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Frequency-Domain Stability Analysis of Retrospective-Cost Adaptive Control for Systems with Unknown Nonminimum-Phase Zeros

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Abstract— **We develop a multi-input, multi-output direct adaptive controller for discrete-time, possibly nonminimum-phase, systems with unknown nonminimum-phase zeros. The adaptive controller requires limited modeling information about the system, specifically, Markov parameters from the control input to the performance variables. Often, only a single Markov parameter is required, even in the nonminimum-phase case. We analysis the stability of the algorithm using a time-and-frequency-domain approach. We demonstrate the algorithm on disturbance-rejection problems, where the disturbance spectra are unknown. This controller is based on a retrospective performance objective, where the controller is updated using either batch or recursive least squares.**

I. Introduction

Unlike robust control, an adaptive controller is self-tuned during operation. This tuning accounts for the actual—and possibly changing—dynamics of the system as well as the nature of the external signals, such as commands and disturbances. Adaptive control may also be required for systems that are difficult to model due to unknown physics or due to the inability to perform sufficiently accurate identification. Adaptive control may depend on prior modeling information, such as bounds on the model order and parameters, or it may entail explicit on-line identification. These approaches are known, respectively, as direct and indirect adaptive control. The key issue then becomes the nature of the modeling information required by the adaptive controller provided either prior to or during operation.

In adaptive control, the controller is tuned to the actual plant during operation. However, this ability comes at a cost. Adaptive control algorithms may require restrictive assumptions, such as full-state feedback, positive realness, minimum-phase zeros, matched disturbances, as well as information on the sign of the high frequency gain, relative degree, or zero locations [1–4]. In particular, the starting point for the present paper is the retrospective cost adaptive control (RCAC) approach [5–8]. This direct adaptive control approach is applicable to MIMO (output feedback) plants that are possibly unstable and nonminimum phase (NMP) with uncertain command and disturbance spectra. The modeling information required by RCAC in [5–8] is the first nonzero Markov parameter and locations of the NMP zeros, if any. Alternatively, a collection of Markov parameters can be used as long as a sufficient number is available to capture the NMP zero locations.

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The present paper extends prior RCAC results by describing a modification of RCAC that does not require knowledge of the locations of the NMP zeros. Instead, this extension requires knowledge of a limited number of Markov parameters; typically only one Markov parameter is needed. The significant aspect of this extension is the fact that knowledge of the NMP zeros is not needed. This extension thus increases the applicability of the method to systems with unknown NMP zeros, as well as systems with NMP zeros that may be changing slowly due to aging or due to a slowly varying linearization of a nonlinear plant.

The algorithm developed in the present paper is analased using time-and-frequency-domain methods and is demonstrated on a few SISO. In all cases, the number of Markov parameters that are used is not sufficient to determine the NMP zeros of the system. Consequently, these examples demonstrate the ability to control MIMO NMP systems with unknown NMP zeros.

II. Problem Formulation

Consider the MIMO discrete-time system

$$x(k+1) = Ax(k) + Bu(k) + D_1 w(k), \quad (1)$$
$$y(k) = Cx(k) + D_2 w(k), \quad (2)$$
$$z(k) = E_1 x(k) + E_0 w(k), \quad (3)$$

where $x(k) \in \mathbb{R}^n$, $y(k) \in \mathbb{R}^{l_y}$, $z(k) \in \mathbb{R}^{l_z}$, $u(k) \in \mathbb{R}^{l_u}$, $w(k) \in \mathbb{R}^{l_w}$, and $k \geq 0$. Our goal is to develop an adaptive output feedback controller that minimizes the performance variable z in the presence of the exogenous signal w with minimal modeling information about the dynamics and w. The block diagram for (1)-(3) is shown in Figure 1, where $G(\mathbf{q}) = [G_{zw}(\mathbf{q}) \ G_{zu}(\mathbf{q})]$ and

$$z(k) = G_{zw}(\mathbf{q})w(k) + G_{zu}(\mathbf{q})u(k), \quad (4)$$

where $\mathbf{q}$ is the forward-shift operator. Note that w can represent either a command signal to be followed, an external disturbance to be rejected, or both. The system (1)–(3) can represent a sampled-data application arising from a continuous-time system with sample and hold operations.

