2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Adaptive Twist Sliding Mode Control: a Lyapunov Design

Jose Kochalummoottil, Yuri B. Shtessel, Jaime A. Moreno, and Leonid Fridman

Abstract— **A novel adaptive-gain twist sliding mode controller is proposed. The disturbance term is assumed to be bounded with unknown bounds. The proposed Lyapunov-based approach consists in using dynamically adaptive control gains that ensure the establishment, in finite time, of a real second order sliding mode. Also the adaptation algorithm doesn't overestimate the values of the control gain. A numerical example confirms the efficacy of the proposed adaptive-gain twisting control.**

I. INTRODUCTION

SLIDING mode control is a very popular choice when it comes to dealing with matched disturbances and uncertainties.. The approach is based on keeping a suitably chosen constraint by 'brute force', i.e. by responding immediately to any deviation of the system from the constraint by steering it back by a sufficiently energetic effort [1]-[5], [10]. However in-spite of it being very robust and accurate, the standard sliding mode can be implemented only if the relative degree of the sliding variable is equal to one. On the other hand, Higher order sliding mode control ([6-8]) can be applied to systems with arbitrary relative degree. Also while the standard sliding mode precision is proportional to the time interval between measurements, the r-sliding mode realization can provide up to the rth order of sliding precision with respect to the switching delay [5].

A powerful second order sliding mode control (2-SMC) algorithm is the twisting control law [9], [14] that handles a relative degree equal to two. The state variable and its derivative is driven to zero in finite time by means of discontinuous control in the presence of smooth matched disturbances with bounded gradient, when ***this boundary is known***. Hence the boundary of the disturbance gradient is required to be known which is a serious drawback, since in many practical cases, the bound cannot be easily estimated [11].

Contribution. In this work, we propose the novel ***adaptive*** gain Twist control law that continuously drives the state variable and its derivative to a bounded domain (or zero if the adaptive gains are allowed to be overestimated) in finite time in the presence of the bounded disturbance with the ***unknown boundary***. The derivation and the proof is based on the recently proposed Lyapunov function for twisting controller [13] as well as on the adaptation technique developed for the derivation and the proof of adaptive Supertwist control in [12]. The structure of this paper is as follows. The problem is formulated in Section II, and the control structure is discussed in Section III. The derivation and the proof of the proposed adaptive twist 2-SMC algorithm are presented in Section IV. Section V contains a simulation example. The conclusions can be found in Section VI.

J. Kochalummoottil is with the University of Alabama in Huntsville, Huntsville, AL 35899, USA, (e-mail: jok0001@uah.edu).

Y. B. Shtessel is with the University of Alabama in Huntsville, Huntsville, AL 35899, USA, (tel: +(256) 824-6164; fax: +(256)824-6803; e-mail: shtessel@eng.uah.edu).

J. A. Moreno is with Electrica Y Computacion, Instituto de Ingeneria, UNAM, D.F., Mexico, (e-mail: jmorenop@iingen.unam.mx).

L.Fridman is with Departamento de Ingenier´ıa de Control, Universidad Nacional Aut´onoma de M´exico, UNAM, Coyoacan, 04510, D.F., Mexico,(e-mail: lfridman@servidor.unam.mx).

II. PROBLEM FORMULATION

The closed loop system is described by

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= u + \delta(t,x,y) \end{aligned}\right\} \qquad (1)$$

where $x, y \in \mathbb{R}$ are scalar state variables, $\delta(t,x,y)$ is a bounded disturbance, whose finite boundary

$$\left|\delta(t,x,y)\right| \le D > 0 \qquad (2)$$

exists but ***is not known.***

In this work we are looking for an adaptive-gain Twisting algorithm that is able to address this problem via generating a control function, whose gains are adapted to the ***unknown*** perturbation with the unknown boundary.

III. CONTROL STRUCTURE

The following Twisting control algorithm [6] is considered.

$$u = -\alpha\left(\operatorname{sgn}(x) + 0.5\operatorname{sgn}(y)\right), \qquad (3)$$

where the adaptive gain

$$\alpha = \alpha(t,x,y) \qquad (4)$$

is to be defined.

