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On Decomposition of Linear-Quadratic Optimal Control Problems for Two-Steps Descriptor Systems

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Abstract— **We study the linear-quadratic optimal control problem with the state equation consisting of two sequentially acting descriptor systems. Matching conditions for trajectories at the switch point are absent. However, the minimized functional depends on values of a state trajectory at the left and right sides of the switch point. State trajectories have partially fixed left and right points and, in general, they are discontinuous functions. We present the algorithm for solving this problem, which avoids the use of boundary value problems. It is based on the sequential solving of eight initial value problems for differential-algebraic equations. The formula for the minimal value of the performance index is also given.**

I. INTRODUCTION

Mixed systems of differential and algebraic equations arise in modeling of electrical, mechanical, chemical, economic, and biological systems. These systems are known as differential-algebraic equations (DAEs) or descriptor systems. There is a vast literature devoted to theoretical and numerical analysis of these systems (see, for instance, [1–3] and references therein). Optimal control problems have been considered, e.g., in [4–7]. Note that discontinuous systems are often used in control problems (see for details [8, 9]).

It is known that control optimality conditions for a control in a programme form result in boundary value problems. However, it is possible to avoid the use of boundary value problems in classic linear–quadratic optimal control problems by applying a feedback control (see, for instance, [10]). Analogous results for descriptor systems and for DAEs with properly formulated leading term were obtained in [11] and [7], respectively. In the case of continuous state trajectories, optimal feedback controls for two linear–quadratic problems with intermediate points in the performance index are presented in [12, 13].

This paper deals with the problem of minimizing the functional

$$J(u,x)=\frac{1}{2}\Big\{\big\langle C_1E_1x_1(t_1)-C_2E_2x_2(t_1),\; F(C_1E_1x_1(t_1)-C_2E_2x_2(t_1))\big\rangle + \sum_{j=1}^{2}\int_{t_{j-1}}^{t_j}\Big(\langle x_j(t),W_j(t)x_j(t)\rangle+\langle u_j(t),R_j(t)u_j(t)\rangle\Big)dt\Big\} \tag{1}$$

with respect to the trajectories of the following system

$$(E_jx_j)'(t)=A_j(t)x_j(t)+B_j(t)u_j(t), \tag{2}$$

$$t_{j-1}\le t\le t_j,\quad j=1,2,$$

$$E_1x_1(0)=a,\quad E_2x_2(T)=b. \tag{3}$$

Here the prime denotes the differentiation with respect to t; $0=t_0<t_1<t_2=T$; $a\in \mathrm{Im}E_1$, $b\in \mathrm{Im}E_2$ and t_j are fixed; $x_j(t)\in X_j$, $u_j(t)\in U_j$; $E_j, A_j(t), W_j(t)\in L(X_j)$, $B_j(t)\in L(U_j,X_j)$, $R_j(t)\in L(U_j)$ for all $t\in[t_{j-1},t_j]$; $C_j\in L(X_j,Y)$, $F\in L(Y)$; $KerE_j\neq X_j$; X_j, U_j, and Y are real finite-dimensional Euclidean spaces. As usually, $L(X,Z)$ means a set of linear bounded operators acting from X into Z, $L(X):=L(X,X)$. Further, the operators F, $W_j(t)$, and $R_j(t)$ are symmetric, F and $W_j(t)$ are non-negative definite, $R_j(t)$ is positive definite for all $t\in[t_{j-1},t_j]$; the operators F, C_j, and E_j are independent of t, however, the other operators depend continuously on t in the corresponding segments $[t_{j-1},t_j], j=1,2$; $\langle\cdot,\cdot\rangle$ denotes an inner product in an appropriate space.

Admissible controls are piecewise continuous functions formed from continuous functions $u_1(\cdot)$ and $u_2(\cdot)$ defined on the segments $[0,t_1]$ and $[t_1,T]$ respectively. Corresponding trajectories are also piecewise continuous functions formed from continuous functions $x_1(\cdot)$ and $x_2(\cdot)$ defined on the segments $[0,t_1]$ and $[t_1,T]$ respectively.

If E_1 and E_2 are identity operators, the problem (1)–(3) is a particular case of the problem considered in the paper [14]. For this case of the problem (1)–(3), necessary and sufficient control optimality conditions as well as the unique solvability are established in [14].

It should be noted that control optimality conditions for a control in a programme form for various nonlinear control problems for discontinuous systems have been given, for instance, in [8, 15]. In order to use these control optimality conditions it is necessary to solve boundary value problems.