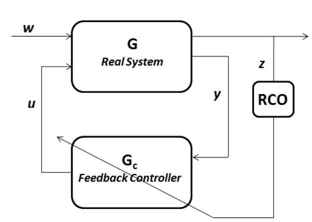

Fig. 1. Disturbance-rejection and command-following architecture.

If $D_1 = 0$ and $E_0 \neq 0$, then the objective is to have the output $E_1 x$ follow the command signal $-E_0 w$. On the other hand, if $D_1 \neq 0$ and $E_0 = 0$, then the objective is to reject the disturbance w from the performance measurement $E_1 x$. Furthermore, if $D_1 = \begin{bmatrix} \hat{D}_1 & 0 \end{bmatrix}$, $E_0 = \begin{bmatrix} 0 & \hat{E}_0 \end{bmatrix}$, and $w(k) = \begin{bmatrix} w_1(k)^{\rm T} & w_2(k)^{\rm T} \end{bmatrix}^{\rm T}$, then the objective is to have $E_1 x$ follow the command $-\hat{E}_0 w_2$ while rejecting the disturbance w_1. Lastly, if D_1 and E_0 are empty matrices, then the objective is output stabilization, that is, convergence of z to zero.

III. Retrospective Surrogate Cost

For $i \geq 1$, define the Markov parameter of G_{zu} given by

$$H_i \triangleq E_1 A^{i-1} B. \quad (5)$$

For example, $H_1 = E_1 B$ and $H_2 = E_1 AB$. Let r be a positive integer. Then, for all $k \geq r$,

$$x(k) = A^r x(k-r) + \sum_{i=1}^{r} A^{i-1} B u(k-i) + \sum_{i=1}^{r} A^{i-1} D_1 w(k-i), \quad (6)$$

and thus

$$z(k) = E_1 A^r x(k-r) + \sum_{i=1}^{r} E_1 A^{i-1} D_1 w(k-i) + E_0 w(k) + \bar{H}\bar{U}(k-1), \quad (7)$$

where

$$\bar{H} \triangleq \begin{bmatrix} H_1 & \cdots & H_r \end{bmatrix} \in \mathbb{R}^{l_z \times r l_u}$$

and

$$\bar{U}(k-1) \triangleq \begin{bmatrix} u(k-1) \\ \vdots \\ u(k-r) \end{bmatrix}.$$

Next, we rearrange the columns of $\bar{H}$ and the components of $\bar{U}(k-1)$ and partition the resulting matrix and vector so that

$$\bar{H}\bar{U}(k-1) = \mathcal{H}' U'(k-1) + \mathcal{H} U(k-1), \quad (8)$$

where $\mathcal{H}' \in \mathbb{R}^{l_z \times (r l_u - l_U)}$, $\mathcal{H} \in \mathbb{R}^{l_z \times l_U}$, $U'(k-1) \in \mathbb{R}^{r l_u - l_U}$, and $U(k-1) \in \mathbb{R}^{l_U}$. Then, we can rewrite (7) as

$$z(k) = \mathcal{S}(k) + \mathcal{H} U(k-1), \quad (9)$$

where

$$\mathcal{S}(k) \triangleq E_1 A^r x(k-r) + \sum_{i=1}^{r} E_1 A^{i-1} D_1 w(k-i) + E_0 w(k) + \mathcal{H}' U'(k-1). \quad (10)$$

Next, for $j = 1, \ldots, s$, we rewrite (9) with a delay of k_j time steps, where $0 \leq k_1 \leq k_2 \leq \cdots \leq k_s$, in the form

$$z(k-k_j) = \mathcal{S}_j(k-k_j) + \mathcal{H}_j U_j(k-k_j-1), \quad (11)$$

where (10) becomes

$$\mathcal{S}_j(k-k_j) \triangleq E_1 A^r x(k-k_j-r) + \sum_{i=1}^{r} E_1 A^{i-1} D_1 w(k-k_j-i) + E_0 w(k-k_j) + \mathcal{H}_j' U_j'(k-k_j-1)$$

and (8) becomes

$$\bar{H}\bar{U}(k-k_j-1) = \mathcal{H}_j' U_j'(k-k_j-1) + \mathcal{H}_j U_j(k-k_j-1), \quad (12)$$

where $\mathcal{H}_j' \in \mathbb{R}^{l_z \times (r l_u - l_{U_j})}$, $\mathcal{H}_j \in \mathbb{R}^{l_z \times l_{U_j}}$, $U_j'(k-k_j-1) \in \mathbb{R}^{r l_u - l_{U_j}}$, and $U_j(k-k_j-1) \in \mathbb{R}^{l_{U_j}}$. Now, by stacking $z(k-k_1), \ldots, z(k-k_s)$, we define the *extended performance*

$$Z(k) \triangleq \begin{bmatrix} z(k-k_1) \\ \vdots \\ z(k-k_s) \end{bmatrix} \in \mathbb{R}^{s l_z}. \quad (13)$$

