g_{22} \end{bmatrix} > \begin{bmatrix} 0 & -\Im_k^T P_1 \\ -P_1 \Im_k & 0 \end{bmatrix} \quad (36)$$

where

$g_{11} = x_k^T P_1 x_k + \alpha w_k^T w_k - z_k^T z_k$, $g_{22} = P_1$, and

$g_{12} = \left(A_c x_k + F w_k \right)^T P_1$

Using (17) and (14), the following is a sufficient condition for (36).

$$\begin{bmatrix} g_{11} - b_1^{-1} \left(A_f x_k + F_f w_k \right)^T \left(A_f x_k + F_f w_k \right) & g_{12} \\ g_{12}^T & g_{22} - b_1 P_1^2 \end{bmatrix} > 0 \quad (37)$$

Applying Schur's complement to (37), the following two conditions are obtained :

$$g_{11} - b_1^{-1} \left(A_f x_k + F_f w_k \right)^T \left(A_f x_k + F_f w_k \right) - g_{12} \left(g_{22} - b_1 P_1^2 \right)^{-1} g_{12}^T > 0 \quad (38)$$

and

$$g_{22} - b_1 P_1^2 > 0 \quad (39)$$

Substitute the expressions of g_{11}, g_{12}, and g_{22} and the expression of z_k shown in (4) in (38), and then rearrange the obtained expression into a quadratic form as shown in (40)

$$\begin{bmatrix} x_k^T & w_k^T \end{bmatrix} \begin{bmatrix} s_{11} & s_{12} \\ s_{12}^T & s_{22} \end{bmatrix} \begin{bmatrix} x_k^T \\ w_k^T \end{bmatrix} > 0 \quad (40)$$

where $s_{11} = P_1 - C_z^T C_z - b_1^{-1} A_f^T A_f - A_c^T P_1 \left(P_1 - b_1 P_1^2 \right) P_1 A_c$,

$s_{12} = -C_z^T D_z - b_1^{-1} A_f^T A_f - A_c^T P_1 \left(P_1 - b_1 P_1^2 \right) P_1 F$, and

$s_{22} = -D_z^T D_z - b_1^{-1} F_f^T F_f - F^T P_1 \left(P_1 - b_1 P_1^2 \right) P_1 F + \alpha I$

Inequality (40) implies that the matrix $\begin{bmatrix} s_{11} & s_{12} \\ s_{12}^T & s_{22} \end{bmatrix} > 0$ which can be rewritten as follows

$$\begin{bmatrix} P_1 - C_z^T C_z - b_1^{-1} A_f^T A_f & -C_z^T D_z - b_1^{-1} A_f^T F_f \\ -D_z^T C_z - b_1^{-1} F_f^T A_f & -D_z^T D_z - b_1^{-1} F_f^T F_f + \alpha I \end{bmatrix} - \begin{bmatrix} A_c^T P_1 \\ F^T P_1 \end{bmatrix} \left(P_1 - b_1 P_1^2 \right)^{-1} \begin{bmatrix} P_1 A_c & P_1 F \end{bmatrix} > 0 \quad (41)$$

Apply Schur's complement to (41) and pre and post multiply the obtained matrix by $\begin{bmatrix} P_1^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & P_1^{-1} \end{bmatrix}$ to obtain the following condition.

$$\begin{bmatrix} P_1^{-1} - P_1^{-1} C_z^T C_z P_1^{-1} & -P_1^{-1} C_z^T D_z & P_1^{-1} A_c^T \\ -D_z^T C_z P_1^{-1} & -D_z^T D_z + \alpha I & F^T \\ A_c P_1^{-1} & F & P_1^{-1} - b_1 I \end{bmatrix} - \begin{bmatrix} P_1^{-1} A_f^T \\ F_f^T \\ 0 \end{bmatrix} \left(b_1^{-1} I \right) \begin{bmatrix} A_f P_1^{-1} & F_f & 0 \end{bmatrix} > 0 \quad (42)$$

Note that condition (39) is implicitly satisfied when (42) holds. Therefore, it would be redundant to add (39) to the set of conditions under which the controller exists.