In this paper, we present the algorithm for the solving of the problem (1)–(3) which avoids the solving of boundary value problems. The formula for the minimal value of the performance index is also obtained. We emphasize that the state equation in the considered problem (1)–(3) consists of two sequentially acting systems. Matching conditions for trajectories at the switch point are absent, however, the minimized functional depends on values of the state trajectory at the left and right sides of the switch point. State trajectories have partially fixed left and right points and, in general, they are discontinuous functions. Note that we do not need any assumptions on the index of the system (2).

The paper is organized as follows. In section II we give the decomposition of DAE, which follows from a control optimality condition, to independent DAEs. At first, we

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present an optimal control using a state variable. Then the solvability of the appearing problems is discussed and the formula for the minimal value of the performance index is derived. In the end of this section the decomposition of the systems for finding an optimal control is described. In section III we present the main result (Theorem 3.1) on reducing the problem (1)–(3) to eight initial value problems for DAEs. This result is based on a decomposition of boundary values. An illustrative example is presented in the next section. Section V contains concluding remarks.

II. DECOMPOSITION OF DIFFERENTIAL-ALGEBRAIC EQUATION FOLLOWING FROM CONTROL OPTIMALITY CONDITION

A. Presentation for Optimal Control Using a State Variable

Further the superscript * with an operator notation denotes the adjoint operator. For brevity, the argument will be sometimes omitted.

Lemma 2.1. *Let $x_{1*}(\cdot)$ and $x_{2*}(\cdot)$ be components of a trajectory for the system (2), (3) corresponding to the control $u(\cdot) = u_*(\cdot)$ composed of the functions $u_{1*}(\cdot)$ and $u_{2*}(\cdot)$, where*

$$u_{j*}(t) = R_j(t)^{-1}B_j(t)^*\psi_j(t), \quad t_{j-1} \le t \le t_j, \ j = 1,2, \tag{4}$$

and $\psi_j(\cdot)$ solve the following problem

$$(E_j^*\psi_j)'(t) = W_j(t)x_{j*}(t) - A_j(t)^*\psi_j(t), \quad t_{j-1} \le t \le t_j, \tag{5}$$

$$E_j^*\psi_j(t_1) = -\tilde{C}_j^* F\big(\tilde{C}_1 x_{1*}(t_1) - \tilde{C}_2 x_{2*}(t_1)\big), \tag{6}$$

$$\tilde{C}_j := C_j E_j, \quad j = 1,2.$$

Then $u_*(\cdot)$ is the optimal control for the problem (1)–(3).

Note that, in general, the optimal control $u_*(\cdot)$ is discontinuous.

The proof of this lemma is analogous to the proof of Theorem 4.1. in [14] for problems with a state equation resolved with respect to the derivative.

Thus, to find a solution of the problem (1)–(3) we have to solve the boundary value problem with respect to x_j and ψ_j, $j = 1,2$.

Lemma 2.2. *Let the operator–function $K_j(\cdot)$ be a solution of the independent initial value problem*

$$(E_j^*K_j)'(t) = -K_j(t)^*A_j(t) - A_j(t)^*K_j(t) + K_j(t)^*S_j(t)K_j(t) - W_j(t), \tag{7}$$

$$S_j(t) := B_j(t)R_j(t)^{-1}B_j(t)^*, \quad t_{j-1} \le t \le t_j,$$

$$E_j^*K_j(t_1) = 0, \tag{8}$$

$$j = 1,2.$$

Let also the functions $x_{j*}(\cdot)$ and $\varphi_j(\cdot)$ solve the following boundary value problem

$$(E_jx_j)'(t) = (A_j(t) - S_j(t)K_j(t))x_j(t) - S_j(t)\varphi_j(t), \tag{9}$$

$$(E_j^*\varphi_j)'(t) = -(A_j(t) - S_j(t)K_j(t))^*\varphi_j(t), \tag{10}$$

$$t_{j-1} \le t \le t_j,$$

$$E_1x_1(0) = a, \quad E_2x_2(T) = b, \tag{11}$$

$$E_j^*\varphi_j(t_1) = \tilde{C}_j^* F\big(\tilde{C}_1x_1(t_1) - \tilde{C}_2x_2(t_1)\big), \tag{12}$$

$$j = 1,2.$$

Then the functions

$$\psi_1 = -K_1x_{1*} - \varphi_1, \quad \psi_2 = -K_2x_{2*} - \varphi_2 \tag{13}$$

solve the system (5), (6) and the components of the optimal control are given by

$$u_{j*}(t) = -R_j(t)^{-1}B_j(t)^*\big(K_j(t)x_{j*}(t) + \varphi_j(t)\big), \quad j = 1,2. \tag{14}$$