Therefore,

$$Z(k) \triangleq \tilde{\mathcal{S}}(k) + \tilde{\mathcal{H}}\tilde{U}(k-1), \quad (14)$$

where

$$\tilde{\mathcal{S}}(k) \triangleq \begin{bmatrix} \mathcal{S}_1(k-k_1) \\ \vdots \\ \mathcal{S}_s(k-k_s) \end{bmatrix} \in \mathbb{R}^{s l_z}, \quad (15)$$

$\tilde{U}(k-1)$ has the form

$$\tilde{U}(k-1) \triangleq \begin{bmatrix} u(k-q_1) \\ \vdots \\ u(k-q_{l_{\tilde{U}}}) \end{bmatrix} \in \mathbb{R}^{l_{\tilde{U}}}, \quad (16)$$

where, for $i = 1, \ldots, l_{\tilde{U}}$, $k_1 \leq q_i \leq k_s + r$, and $\tilde{\mathcal{H}} \in \mathbb{R}^{s l_z \times l_{\tilde{U}}}$ is constructed according to the structure of $\tilde{U}(k-1)$. The vector $\tilde{U}(k-1)$ is formed by stacking $U_1(k-k_1-1), \ldots, U_s(k-k_s-1)$ and removing copies of repeated components.

Next, we define the *surrogate performance*

$$\hat{z}(k-k_j) \triangleq \mathcal{S}_j(k-k_j) + \mathcal{H}_j \hat{U}_j(k-k_j-1), \quad (17)$$

where the past controls $U_j(k-k_j-1)$ in (11) are replaced by the surrogate controls $\hat{U}_j(k-k_j-1)$. In analogy with (13), the *extended surrogate performance* for (17) is defined as

$$\hat{Z}(k) \triangleq \begin{bmatrix} \hat{z}(k-k_1) \\ \vdots \\ \hat{z}(k-k_s) \end{bmatrix} \in \mathbb{R}^{s l_z} \quad (18)$$

and thus is given by

$$\hat{Z}(k) = \tilde{\mathcal{S}}(k) + \tilde{\mathcal{H}}\hat{\tilde{U}}(k-1), \quad (19)$$

where the components of $\hat{\tilde{U}}(k-1) \in \mathbb{R}^{l_{\tilde{U}}}$ are the components of $\hat{U}_1(k-k_1-1), \ldots, \hat{U}_s(k-k_s-1)$ ordered in the same way as the components of $\tilde{U}(k-1)$. Subtracting (14) from

(19) yields

$$\hat{Z}(k) = Z(k) - \tilde{\mathcal{H}}\tilde{U}(k-1) + \tilde{\mathcal{H}}\hat{\tilde{U}}(k-1). \quad (20)$$

Finally, we define the *retrospective cost function*

$$J(\hat{\tilde{U}}(k-1), k) \triangleq \hat{Z}^{\mathrm{T}}(k)R(k)\hat{Z}(k), \quad (21)$$

where $R(k) \in \mathbb{R}^{l_z s \times l_z s}$ is a positive-definite performance weighting. The goal is to determine refined controls $\hat{\tilde{U}}(k-1)$ that would have provided better performance than the controls $U(k)$ that were applied to the system. The refined control values $\hat{\tilde{U}}(k-1)$ are subsequently used to update the controller.