Again, apply Schur's complement to (42) and rearrange the result obtained so that it would have the following form:

$$\begin{bmatrix} P_1^{-1} & 0 & P_1^{-1}A_c^T & P_1^{-1}A_f^T \\ 0 & \alpha I & F^T & F_f^T \\ A_cP_1^{-1} & F & P_1^{-1}-b_1 I & 0 \\ A_fP_1^{-1} & F_f & 0 & I \end{bmatrix} - \begin{bmatrix} P_1^{-1}C_z^T \\ D_z^T \\ 0 \\ 0 \end{bmatrix}(I)\begin{bmatrix} C_zP_1^{-1} & D_z & 0 & 0 \end{bmatrix} > 0 \quad (43)$$

After applying Schur's complement to (43) and substituting for A_f and A_c with their corresponding expressions, let $Q_1 = P_1^{-1}$ and $Y = KQ_1$ in order to obtain condition (11). Thus, the proof of theorem 1 is concluded.

Given $(\alpha_x, \alpha_w, \beta, R, N)$ and a nonlinear system satisfying a conic inequality as in (2), conditions (8) through (11) represent a set of LMIs when the value of γ is fixed. Thus, the problem is transformed into a feasibility problem which can be used to solve for the unknowns Q_1, Q_2, Y, b_1, δ, and α. The controller gain is $K = YQ_1^{-1}$. A numerical example is presented in the following section to demonstrate the applicability of the design criterion developed.

V. Numerical Example

Consider the following open loop discretized state-space model corresponding to Chua's circuit [13]:

$$\begin{cases} x_{k+1}^1 = 1 - T\alpha_C(1+b)x_k^1 + T\alpha_C x_k^2 \\ \qquad\qquad + 0.5T\alpha_c(a-b)\left(\left|x_k^1+1\right| - \left|x_k^1-1\right|\right) \\ x_{k+1}^2 = Tx_k^1 + (1-T)x_k^2 + Tx_k^3 \\ x_{k+1}^3 = -T\beta_C x_k^2 + (1-T\mu)x_k^3 \end{cases} \quad (44)$$

where x_k^i is the i^{th} state variable, $\alpha_C = 9.1$, $\beta_C = 16.5811$, $\mu = 0.138083$, $a = -1.3659$, $b = -0.7408$.

System (44) can be rewritten in a matrix form resembling that of the class of nonlinear systems considered in the design criteria with an added control and disturbance inputs.

$$x_{k+1} = Ax_k + Bu_k + Fw_k + \Im_k \quad (45)$$

where

$$A = \begin{bmatrix} 1-T\alpha_C(1+b) & T\alpha_C & 0 \\ T & 1-T & T \\ 0 & -T\beta_C & 1-T\mu \end{bmatrix},\ B = T\begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix},$$

$$x_k = \begin{bmatrix} x_k^1 \\ x_k^2 \\ x_k^3 \end{bmatrix}\ F = T\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

$$\Im_k = \begin{bmatrix} 0.5T\alpha_c(a-b)\left(\left|x_k^1+1\right| - \left|x_k^1-1\right|\right) \\ 0 \\ 0 \end{bmatrix}$$

The disturbance input is of **finite energy and with known dynamics** described by (7) where $\Phi = 0.9$.

Since $\left|x_k^1+1\right| - \left|x_k^1-1\right| \le \left|2x_k^1\right|$, then

$$\Im_k^T \Im_k \le \left(T\alpha_c(a-b)x_k^1\right)^2 \quad (46)$$

Inequality (46) can be rewritten in the following form:

$$\Im_k^T \Im_k \le \left(C_f x_k + D_f u_k + F_f w_k\right)^T \left(C_f x_k + D_f u_k + F_f w_k\right)$$

where $C_f = \begin{bmatrix} T\alpha_c(a-b) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $D_f = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, and $F_f = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Moreover, the weights in the performance output z_k shown in (4) are assigned as $C_z = \begin{bmatrix} 0.01 & 0.01 & 0.01 \end{bmatrix}$ and $D_z = 0.01$. Assume that $\alpha_x = 1.1$, $\alpha_w = 0.1$, $N = 25$, $T = 0.05$, $R = I$, $w_0 = 0.09$, and $x_0 = \begin{bmatrix} 0 & -1.099 & 0 \end{bmatrix}^T$

Starting with a large value of β, we check for the feasibility of the LMIs while varying the value of γ^{-1} over the range $(0,1]$. If the LMIs are infeasible for all values of γ, we increase the value of β and check again; otherwise the value of β is decreased. For the given system and the set of parameters considered, a solution for the controller gain is found for $\gamma = 1.0101$ and $\beta = 8.72$ where $K = \begin{bmatrix} -3.7078 & -4.3354 & -0.0003 \end{bmatrix}$. Moreover, the H_∞ bound is found to be $\alpha = 45.404$.