Proof: Taking into account (7) and (8) we obtain that the operators $E_j^*K_j(t)$ are symmetric, i.e.,

$$E_j^*K_j(t) = K_j(t)^*E_j, \quad t_{j-1} \le t \le t_j, \quad j = 1,2. \tag{15}$$

Combining (15) with (13), we get

$$(E_j^*\psi_j)' = -(E_j^*K_j)'x_j - K_j^*(E_jx_j)' - (E_j^*\varphi_j)'.$$

Using this relation and making immediate substitutions, we prove the lemma. ■

In this lemma, we do not avoid the solving of a boundary value problem, however, equation (10) for φ_j does not depend on x_j, i.e. it is simpler than the equation (5) for ψ_j.

B. Solvability of Auxiliary Problems from Lemma 2.2

Equation (7) is a differential-algebraic operator Riccati equation. The solvability of this equation with given condition (8) has been considered in [16]. The Riccati DAE of a more general form has been studied in [7].

Let us denote by P_j and Q_j the projectors of the space X_j onto $\mathrm{Ker}E_j$ and $\mathrm{Ker}E_j^*$ corresponding to the orthogonal decompositions $X_j = \mathrm{Ker}E_j \bigoplus \mathrm{Im}E_j^* = \mathrm{Ker}E_j^* \bigoplus \mathrm{Im}E_j$, respectively. Let us also denote by E_j^+ the inverse operator for the operator $(I - Q_j)E_j(I - P_j) : \mathrm{Im}E_j^* \longrightarrow \mathrm{Im}E_j$. Here and further I is an identity operator.

Relations (15) imply

$$(I - Q_j)K_j(t)P_j = 0, \quad t_{j-1} \le t \le t_j, \quad j = 1,2. \tag{16}$$

As it is established in [16], $Q_jK_j(t)P_j$ satisfies an algebraic operator Riccati equation. Let us assume that the operators $P_jW_j(t)P_j : \mathrm{Ker}E_j \longrightarrow \mathrm{Ker}E_j$ are positive definite and the pairs $(Q_jA_j(t)P_j, Q_jB_j(t))$ are controllable for all $t \in [t_{j-1}, t_j]$, $j = 1,2$. Then the equation for $Q_jK_j(t)P_j$ has a unique symmetric positive definite solution and the operators

$$Q_j\big(A_j(t) - S_j(t)K_j(t)\big)P_j, \quad t_{j-1} \le t \le t_j, \quad j = 1,2, \tag{17}$$

are stable (see, e.g., [10]). It is proved in [16] that under these conditions the problem (7), (8) is solvable.

Lemma 2.3. *The boundary value problem (9)–(12)*

has a unique solution.

Proof: It suffices to prove that the system (9)–(12) with trivial conditions

$$a = 0, \quad b = 0 \tag{18}$$

has only the trivial solution.

Let us multiply scalarly equations (9) and (10) by $\varphi_j(t)$ and $x_j(t)$, respectively. Summing up the obtained relations, we get

$$\langle E_j x_j, \varphi_j \rangle' (t) = -\big\langle S_j(t)\varphi_j(t), \varphi_j(t)\big\rangle, \quad j = 1, 2.$$

Further, we integrate these equalities over the corresponding segment $[t_{j-1}, t_j]$. Summing up the obtained equalities with $j = 1$ and $j = 2$ and taking into account (11), (12), and (18), we obtain

$$\Big\langle \tilde{C}_1 x_1(t_1) - \tilde{C}_2 x_2(t_1), F(\tilde{C}_1 x_1(t_1) - \tilde{C}_2 x_2(t_1))\Big\rangle$$
$$+ \sum_{j=1}^{2} \int_{t_{j-1}}^{t_j} \langle S_j(t)\varphi_j(t), \varphi_j(t)\rangle \, dt = 0.$$

Since the operators F, $S_1(t)$, and $S_2(t)$ are positive semi-definite, it follows from the last equality that

$$F(\tilde{C}_1 x_1(t_1) - \tilde{C}_2 x_2(t_1)) = 0,$$
$$S_j(t)\varphi_j(t) = 0, \quad j = 1, 2.$$

Combining the last equalities with (9)–(12) and (15) and noting that the operators (17) are invertible, we prove the claim. ■