IV. Cost Function Optimization with Adaptive Regularization

To ensure that (21) has a global minimizer, we consider the regularized cost

$$\bar{J}(\hat{\tilde{U}}(k-1), k) \triangleq \hat{Z}^{\mathrm{T}}(k)R(k)\hat{Z}(k) + \eta(k)\hat{\tilde{U}}^{\mathrm{T}}(k-1)\hat{\tilde{U}}(k-1), \quad (22)$$

where $\eta(k) \geq 0$. Substituting (20) into (22) yields

$$\bar{J}(\hat{\tilde{U}}(k-1), k) = \hat{\tilde{U}}(k-1)^{\mathrm{T}}\mathcal{A}(k)\hat{\tilde{U}}(k-1) + \mathcal{B}(k)\hat{\tilde{U}}(k-1) + \mathcal{C}(k), \quad (23)$$

where

$$\mathcal{A}(k) \triangleq \tilde{\mathcal{H}}^{\mathrm{T}}R(k)\tilde{\mathcal{H}} + \eta(k)I_{l_{\tilde{U}}}, \quad (24)$$

$$\mathcal{B}(k) \triangleq 2\tilde{\mathcal{H}}^{\mathrm{T}}R(k)[Z(k) - \tilde{\mathcal{H}}\tilde{U}(k-1)], \quad (25)$$

$$\mathcal{C}(k) \triangleq Z^{\mathrm{T}}(k)R(k)Z(k) - 2Z^{\mathrm{T}}(k)R(k)\tilde{\mathcal{H}}\tilde{U}(k-1) + \tilde{U}^{\mathrm{T}}(k-1)\tilde{\mathcal{H}}^{\mathrm{T}}R(k)\tilde{\mathcal{H}}\tilde{U}(k-1). \quad (26)$$

If either $\tilde{\mathcal{H}}$ has full column rank or $\eta(k) > 0$, then $\mathcal{A}(k)$ is positive definite. In this case, $\bar{J}(\hat{\tilde{U}}(k-1), k)$ has the unique global minimizer

$$\hat{\tilde{U}}(k-1) = -\frac{1}{2}\mathcal{A}^{-1}(k)\mathcal{B}(k). \quad (27)$$

V. Controller Construction

The control $u(k)$ is given by the strictly proper time-series controller of order n_{c} given by

$$u(k) = \sum_{i=1}^{n_{\mathrm{c}}} M_i(k)u(k-i) + \sum_{i=1}^{n_{\mathrm{c}}} N_i(k)y(k-i), \quad (28)$$

where, for all $i = 1, \ldots, n_{\mathrm{c}}$, $M_i(k) \in \mathbb{R}^{l_u \times l_u}$ and $N_i(k) \in \mathbb{R}^{l_u \times l_y}$. The control (28) can be expressed as

$$u(k) = \theta(k)\phi(k-1), \quad (29)$$

where

$$\theta(k) \triangleq [M_1(k) \ \cdots \ M_{n_{\mathrm{c}}}(k) \ N_1(k) \ \cdots \ N_{n_{\mathrm{c}}}(k)] \in \mathbb{R}^{l_u \times n_{\mathrm{c}}(l_u + l_z)} \quad (30)$$

and

$$\phi(k-1) \triangleq \begin{bmatrix} u(k-1) \\ \vdots \\ u(k-n_{\mathrm{c}}) \\ y(k-1) \\ \vdots \\ y(k-n_{\mathrm{c}}) \end{bmatrix} \in \mathbb{R}^{n_{\mathrm{c}}(l_u + l_y)}. \quad (31)$$

A. Recursive Least Squares Update of $\theta(k)$

Let d be a positive integer such that $\tilde{U}(k-1)$ contains $u(k-d)$. Next, we define the cumulative cost function

$$J_{\mathrm{R}}(\theta(k)) \triangleq \sum_{i=d+1}^{k} \lambda^{k-i} \|\phi^{\mathrm{T}}(i-d-1)\theta^{\mathrm{T}}(i-1) - \hat{u}^{\mathrm{T}}(i-d)\|^2, \quad (32)$$

where $\|\cdot\|$ is the Euclidean norm, and $\lambda(k) \in (0, 1]$ is the forgetting factor. Minimizing (32) yields

$$\theta^{\mathrm{T}}(k) \triangleq \theta^{\mathrm{T}}(k-1) + \beta(k)P(k-1)\phi(k-d-1) \cdot [\phi^{\mathrm{T}}(k-d)P(k-1)\phi(k-d-1) + \lambda(k)]^{-1} \cdot [\phi^{\mathrm{T}}(k-d-1)\theta^{\mathrm{T}}(k-1) - \hat{u}^{\mathrm{T}}(k-d)], \quad (33)$$