In order to examine the FTB property of the closed-loop system, the controller is applied for a time interval of length N and then removed afterwards. On the other hand, the H_∞ property of the close-loop system is examined by constantly applying the controller over time until the steady state is reached. Fig. 1 shows the states of the system for the open loop system, the system with finite-time control, and the system with H_∞ control. For a better grasp of the performance of the controller designed, the norm of the state vector is shown in Fig. 2 for the three different cases.

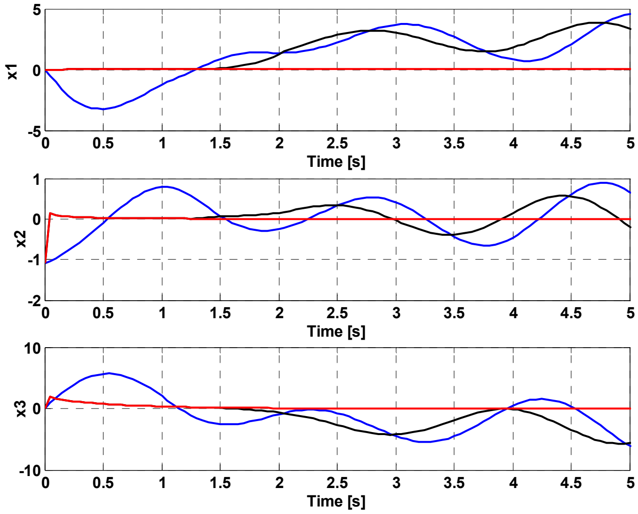

Fig. 1 State variables of the system for three different cases. Finite-time controller ( ). H_∞ and FT controller ( ). Open Loop ( )

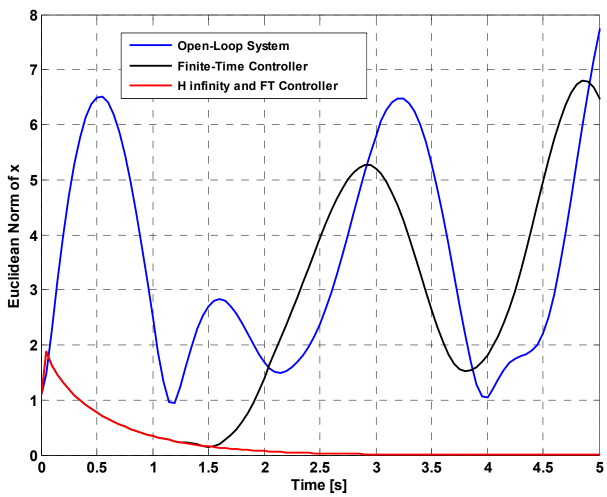

Fig. 2 Progression of $\|x_k\|$ over time for the three different cases considered.

As seen in Fig.1 and Fig.2, when the controller is applied for a finite-time and then removed, the state of the system is maintained within the imposed bound during the transient response and reverses to the open-loop case. That is, when the controller is removed, the state of the system is defined by the open loop system model as expected. On the other hand, when the same controller is not removed after N=25 steps, the controller drives the state of the system to zero as should be expected from an H_∞ and FTB controller satisfying the bound on the performance output norm.

VI. Conclusion

A finite-time state-feedback control design with an H_∞ property for a class of nonlinear systems with conic type nonlinearities and additive disturbances is presented. Conditions under which the controller exists are derived. A solution for the controller gain is obtained by transforming the conditions into LMIs. The controller obtained is robust for all nonlinearities satisfying the conic inequality and for all admissible disturbances. A numerical example based on Chua's circuit is used to illustrate the applicability and effectiveness of the control design proposed.

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