C. Minimal Value of Performance Index

Lemma 2.4. *The minimal value of the performance index (1) is calculated by the formula*

$$J(u_*, x_*) = \frac{1}{2}\Big(\big\langle a, K_1(0)E_1^+ a + \varphi_1(0)\big\rangle$$
$$- \big\langle b, K_2(T)E_2^+ b + \varphi_2(T)\big\rangle \Big). \tag{19}$$

Proof: Substituting into (1) the relations for $W_j(t)x_{j*}(t)$ and $R_j(t)u_{j*}(t)$ obtained from (5) and (4), respectively, we get

$$J(u_*, x_*) = \frac{1}{2}\Big\{ \big\langle \tilde{C}_1 x_{1*}(t_1) - \tilde{C}_2 x_{2*}(t_1),$$
$$F(\tilde{C}_1 x_{1*}(t_1) - \tilde{C}_2 x_{2*}(t_1))\big\rangle$$
$$+ \sum_{j=1}^{2} \int_{t_{j-1}}^{t_j} \Big(\big\langle x_{j*}(t), (E_j^*\psi_j)'(t) + A_j(t)^*\psi_j(t)\big\rangle$$
$$+ \big\langle u_{j*}(t), B_j(t)^*\psi_j(t)\big\rangle \Big) dt \Big\} =$$
$$= \frac{1}{2}\Big\{ \big\langle \tilde{C}_1 x_{1*}(t_1) - \tilde{C}_2 x_{2*}(t_1),$$
$$F(\tilde{C}_1 x_{1*}(t_1) - \tilde{C}_2 x_{2*}(t_1))\big\rangle$$
$$+ \sum_{j=1}^{2} \int_{t_{j-1}}^{t_j} \Big(\big\langle x_{j*}(t), (E_j^*\psi_j)'(t)\big\rangle$$
$$+ \big\langle \psi_j(t), A_j(t)x_{j*}(t) + B_j(t)u_{j*}(t)\big\rangle \Big) dt \Big\}.$$

Using (2), we derive

$$J(u_*, x_*) = \frac{1}{2}\{ \Big\langle \tilde{C}_1 x_{1*}(t_1) - \tilde{C}_2 x_{2*}(t_1),$$
$$F(\tilde{C}_1 x_{1*}(t_1) - \tilde{C}_2 x_{2*}(t_1))\Big\rangle$$
$$+ \sum_{j=1}^{2} \langle E_j x_{j*}(t), \psi_j(t)\rangle|_{t_{j-1}}^{t_j} \}.$$

Combining this with (3), (5), and (13), we arrive at (19). ■

D. System Decomposition

In this subsection we give the transformation such that the solving of the system (9), (10) for x_j, φ_j results in the solving of six independent DAEs.

Let

$$M_j(t) := A_j(t) - S_j(t)K_j(t), \quad j = 1, 2.$$

Then (9), (10) take the form

$$(E_j x_j)'(t) = M_j(t)x_j(t) - S_j(t)\varphi_j(t), \tag{20}$$

$$(E_j^*\varphi_j)'(t) = -M_j(t)^*\varphi_j(t). \tag{21}$$

Lemma 2.5. *Let $V_j(\cdot)$ be a solution of the operator equation*

$$(E_j V_j)'(t) = M_j(t)V_j(t) + V_j(t)^* M_j(t)^* + S_j(t),$$
$$t_{j-1} \le t \le t_j, \quad j = 1, 2, \tag{22}$$

such that $E_j V_j$ is pointwise symmetric. Let also $\varphi_j(\cdot)$ and $x_j(\cdot)$ be solutions of equations (10) and (9), respectively.

Then $z_j(\cdot)$ given by

$$z_j(t) = x_j(t) + V_j(t)\varphi_j(t), \tag{23}$$

is a solution of the equation

$$(E_j z_j)'(t) = M_j(t)z_j(t), \quad j = 1, 2. \tag{24}$$

Proof: Immediately differentiating $E_j z_j$ where z_j is defined by (23), using (20)–(22), the symmetry of $E_j V_j(t)$ and the equality

$$x_j(t) = z_j(t) - V_j(t)\varphi_j(t), \tag{25}$$

we obtain (24). ■

III. ALGORITHM FOR SOLVING PROBLEM (1)-(3)

Now we are going to show that the solving of the boundary value problem (9)–(12) can be reduced to the solving of six successively solved independent initial value problems for DAEs.

A. Some Properties of Solutions of Equations (22)

Consider solutions of equations (22) satisfying the following conditions

$$E_1V_1(0) = 0, \quad E_2V_2(T) = 0. \tag{26}$$

Then, due to (23) and (3), we get

$$E_1z_1(0) = a, \quad E_2z_2(T) = b. \tag{27}$$

Hence we obtained the independent initial value problems (24), (27) for z_1 and z_2.