where $\beta(k)$ is either 0 or 1. When $\beta(k)$ is 1, the controller is allowed to adapt, when $\beta(k)$ is 0, the controller adaption is off. The error covariance is updated by

$$P(k) \triangleq (1-\beta(k))P(k-1) + \beta(k)\lambda^{-1}(k)P(k-1) - \beta(k)\lambda^{-1}(k)P(k-1)\phi(k-d-1) \cdot [\phi^{\mathrm{T}}(k-d-1)P(k-1)\phi(k-d) + \lambda(k)]^{-1} \cdot \phi^{\mathrm{T}}(k-d-1)P(k-1). \quad (34)$$

We initialize the error covariance matrix as $P(0) = \gamma I$, where $\gamma > 0$.

VI. Stability Analysis

A. Conditions for Convergence of $z(k) - \hat{z}(k)$ to Zero

Consider the *retrospective system*

$$\hat{x}(k+1) = Ax(k) + B\hat{u}(k) + D_1 w(k), \quad (35)$$

$$\hat{z}(k) = E_1\hat{x}(k) + E_0 w(k), \quad (36)$$

which is obtained by replacing $u(k)$ in (1) with $\hat{u}(k)$. The *extended retrospective system* is given by

$$\hat{X}(k+1) = \tilde{A}X(k) + \tilde{B}\hat{\tilde{U}}(k) + \tilde{B}'\hat{\tilde{U}}'(k) + \tilde{D}_1 W(k), \quad (37)$$

$$\hat{Z}(k) = \tilde{E}_1\hat{X}(k) + \tilde{E}_0 W(k), \quad (38)$$

where $\hat{X}(k) \in \mathbb{R}^{sn}$, $W(k) \in \mathbb{R}^{sl_w}$, $X(k) \in \mathbb{R}^{sn}$, $\tilde{U}'(k-1) \in \mathbb{R}^{l_{\tilde{U}'}}$ and

$$\hat{X}(k) \triangleq \begin{bmatrix} \hat{x}(k-k_1) \\ \vdots \\ \hat{x}(k-k_s) \end{bmatrix}, \ W(k) \triangleq \begin{bmatrix} w(k-k_1) \\ \vdots \\ w(k-k_s) \end{bmatrix}, \quad (39)$$

$$X(k) \triangleq \begin{bmatrix} x(k-k_1) \\ \vdots \\ x(k-k_s) \end{bmatrix}, \ \tilde{U}'(k-1) \triangleq \begin{bmatrix} u(k-q_1') \\ \vdots \\ u(k-q_{l_{\tilde{U}'}}') \end{bmatrix}, \tag{40}$$

$\tilde{A} \triangleq I_s \otimes A \in \mathbb{R}^{sn\times sn}$, $\tilde{D_1} \triangleq I_s \otimes D_1 \in \mathbb{R}^{sn\times sl_w}$, $\tilde{E}_0 \triangleq I_s \otimes E_1 \in \mathbb{R}^{sl_z\times sl_w}$, $\tilde{E}_1 \triangleq I_s \otimes E_1 \in \mathbb{R}^{sl_z\times sn}$, and $\otimes$ is the Kronecker product. The matrices $\tilde{B} \in \mathbb{R}^{sn\times l_{\tilde{U}}}$ and $\tilde{B}' \in \mathbb{R}^{sn\times l_{\tilde{U}'}}$ are block-row matrices with block entries B and $0_{n\times l_u}$ such that

$$\tilde{B}\hat{\tilde{U}}(k) + \tilde{B}'\hat{\tilde{U}}'(k) = \begin{bmatrix} B\hat{u}(k-k_1) \\ \vdots \\ B\hat{u}(k-k_s) \end{bmatrix} \in \mathbb{R}^{sl_u}, \tag{41}$$

where $\hat{\tilde{U}}'(k)$ is formed by replacing the entries $u(k-q_i')$ of $\tilde{U}'(k)$ by $\hat{u}(k-q_i')$ for $i = 1, \ldots, l_{\tilde{U}'}$.

The following result gives conditions under which $\hat{Z}(k) = 0$.