The adaptation process consists of dynamically increasing the control gain $\alpha(t)$ such that the variable x and its derivative y converge to the equilibrium point $x = y = 0$ in the 2-sliding mode (2-SMC) in finite time in the presence of the bounded perturbation with the unknown bound. Thereafter the gain $\alpha(t)$ starts to reduce. This gain reduction gets reversed as soon as the system trajectories again start deviating from the equilibrium. In order to avoid the control gain $\alpha(t)$ from being over- estimated, a detector that reveals the beginning of the destruction of the 2-SMC is constructed and incorporated in the Adaptive Twist control law. This detection mechanism is designed by introducing a domain

$M:\{x,y:N(x,y)\le\mu\}$, where $N=N(x,y)=\dfrac{x^2}{a}+\dfrac{y^2}{b}$ is an ellipse, so that as soon as this domain is reached, the gain $\alpha(t)$ starts dynamically reducing. However, as soon as the system trajectories leave the domain, the control gain $\alpha(t)$ starts increasing in order to force the trajectories back to the domain in finite time. In the case when $a=b$, the domain is represented by a circle of radius $r=\sqrt{a\mu}$.

IV. MAIN RESULTS

The main result of the paper is formulated in the following theorem.

Theorem 1. Consider system (1), where the perturbation $\delta(t,x,y)$ satisfies (2) for some unknown constant $D>0$. Then for any initial conditions $x(0),y(0)$, a real 2-sliding mode is established in the domain $M:\{x,y:N(x,y)\le\eta\}$, $\eta>\mu$ in **finite time** via Twist control (3) with the adaptive gain

$$\dot{\alpha}=\begin{cases}\dfrac{\dfrac{\omega_1}{\sqrt{2\gamma_1}}}{\dfrac{1}{\gamma_1}-\dfrac{2\alpha x^2+|x|y^2}{\left|\alpha-\alpha^*\right|^3}}\operatorname{sgn}(N(x,y)-\mu), if\ \alpha\ge\alpha_{\min}\\ \chi,\quad \alpha<\alpha_{\min}\end{cases}\tag{5}$$

with the establishment of the following conditions $(a)\ \alpha>2D$ and $(b)\ 0<\gamma<\dfrac{4\sqrt{2}}{3}\sqrt{\alpha}(0.5\alpha-D)$

where $\gamma_1,\omega_1,\mu,\chi,\alpha_{\min}$ are arbitrary positive constants, and α^* is a sufficiently large constant.

Proof. Consider the following Lyapunov function [12-13].

$$V(x,y,\alpha)=V_0(x,y)+\frac{1}{4\gamma_1}\left(\alpha-\alpha^*\right)^4\tag{6}$$

where

$$V_0(x,y)=\alpha^2x^2+\gamma|x|^{3/2}\operatorname{sgn}(x)y+\alpha|x|y^2+\frac{1}{4}y^4,\tag{7}$$

and $\alpha=\alpha(t,x,y)$ is the adaptive gain, while $\alpha^*\gg0$ is a large value and $\gamma>0$.

The proof is split into two steps. In the first step, we show that $V_0(x,y)$ is finite time convergent, for which the function $V_0(x,y)$ has to be simplified as follows.

$$\begin{aligned}V_0(x,y)&=\alpha^2x^2+\gamma|x|^{3/2}\operatorname{sgn}(x)y+\alpha|x|y^2+\frac{1}{4}y^4\\&=|x|\{\alpha^2|x|+\gamma|x|^{1/2}\operatorname{sgn}(x)y+\alpha y^2\}+\frac{1}{4}y^4\\&=|x|z^TAz+\frac{1}{4}y^4\end{aligned}\tag{8}$$

where $z=[|x|^{1/2}s\operatorname{gn}(x)\quad y]^T$ and $A=\begin{bmatrix}\alpha^2 & \frac{\gamma}{2}\\ \frac{\gamma}{2} & \alpha\end{bmatrix}$.

For the matrix A to be positive definite,

$$\gamma<2\alpha_{\min}^{3/2}\tag{9}$$

Since $\lambda_{\min}\{A\}\|z\|^2\le z^TAz\le\lambda_{\max}\{A\}\|z\|^2$, we can write

$$V_0(x,y)\le|x|\lambda_{\max}\{A\}\|z\|^2+\frac{1}{4}y^4.\tag{9a}$$

Let

$$k_1=\left[|x|,\ |y|^2\right]^T.\tag{10}$$

Therefore

$$\|k_1\|^2=|x|^2+|y|^4\le(|x|^{1/2}+|y|)^4.\tag{11}$$

Hence,

$$V_0(x,y)\le\lambda_{\max}(A)\{|x|^2+|x||y|^2\}+\frac{1}{4}y^4\le k_1^TP_1k_1\tag{12}$$

where

$$P_1=\begin{bmatrix}\lambda_{\max}(A) & \dfrac{\lambda_{\max}(A)}{2}\\ \dfrac{\lambda_{\max}(A)}{2} & \dfrac{1}{4}\end{bmatrix}$$