Lemma 3.1. *If the operators E_1 and E_2 are invertible, then the operators $V_1(t)(E_1^*)^{-1}$, $0 \le t \le t_1$, are non-negative definite and the operators $V_2(t)(E_2^*)^{-1}$, $t_1 \le t \le T$, are non-positive definite.*

Proof: Since the operator $S_j(t)$ is symmetric, it follows from (22) and (26) that the operator $E_jV_j(t)$ and hence the operator $V_j(t)(E_j^*)^{-1}$ is symmetric too. Next we use the formula from [17, p.151] for solutions of the equations

$$(V_j(E_j^*)^{-1})'(t) = (E_j)^{-1}M_j(t)V_j(t)(E_j^*)^{-1}$$
$$+V_j(t)(E_j^*)^{-1}((E_j)^{-1}M_j(t))^* + (E_j)^{-1}S_j(t)(E_j^*)^{-1},$$
$$t_{j-1} \le t \le t_j, \quad j = 1, 2,$$

with the conditions

$$V_1(0)(E_1^*)^{-1} = 0, \quad V_2(T)(E_2^*)^{-1} = 0.$$

Namely, the solutions of the last two problems have the form

$$V_1(t)(E_1^*)^{-1} =$$
$$\int_0^t U_{(E_1)^{-1}M_1}(t,\tau)\widetilde{S}_1(\tau)U_{-((E_1)^{-1}M_1)^*}(\tau,t)d\tau, \tag{28}$$
$$0 \le t \le t_1,$$
$$V_2(t)(E_2^*)^{-1} =$$
$$\int_T^t U_{(E_2)^{-1}M_2}(t,\tau)\widetilde{S}_2(\tau)U_{-((E_2)^{-1}M_2)^*}(\tau,t)d\tau, \tag{29}$$
$$t_1 \le t \le T.$$

Here $\widetilde{S}_j(t) = E_j^{-1}S_j(t)(E_j^*)^{-1}$, $j = 1, 2$, and $U_A(t,\tau)$ denotes the transition operator of the equation $Y' = A(t)Y$.

Using the formula for the fundamental matrix of the adjoint equation [18, p.62], we get $U_{-A^*}(\tau,t) = (U_A(t,\tau))^*$. Noting that the operators $\widetilde{S}_1$ and $\widetilde{S}_2$ are non-negative definite, the claim clearly follows from (28) and (29). ■

Now let us consider the case when the operators E_1 and E_2 may be singular.

Let us represent the operators $V_j(t)$ in the following form

$$V_j(t) = V_{j1}(t) + V_{j2}(t) + V_{j3}(t) + V_{j4}(t), \tag{30}$$

where

$$V_{j1}(t) := (I - P_j)V_j(t)(I - Q_j),$$
$$V_{j2}(t) := (I - P_j)V_j(t)Q_j,$$
$$V_{j3}(t) := P_jV_j(t)(I - Q_j), \quad V_{j4}(t) := P_jV_j(t)Q_j.$$

Since $E_jV_j(t)$ is symmetric, we get

$$V_{j2}(t) = 0. \tag{31}$$

Combining (30), (31) with (22), (26), we obtain the following system

$$(I-Q_j)E_j(I-P_j){V_{j1}}'(t) = (I-Q_j)M_j(t)(I-P_j)V_{j1}(t)$$
$$+(I-Q_j)M_j(t)P_jV_{j3}(t)+V_{j1}(t)^*((I-Q_j)M_j(t)(I-P_j))^*$$
$$+V_{j3}(t)^*((I-Q_j)M_j(t)P_j)^*+(I-Q_j)S_j(t)(I-Q_j), \tag{32}$$
$$0 = (I-Q_j)M_j(t)P_jV_{j4}(t) + V_{j1}(t)^*(Q_jM_j(t)(I-P_j))^*$$
$$+V_{j3}(t)^*(Q_jM_j(t)P_j)^* + (I-Q_j)S_j(t)Q_j, \tag{33}$$
$$0 = Q_jM_j(t)P_jV_{j4}(t)+V_{j4}(t)^*(Q_jM_j(t)P_j)^*+Q_jS_j(t)Q_j, \tag{34}$$
$$\begin{array}{c}(I-Q_1)E_1(I-P_1)V_{11}(0) = 0,\\ (I-Q_2)E_2(I-P_2)V_{21}(T) = 0.\end{array} \tag{35}$$

Lemma 3.2. *The operators $V_{11}(t)E_1^{*+}$, $0 \le t \le t_1$, are non-negative definite and the operators $V_{21}(t)E_2^{*+}$, $t_1 \le t \le T$, are non-positive definite.*

Proof: Taking into account the stability of the operators (17) we can find from (34) the pointwise symmetric operator $V_{j4}(t)$ (see, e.g., [10, p. 215]) and resolve equation (33) with respect to $V_{j3}(t)^*$. Substituting the obtained expression for $V_{j3}(t)^*$ into (32), we get for the operator $V_{j1}(t)$ the following equation

$$\begin{array}{c}(I-Q_j)E_j(I-P_j)V_{j1}'(t) = D_j(t)V_{j1}(t)\\ +V_{j1}(t)^*D_j(t)^* + F_j(t),\\ t_{j-1} \le t \le t_j, \quad j = 1, 2,\end{array} \tag{36}$$

where

$$D_j(t) :=$$
$$(I-Q_j)(I-M_j(t)P_j(Q_jM_j(t)P_j)^{-1}Q_j)M_j(t)(I-P_j),$$
$$F_j(t) := (I-Q_j)(S_j(t) - M_j(t)P_j(Q_jM_j(t)P_j)^{-1}\cdot$$
$$(V_{j4}(t)P_jM_j(t)^* + Q_jS_j(t)) - (M_j(t)P_j(Q_jM_j(t)P_j)^{-1}\cdot$$
$$(V_{j4}(t)P_jM_j(t)^* + Q_jS_j(t)))^*)(I-Q_j).$$

Now let us prove that the pointwise symmetric operator $F_j(t)$ is pointwise non-negative definite. Using (34), we obtain

$$-(I-Q_j)M_jP_j(Q_jM_jP_j)^{-1}V_{j4}P_jM_j^*(I-Q_j)$$
$$-(I-Q_j)M_jP_jV_{j4}((Q_jM_jP_j)^*)^{-1}((I-Q_j)M_jP_j)^*$$
$$= -(I-Q_j)M_jP_j(Q_jM_jP_j)^{-1}(V_{j4}(Q_jM_jP_j)^*$$
$$+Q_jM_jP_jV_{j4})((Q_jM_jP_j)^*)^{-1}((I-Q_j)M_jP_j)^*$$
$$= (I-Q_j)M_jP_j(Q_jM_jP_j)^{-1}Q_jS_jQ_j\cdot$$
$$((Q_jM_jP_j)^*)^{-1}((I-Q_j)M_jP_j)^*.$$

In view of the last equality we derive the relation

$$F_j = (I - Q_j)(S_j - G_j S_j - S_j G_j^* + G_j S_j G_j^*)(I - Q_j),$$

where

$$G_j = M_j P_j (Q_j M_j P_j)^{-1} Q_j.$$

The non-negative definiteness of the operator F_j follows from the same property of the operator S_j since

$$\Big\langle \big(S_j - G_j S_j - S_j G_j^* + G_j S_j G_j^*\big)x, x \Big\rangle$$
$$= \Big\langle (I - G_j^*)x, S_j(I - G_j^*)x \Big\rangle, \quad x \in X_j.$$

Therefore, for the operators $V_{11}(t)$, $V_{21}(t)$ we obtained the problems (36), (35) of the form (22), (26) with the invertible operators at the derivative. Hence, Lemma 6.1 completes the proof. ■

B. Decomposition of Boundary Values

The functions $\varphi_j(\cdot)$ must satisfy (12). Using (25), the symmetry of $E_j V_j$, and introduced notations, we derive from (12) the following system

$$(I + \tilde{C}_1^* F \tilde{C}_1 \tilde{V}_{11}(t_1)) E_1^* \varphi_1(t_1) - \tilde{C}_1^* F \tilde{C}_2 \tilde{V}_{21}(t_1) E_2^* \varphi_2(t_1)$$
$$= \tilde{C}_1^* F(\tilde{C}_1 z_1(t_1) - \tilde{C}_2 z_2(t_1)),$$
$$\tilde{C}_2^* F \tilde{C}_1 \tilde{V}_{11}(t_1) E_1^* \varphi_1(t_1) + (I - \tilde{C}_2^* F \tilde{C}_2 \tilde{V}_{21}(t_1)) E_2^* \varphi_2(t_1)$$
$$= \tilde{C}_2^* F(\tilde{C}_1 z_1(t_1) - \tilde{C}_2 z_2(t_1)), \qquad (37)$$

where

$$\tilde{V}_{j1}(t_1) := V_{j1}(t_1) E_1^{*+}, \;\; j = 1, 2.$$

Assume that the functions z_1 and z_2 are found by solving two independent initial value problems from (24), (27).