Since

$$\lambda_{\min}\{P_1\}\|k_1\|^2\le k_1^TP_1k_1\le\lambda_{\max}\{P_1\}\|k_1\|^2\tag{13}$$

then equation (12) can be rewritten using (11) and (13) as

$$V_0(x,y)\le\lambda_{\max}\{P_1\}\|k_1\|^2\le\lambda_{\max}\{P_1\}(|x|^{1/2}+|y|)^4\tag{14}$$

Now in order to show that $V_0(x,y)$ is finite time convergent, we determine its derivative as

$$\begin{aligned}\dot{V}_0(x,y)=\ &2\alpha^2x\dot{x}+\gamma|x|^{3/2}\operatorname{sgn}(x)\dot{y}+\frac{3}{2}\gamma|x|^{1/2}\dot{x}y\\&+\alpha\dot{x}\operatorname{sgn}(x)y^2+2\alpha|x|y\dot{y}+y^3\dot{y}\end{aligned}\tag{15}$$

Equation (15) is expanded using system's equations (1)-(3):

$$\begin{aligned}\dot{V}_0(x,y)=\ &\frac{3}{2}\gamma|x|^{1/2}y^2-\alpha\gamma|x|^{3/2}-\alpha^2|x||y|-0.5\alpha|y|^3\\&+\delta y^3+\delta\gamma|x|^{3/2}\operatorname{sgn}(x)+2\alpha|x|y\delta\\&-0.5\alpha\gamma|x|^{3/2}\operatorname{sgn}(x)\operatorname{sgn}(y)\end{aligned}\tag{16}$$

$$\begin{aligned}\dot{V}_0(x,y)\le\ &\frac{3}{2}\gamma|x|^{1/2}y^2-\alpha\gamma|x|^{3/2}-\alpha^2|x||y|-0.5\alpha|y|^3++\delta|y|^3\\&+\delta\gamma|x|^{3/2}+2\alpha|x||y|\delta+0.5\alpha\gamma|x|^{3/2}\\\le\ &\frac{3}{2}\gamma|x|^{1/2}-\gamma|x|^{3/2}\{0.5\alpha-\delta\}-2\alpha|x||y|\{0.5\alpha-\delta\}\\&-|y|^3\{0.5\alpha-\delta\}\end{aligned}\tag{16a}$$

Also equation (16a) can be expressed as

$$\dot{V}_0(x,y) \le -K\{|x|^{3/2} + |x|^{1/2} y^2 + |x||y| + |y|^3\} \le -\frac{K}{3}\{|x|^{1/2} + |y|\}^3 \tag{17}$$

where

$$K = \min\,[\frac{3}{2}\gamma, \gamma\{0.5\alpha-\delta\}, 2\alpha\{0.5\alpha-\delta\}, \{0.5\alpha-\delta\}]. \tag{18}$$

Equation (16a) can be further simplified as $\dot{V}_0(x,y) \le -\gamma|x|^{3/2}\{0.5\alpha-\delta\} - |y|\, B^T PB$

where $B = [|x|^{1/2}\ \ |y|]^T$, and hence the matrix P is

$$P = \begin{bmatrix} 2\alpha(0.5\alpha-\delta) & -\frac{3}{4}\gamma \\ -\frac{3}{4}\gamma & 0.5\alpha-\delta \end{bmatrix} \tag{19}$$

If the following conditions hold:

$$(a)\ \alpha > 2D \tag{20a}$$

$$(b)\ \ 0 < \gamma < \frac{4\sqrt{2}}{3}\sqrt{\alpha}(0.5\alpha - D) \tag{20b}$$

then the matrix P is positive definite and $\dot{V}_0(x,y)$ is negative definite.