If the operator

$$\begin{pmatrix} I + \tilde{C}_1^* F \tilde{C}_1 \tilde{V}_{11}(t_1) & -\tilde{C}_1^* F \tilde{C}_2 \tilde{V}_{21}(t_1) \\ \tilde{C}_2^* F \tilde{C}_1 \tilde{V}_{11}(t_1) & I - \tilde{C}_2^* F \tilde{C}_2 \tilde{V}_{21}(t_1) \end{pmatrix}, \qquad (38)$$

acting in $\mathrm{Im}E_1^* \bigoplus \mathrm{Im}E_2^*$, is invertible, then we can uniquely determine the values $E_1^* \varphi_1(t_1)$ and $E_2^* \varphi_2(t_1)$ from (37).

Let us rewrite the operator (38) in the form

$$I + \tilde{F}V,$$

where

$$\tilde{F} := \begin{pmatrix} \tilde{C}_1^* F \tilde{C}_1 & \tilde{C}_1^* F \tilde{C}_2 \\ \tilde{C}_2^* F \tilde{C}_1 & \tilde{C}_2^* F \tilde{C}_2 \end{pmatrix},$$

$$V := \begin{pmatrix} \tilde{V}_{11}(t_1) & 0 \\ 0 & -\tilde{V}_{21}(t_1). \end{pmatrix}.$$

It is easy to check that the non-negative term outside the integral in the performance index (1) admits the representation

$$\left\langle \tilde{F} \begin{pmatrix} (I - P_1)x_1(t_1) \\ -(I - P_2)x_2(t_1) \end{pmatrix}, \begin{pmatrix} (I - P_1)x_1(t_1) \\ -(I - P_2)x_2(t_1) \end{pmatrix} \right\rangle.$$

Therefore, the operator $\tilde{F}$ is non-negative definite.

Further, suppose that the operator $\tilde{F}$ acting in $\mathrm{Im}E_1^* \bigoplus \mathrm{Im}E_2^*$ is invertible. Hence, this operator is positive definite.

Lemma 3.3 *The operator (38) is invertible.*

Proof: Assume that there is $x \in \mathrm{Im}E_1^* \bigoplus \mathrm{Im}E_2^*$ such that $x \neq 0$ and

$$(I + \tilde{F}V)x = 0.$$

Since $\tilde{F}$ is invertible, the last equality is equivalent to

$$(\tilde{F}^{-1} + V)x = 0.$$

By Lemma 3.2, the operator V is non-negative definite. Since the operator $\tilde{F}^{-1}$ is positive definite, we get $x = 0$. This contradiction proves the lemma. ■

So, we can find the values $E_1^* \varphi_1(t_1)$ and $E_2^* \varphi_2(t_1)$ from the system (37) and we have two independent initial value problems for DAEs for finding φ_1 and φ_2.

C. Main Result

Combining Lemmas 2.2, 2.4, 2.5, and 3.3, we arrive at the following statement.

Theorem 3.1. *The solving of the problems of the form (1)–(3) is equivalent to the successive solving of eight initial value problems for DAEs for finding K_j, V_j, z_j, and φ_j, $j = 1, 2$: (7), (8); (22), (26); (24), (27); and (10), (37), respectively.*

In addition, the optimal trajectory, the optimal control, and the minimal value of the performance index are given by the formulas (25), (14), and (19), respectively.

The similar theorem is presented in [19] for problems with a state equation resolved with respect to the derivative.

Note that we have established the solvability of the problems (7), (8) and (22), (26) for the operator DAEs. In view of the stability of the operators (17) linear DAEs (24) and (10) have index one. Therefore the problems (24), (27) and (10), (37) have unique solutions.

Now we are formulating the obtained result as the algorithm for solving problem (1)–(3). 1) To solve problems (7), (8) for $K_j, j = 1, 2$. 2) To solve problems (22), (26) for $V_j, j = 1, 2$. 3) To solve problems (24), (27) for $z_j, j = 1, 2$. 4) To solve problem (37) for finding $E_j^* \varphi_j(t_1), j = 1, 2$. 5) To solve problems (10) for $\varphi_j, j = 1, 2$, by using the values $E_j^* \varphi_j(t_1)$ obtained in the previous point. 6) To find the optimal trajectory by using (25). 7) To find the optimal control by using (14). 8) To calculate the minimal value of performance index (1) by using (19).