It can be observed that in view of (14), eq. (17) can be modified as

$$\dot{V}_0(x,y) = -\frac{K}{3}\{|x|^{1/2} + |y|\}^3 \le -\frac{K}{3\lambda_{\max}^{3/4}(P_1)} V_0^{3/4}(x,y) \tag{21}$$

For the second step of the proof to prove the finite time convergence, we compute the derivative of the entire Lyapunov function candidate presented in (6) for the system given by (1) - (3).

$$\dot{V}(x,y,\alpha) = \dot{V}_0(x,y,\alpha) + \frac{1}{\gamma_1}\left(\alpha-\alpha^*\right)^3 \dot{\alpha} \tag{22}$$

Note that in the previous calculation of $\dot{V}_0$ in eq. (21) it was assumed (implicitly) that α was constant. However α is a time dependent, and so the true derivative of $\dot{V}_0$ is calculated as

$$\dot{V}_0(x,y,\alpha) \le -\frac{K}{3\lambda_{\max}^{3/4}(P_1)} V_0^{3/4}(x,y,\alpha) + \left(2\alpha x^2 + |x| y^2\right)\dot{\alpha} \tag{23}$$

Let $R = \dfrac{K}{3\lambda_{\max}^{3/4}(P_1)}$, and therefore (22) becomes

$$\dot{V}(x,y,\alpha) \le -RV_0^{3/4}(x,y,\alpha) + \left(2\alpha x^2 + |x| y^2\right)\dot{\alpha} + \frac{1}{\gamma_1}\left(\alpha-\alpha^*\right)^3 \dot{\alpha} \tag{24}$$

On adding and subtracting the term $\dfrac{\omega_1}{\sqrt{2\gamma_1}}\left|\alpha-\alpha^*\right|^3$ in eq. (24) we get

$$\dot{V}(x,y,\alpha) \le -RV_0^{3/4}(x,y,\alpha) - \frac{\omega_1}{\sqrt{2\gamma_1}}\left|\alpha-\alpha^*\right|^3 + \left(2\alpha x^2 + |x| y^2\right)\dot{\alpha} + \frac{1}{\gamma_1}\left(\alpha-\alpha^*\right)^3\dot{\alpha} + \frac{\omega_1}{\sqrt{2\gamma_1}}\left|\alpha-\alpha^*\right|^3 \tag{25}$$

Applying Jensen's inequality

$$\left(|x|^q + |y|^q\right)^{1/q} \le \left(|x| + |y|\right) \quad , \quad q = \frac{4}{3} > 1$$

we obtain

$$\left[\left(V_0^{3/4}\right)^{4/3} + \left(\left|\alpha-\alpha^*\right|^3\right)^{4/3}\right]^{3/4} = \left[V_0 + \left|\alpha-\alpha^*\right|^4\right]^{3/4} \le \left[V_0^{3/4} + \left|\alpha-\alpha^*\right|^3\right] \tag{26}$$

and in consequence (25) becomes

$$\dot{V}(x,y,\alpha) \le -rV^{3/4}(x,y,\alpha) + \left(2\alpha x^2 + |x| y^2\right)\dot{\alpha} + \frac{1}{\gamma_1}\left(\alpha-\alpha^*\right)^3\dot{\alpha} + \frac{\omega_1}{\sqrt{2\gamma_1}}\left|\alpha-\alpha^*\right|^3 \tag{27}$$

where $r = \min\left(R, \dfrac{\omega_1}{\sqrt{2\gamma_1}}\right)$

Let there exist a positive constant α^* (very large in value) such that $\alpha(t) - \alpha^* < 0 \quad \forall t \ge 0$, assuming that the adaptation law given by eq. (5) makes the adaptive gain $\alpha(t)$ bounded (this assumption will be proven later),

$$\dot{V}(x,y,\alpha) \le -rV^{3/4}(x,y,\alpha) + \left(2\alpha x^2 + |x| y^2\right)\dot{\alpha} - \left|\alpha-\alpha^*\right|^3\left(\frac{1}{\gamma_1}\dot{\alpha} - \frac{\omega_1}{\sqrt{2\gamma_1}}\right) \tag{28}$$

Thus in view of the above assumption, eq. (28) can be reduced to the following:

$$\dot{V}(x,y,\alpha) \le -r\left[V(x,y,\alpha)\right]^{3/4} - \left|\alpha-\alpha^*\right|^3\left(\dot{\alpha}\left\{\frac{1}{\gamma_1} - \frac{2\alpha x^2 + |x| y^2}{\left|\alpha-\alpha^*\right|^3}\right\} - \frac{\omega_1}{\sqrt{2\gamma_1}}\right) \tag{29}$$

$$\Rightarrow \dot{V}(x,y,\alpha) \le -\eta\left[V(x,y,\alpha)\right]^{3/4} + \xi \tag{29a}$$

where

$$\xi = -\left|\alpha-\alpha^*\right|^3\left(\dot{\alpha}\left\{\frac{1}{\gamma_1}-\frac{2\alpha x^2+|x|y^2}{\left|\alpha-\alpha^*\right|^3}\right\}-\frac{\omega_1}{\sqrt{2\gamma_1}}\right) \quad (30)$$

Next, we consider the following two cases.

Case 1. Suppose that $N(x,y) > \mu$. Then in view of (5),

$$\dot{\alpha} = \frac{\frac{\omega_1}{\sqrt{2\gamma_1}}}{\frac{1}{\gamma_1}-\frac{2\alpha x^2+|x|y^2}{\left|\alpha-\alpha^*\right|^3}} \quad (31)$$

In order to avoid singularity in the adaptation law (31), the gain γ_1 must be selected to satisfy inequality

$$\text{(b)}\ 0<\gamma_1=\frac{\left|\alpha-\alpha^*\right|^3}{\max\limits_{x,y\in\Omega}\left(2\alpha x^2+|x|y^2\right)}<\frac{\left|\alpha-\alpha^*\right|^3}{2\alpha x^2+|x|y^2} \quad (32)$$

where Ω is a bounded set.

Condition (32) is equivalent to the requirement $\dot{\alpha}\ge 0$. It is worth noting that in view of (32), the derived adaptive-gain Twist control algorithm is semi-global.

Thus, the gain $\alpha(t)$ increases in accordance with eq. (31) until eq. (20a) is met i.e. $\alpha(t) > D\ \forall t \ge t_1$. It means that the matrix P in eq. (19) becomes positive definite in finite time t_1, and hence $\xi=0$ and $\dot{V}(x,y,\alpha,\beta)\le -r\left[V(x,y,\alpha,\beta)\right]^{3/4}$. This guarantees finite time convergence to the domain $N(x,y)\le\mu$. However the time instant t_1 cannot be exactly identified since the value of D is not known.

Case 2. Next, suppose that $N(x,y)\le\mu$.

The following 2 situations might arise.

(a) $\alpha(t)\ge\alpha_{\min}$

$\alpha(t)$ decreases in accordance with (5) that takes a form

$$\dot{\alpha} = -\frac{\frac{\omega_1}{\sqrt{2\gamma_1}}}{\frac{1}{\gamma_1}-\frac{2\alpha x^2+|x|y^2}{\left|\alpha-\alpha^*\right|^3}} \quad (33)$$

and the term ξ becomes positive. Hence, in view of (29), the derivative of the Lyapunov function candidate becomes sign indefinite and the states x and y may diverge away. As soon as $N(x,y)$ becomes greater than μ (in finite time), the condition that defines **Case 1** holds so that the system's (1) trajectory is reversed due to (31) and the states x and y reaches the domain $N(x,y)\le\mu$ in finite time, and this continues all over again.

(b) $\alpha(t)<\alpha_{\min}$

We can see from eq. (5), that as soon as the argument inside the sign function becomes negative, $\dot{\alpha}<0$ and $\alpha(t)$ starts decreasing. At the time instant $t=t_1$, when $\alpha(t)<\alpha_{\min}$ and attains the value $\alpha^-_{\min}$, $\dot{\alpha}>0$ in accordance with the second part of the eq. (5), and thus $\alpha(t)$ again starts increasing in this fashion $\alpha(t)=\alpha^-_{\min}+\chi t$.

However the moment $\alpha(t)\ge\alpha_{\min}$, $\dot{\alpha}<0$, and $\alpha(t)$ again starts decreasing. This zigzag switching continues till $N(x,y)>\mu$, and the condition that defines **Case 1** holds. Thus the value of the adaptive gain $\alpha(t)$ never goes below $\alpha_{\min}$, $(\alpha_{\min}>0)$ for any time.

Thus, during this adaptation process, the state variables x and y reach the domain $N(x,y)\le\mu$ in finite time, and may again exit this domain for some finite time interval. Since there is no finite time escape, it is guaranteed that the state variables always stay in a larger domain $N(x,y)\le\eta,\ \eta>\mu$ in a real sliding mode.