IV. EXAMPLE

Let us consider a very simple but illustrative example in order to obtain the solutions of auxiliary problems in an explicit analytical form. Namely, consider the following problem of minimizing the functional

$$J(u, x) = \frac{1}{2}\Big(\big(x_{11}(1) + x_{21}(1)\big)^2 +$$

$$+\int_0^1 \left(x_{11}^2+2x_{11}x_{12}+3x_{12}^2+u_1^2\right)dt+\int_1^2 \left(x_{21}^2+8x_{22}^2+u_2^2\right)dt\Big)$$

with respect to the trajectories of the system

$$x'_{11} = x_{11},\ \ 0 = x_{12} + u_1, \quad x_{11}(0) = -1, \quad t \in [0, 1],$$

$$x'_{21} = 0,\ \ 0 = x_{22} - u_2, \quad x_{21}(2) = 1, \quad t \in [1, 2].$$

Clearly, the solution of this problem is $u_{1*} = -e^t/4,\ u_{2*} = 0,\ x_{1*} = \begin{pmatrix} -e^t \\ e^t/4 \end{pmatrix},\ x_{2*} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and $J(u_*, x_*) = (11e^2 - 16e + 13)/16$. Let us illustrate the solving of this problem by reducing it to the initial value problems for eight DAEs.

In this case, the problems (7), (8) for finding $K_j(\cdot) = \begin{pmatrix} K_{j1} & K_{j2} \\ K_{j3} & K_{j4} \end{pmatrix}, j = 1, 2$, have the form

$$K_{12} = 0,\ \ {K_{11}}' = -2K_{11} + K_{13}^2 - 1,$$

$$0 = -K_{13} + K_{13}K_{14} - 1,\ \ 0 = -2K_{14} + K_{14}^2 - 3,$$

$$t \in [0, 1], \quad K_{11}(1) = 0;$$

$$K_{22} = 0,\ \ {K_{21}}' = K_{23}^2 - 1,\ \ 0 = -K_{23} + K_{23}K_{24},$$

$$0 = -2K_{24} + K_{24}^2 - 8, \quad t \in [1, 2], \quad K_{21}(1) = 0.$$

The solutions of these problems are

$$K_1(t) = \begin{pmatrix} 3(e^{2(1-t)} - 1)/8 & 0 \\ 1/2 & 3 \end{pmatrix},$$

$$K_2(t) = \begin{pmatrix} 1-t & 0 \\ 0 & 4 \end{pmatrix}.$$

The problems (22), (26) for finding $V_j(\cdot) = \begin{pmatrix} V_{j1} & V_{j2} \\ V_{j3} & V_{j4} \end{pmatrix}, j = 1, 2$, have the form

$$V_{12} = 0,\ \ {V_{11}}' = 2V_{11},\ \ 0 = -V_{11}/2 - 2V_{13},\ \ 0 = -4V_{14} + 1,$$

$$t \in [0, 1], \quad V_{11}(0) = 0;$$

$$V_{22} = 0,\ \ {V_{21}}' = 0,\ \ 0 = -3V_{23},\ \ 0 = -6V_{24} + 1,$$

$$t \in [1, 2], \quad V_{21}(2) = 0.$$

The solutions of these problems are

$$V_1(t) = \begin{pmatrix} 0 & 0 \\ 0 & 1/4 \end{pmatrix}, \quad V_2(t) = \begin{pmatrix} 0 & 0 \\ 0 & 1/6 \end{pmatrix}.$$

The solutions of problems (24), (27) are

$$z_1(t) = \begin{pmatrix} -e^t \\ e^t/4 \end{pmatrix}, \quad z_2(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The solutions of problems (10), (37) are

$$\varphi_1(t) = \begin{pmatrix} (1-e)e^{1-t} \\ 0 \end{pmatrix}, \quad \varphi_2(t) = \begin{pmatrix} e-1 \\ 0 \end{pmatrix}.$$

Using (25), (14), and (19), we find the optimal trajectory, the optimal control, and the minimal value of the performance index, respectively.

V. CONCLUSIONS

We presented the algorithm for the solving of linear-quadratic optimal control problems for two-steps descriptor systems. The algorithm is based on the sequential solving of eight independent initial value problems for DAEs. The latter enables us to apply numerical methods for solving initial value problems for DAEs (see, for instance, [1-3]) for the solving of considered optimal control problems. The formula for the minimal value of the performance index is also obtained.

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