The size of this larger domain can be estimated as follows. Let's assume that at $t=t_2$ the state vector leaves the domain $N(x,y)\le\mu$. Then, after the control gain $\alpha(t)$ has increased enough in accordance with (31), the state vector enters this domain at $t=t_3$. Upper state boundaries are estimated while the states are outside the domain $N(x,y)\le\mu$. Therefore at $t=t_2$, $x(t_2)$, $y(t_2)\to N\left(x(t_2),y(t_2)\right)=\mu$, and in accordance with eqs. (1)-(3), and (31)

$$\begin{aligned}&|\dot{y}|\le\alpha(t_3)+\beta(t_3)+D\to\\&|y|\le|y(t_2)|+\left(\alpha(t_3)+\beta(t_3)+D\right)(t_3-t_2)=\eta_{11}\end{aligned} \quad (34)$$

Also

$$\begin{aligned}&|\dot{x}|=|y|\le\eta_{11}\to\\&|x|\le|x(t_2)|+\eta_{11}(t_3-t_2)=\eta_{21}\end{aligned} \quad (35)$$

Substituting (34), (35) into $N=N(x,y)$ we obtain

$$N\left(x(t_2),y(t_2)\right)\le N\left(\eta_{11},\eta_{21}\right) \quad (36)$$

Continuing this analysis, we compute a size of the domain of convergence of real 2-sliding mode as

$$\begin{aligned}&N(x,y)\le\eta\\&\eta=\max\left(N(\eta_{11},\eta_{21}),N(\eta_{12},\eta_{22}),\ldots,N(\eta_{1k},\eta_{2k}),\ldots\right)\end{aligned} \quad (37)$$

It is worth noting that eq. (37) proves only the existence of the real sliding mode domain, since the value η exist but is not known due to its dependence on the boundary D of the disturbance δ which is unknown. The Theorem 1 is proven.

It is worth noting that the gain-adaptation law in eq. (5) of the twist 2-SMC algorithm depends on the unknown upper boundary α^* of the gain α which may not be desirable. Assuming this boundary to be arbitrary large, the gain adaptation law (5) can be reduced to

$$\dot{\alpha} = \omega_1 \sqrt{\frac{\gamma_1}{2}} \operatorname{sgn}(N(x,y) - \mu) \tag{38}$$

since

$$\lim_{\alpha^* \to \infty} \frac{2\alpha x^2 + |x| y^2}{|\alpha - \alpha^*|^3} = 0$$

while α, x, y are bounded.

Theorem 2. The adaptive gain $\alpha(t)$ is bounded.

Proof. A solution to eq. (38) in the domain $\mu < N(x,y) \le \eta$ can be generated as

$$\alpha = \alpha(0) + \omega_1 \sqrt{\frac{\gamma_1}{2}} t, \quad 0 \le t \le t_r \tag{39}$$

where t_r is finite reaching time. Inside the domain $N(x,y) \le \mu$ the control gain $\alpha(t)$ is decreasing. Therefore, the gain $\alpha(t)$ is bounded in the real 2-SMC, and hence, Theorem 2 is proven.

It is worth noting that if the term $\operatorname{sgn}(N(x,y) - \mu)$ in the gain adaptation law (5) is eliminated (by making $\mu = 0$), then the adaptive gain law becomes

$$\dot{\alpha} = \frac{\dfrac{\omega_1}{\sqrt{2\gamma_1}}}{\dfrac{1}{\gamma_1} - \dfrac{2\alpha x^2 + |x| y^2}{|\alpha - \alpha^*|^3}} \tag{40}$$

or

$$\dot{\alpha} = \omega_1 \sqrt{\frac{\gamma_1}{2}} \tag{41}$$

This result is formulated in the following corollary.

Corollary 1. For the system given by eqs. (1) and (2) and any initial conditions $x(0), y(0)$, the ideal 2-SMC $x = y = 0$ is reached in finite time via adaptive gain Twist control law given by eqs. (3), (40) or (41).

V. SIMULATION EXAMPLE

Consider a numeric example given by

$$\begin{cases} \dot{x} = y \\ \dot{y} = u + 7\cos(t) \end{cases} \tag{42}$$

The initial conditions have been taken as $x(0) = 10$, $y(0) = 5, \alpha(0) = 5$, while the controller parameters are $\alpha^* = 400, \omega_1 = 10\sqrt{2}, \gamma_1 = 2, \chi = 6, \alpha_{\min} = 2$ and $\mu = 1$.

Also the values of a and b are selected to be equal to 4, and thus the domain of convergence is represented by a circle of radius 2.

It is clear from Figures 1-2 that the state variables converge to the bounded domain in finite time. The maximum bound of the gain α denoted by α^* is taken to be 400 (a large value). Furthermore, there is no overestimation of the control gain $\alpha(t)$ as seen from Figure 4. As soon as the domain $N(x,y) \le 1$ is reached, the gain $\alpha(t)$ starts dynamically reducing until the system trajectories leave the domain. The control gain $\alpha(t)$ then starts to increase that forces the trajectories back to the domain in finite time. However, if $\alpha(t)$ reaches its minimum value while $N(x,y) \le \mu$, then switching takes place as per the second part of eq. (5), and hence $\dot{\alpha} > 0$. This means that $\alpha(t)$ will again start increasing, and this back and forth switching continues till $N(x,y) > \mu$. This can be seen from Figures 4-5, where $\alpha(t)$ switches around $\alpha_{\min} = 2$ during the interval 12.9- 13.2 seconds, while $N(x,y) \le 1$.

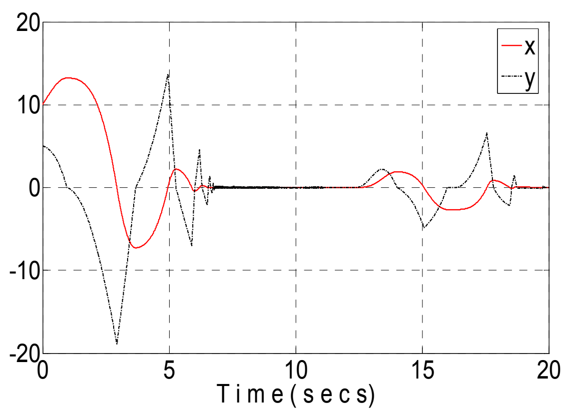

Fig 1. Time history of the state variables

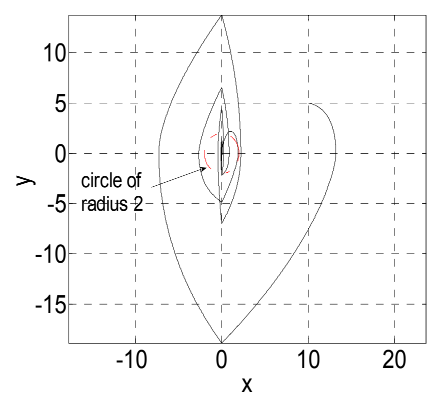

Fig 2. Phase plot of the state variables

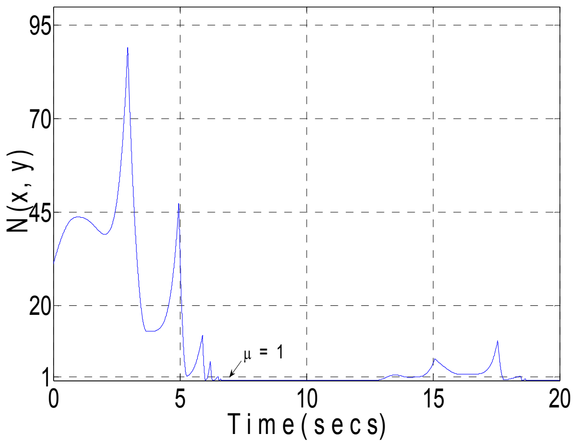

Fig 3. Time history of $N(x,y)$ and μ

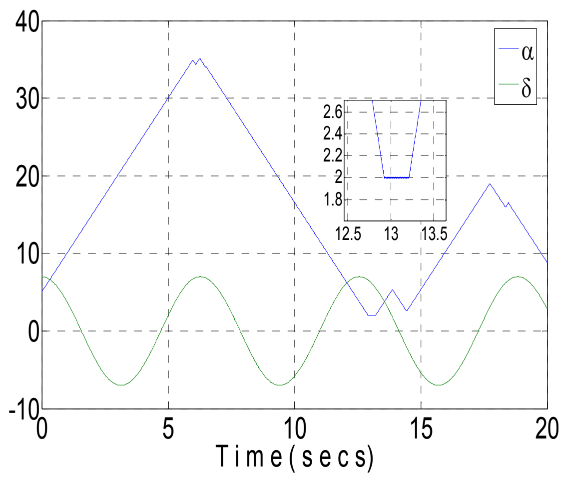

Fig 4. Time history of the gain α and the disturbance δ

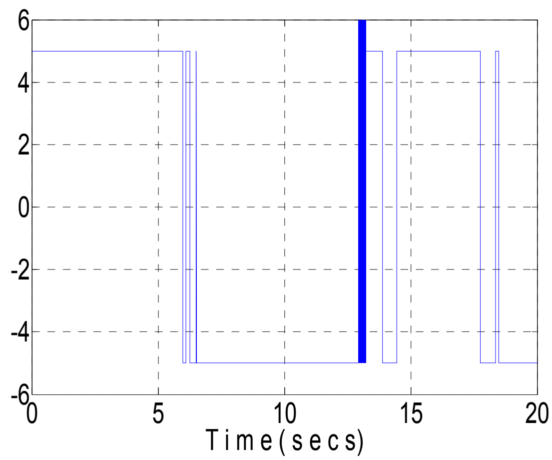

Fig 5. Time history of $\dot{\alpha}$

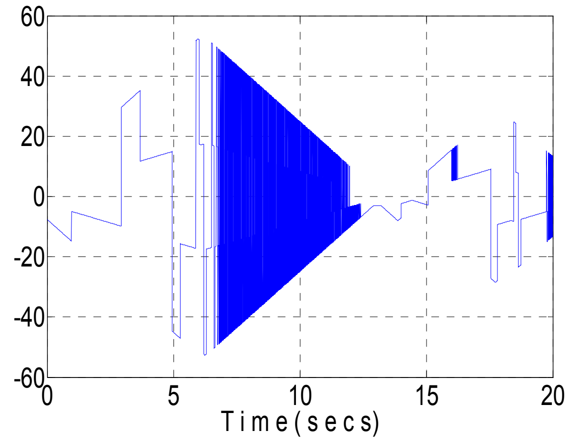

Fig 6. Time history of the control u

VI. Conclusions

A novel finite time convergent adaptive-gain twisting sliding mode control algorithm that is robust to bounded disturbance with the unknown boundary is derived and proved using Lyapunov function technique. An ideal or real second order sliding mode is established in finite time with no overestimation of the control gain. The numerical example demonstrates the efficacy of the controller.

VII. References

[1] C. Edwards and S. Spurgeon, *Sliding Mode Control: Theory and Applications,* Taylor & Francis, Bristol, 1998.

[2] V. Utkin, J. Guldner, and J. Shi, *Sliding Modes in Electromechanical Systems*, Taylor and Francis, London, 1999.

[3] I. Boiko, Discontinuous Systems, Birkhauser, Boston, 2008.

[4] J.-J. Slotine and W. Li., Applied Nonlinear Control, Prentice, 1991.

[5] A. Levant, Arbitrary order sliding modes with finite time convergence, *Proceedings of the 6th IEEE Mediterranean Conference on Control and Systems*, June, Alghero, Italy, 1998.

[6] Levant, A., "Higher-order sliding modes, differentiation and output-feedback control," *International Journal of Control*, *76*, 9/10, 2003, 924-941.

[7] A. Levant, "Homogeneity approach to high-order sliding mode design," *Automatica*, Vol. 41, No. 5, 2005, pp. 823-830.

[8] Y. Shtessel, I. Shkolnikov and A. Levant, "Smooth Second Order Sliding Modes: Missile Guidance Application," *Automatica*, Vol. 43, No.8, 2007, pp. 1470-1476.

[9] J. A. Moreno, and M. Osorio, "A Lyapunov approach to second order sliding mode controller and observers," *Proceedings of 47th IEEE Conference on Decision and Control*, pp. 2856-2861, 2008.

[10] A. F. Filippov, Differential equations with discontinuous right hand side, Kluwer, Dordrecht, The Netherlands, 1988.

[11] F. Plestan, Y. Shtessel, V. Brégeault, and A. Poznyak, "New methodologies for adaptive sliding mode control," *International Journal of Control,* Vol. 83, No. 9, 2010, pp. 1907–1919.

[12] Y.Shtessel, F.Plestan, M.Taleb, "Super-twisting adaptive sliding mode control: a Lyapunov design," *Proceedings of 49th IEEE Conference on Decision and Control*, 2010,pp 5109-5113.

[13] R. Santiesteban, Leonid Fridman, and Jaime A. Moreno, "Finite time convergence analysis for twisting controller via a strict Lyapunov function," *Proceedings of 11th Workshop on Variable Structure Systems*, 2010, pp 1-6.

[14] V.Utkin, "About Second Order Sliding Mode Control, Relative Degree, Finite time Convergence and disturbance rejection," *Proceedings of 11th Workshop on Variable Structure Systems*, 2010.