Fact 6.1: Assume that $\tilde{\mathcal{H}}$ has full column rank, $\eta(k) = 0$, $R(k) = I$, and $Z(k)$ is in the range of $\tilde{\mathcal{H}}$, and let $\hat{\tilde{U}}(k-1)$ be given by (27). Then $\hat{Z}(k) = 0$.

Proof. Since $Z(k)$ is in the range of $\tilde{\mathcal{H}}$, there exists $Q \in \mathbb{R}^{sl_{\tilde{u}}}$ such that $Z(k) = \tilde{\mathcal{H}}Q$. Substituting (27) into (20) yields

$$\begin{aligned}\hat{Z}(k) &= Z(k) + \tilde{\mathcal{H}}(\tilde{\mathcal{H}}^{\rm T}\tilde{\mathcal{H}})^{-1}\tilde{\mathcal{H}}^{\rm T}(-Z(k) + \tilde{\mathcal{H}}\tilde{U}) - \tilde{\mathcal{H}}\tilde{U} \\ &= Z(k) - \tilde{\mathcal{H}}(\tilde{\mathcal{H}}^{\rm T}\tilde{\mathcal{H}})^{-1}\tilde{\mathcal{H}}^{\rm T}Z(k) \\ &= \tilde{\mathcal{H}}Q - \tilde{\mathcal{H}}(\tilde{\mathcal{H}}^{\rm T}\tilde{\mathcal{H}})^{-1}\tilde{\mathcal{H}}^{\rm T}\tilde{\mathcal{H}}Q = 0. \qquad \square\end{aligned}$$

The next result assumes that the recursive-least-squares optimization yields $u(k-d) - \hat{u}(k-d) \to 0$ as $k \to \infty$, that is, $\theta(k)\phi(k-d-1) - \hat{u}(k-d) \to \infty$ as $k \to \infty$.

Fact 6.2: Assume that $\theta(k)$ is updated using (33) and (34), and assume that $\theta(k)\phi(k-d-1) - \hat{u}(k-d) \to 0$ as $k \to \infty$. Then $x(k) - \hat{x}(k) \to 0$ as $k \to \infty$.

Proof. It follows from (1) and (35) that

$$x(k-d+1) - \hat{x}(k-d+1) = Bu(k-d) - B\hat{u}(k-d). \tag{42}$$

It follows from (29) that $u(k-d) = \theta(k-d)\phi(k-d-1)$. Defining $g(k) \triangleq \theta(k)\phi(k-d-1) - \hat{u}(k-d)$, (42) becomes

$$\begin{aligned}x(k-d+1)&-\hat{x}(k-d+1) \\ &= B[\theta(k-d) - \theta(k)]\phi(k-d-1) + Bg(k).\end{aligned} \tag{43}$$

Since $g(k) \to 0$ as $k \to \infty$, it follows from (33) that $\theta(k) - \theta(k-1) \to 0$ as $k \to \infty$. It thus follows from (43) that $x(k-d+1) - \hat{x}(k-d+1) \to 0$ as $k \to \infty$. $\square$

In view of Fact 6.2, we assume henceforth that k is sufficiently large that the difference between $\hat{x}(k)$, $\hat{u}(k)$, $\hat{y}(k)$, and $\hat{z}(k)$ and $x(k)$, $u(k)$, $y(k)$, and $z(k)$, respectively, is negligible. For convenience we set $d = r$. The following analysis focuses on the subsequent behavior of $\hat{x}(k)$, $\hat{u}(k)$, and $\hat{z}(k)$, when $\eta(k) = 0$ and $R(k) = I$.

B. Boundedness of the Internal State

Next, we introduce the *ideal system performance*

$$\begin{aligned}z^*(k) &= E_1A^r x^*(k-r) + \sum_{i=1}^{r} E_1A^{i-1}D_1w(k-i) \\ &\quad + E_0w(k) + \mathcal{H}'U'(k-1) + \mathcal{H}U^*(k-1),\end{aligned} \tag{44}$$

where $x^*(k)$ is the state of the ideal system and $U^*(k-1)$ is defined analogously to $U(k-1)$, with $u(k)$ replaced by $u^*(k)$, where

$$u^*(k) = \theta^*\phi^*(k-1), \tag{45}$$

$$\phi^*(k-1) \triangleq \big[u^{*\rm T}(k-1) \ \cdots \ u^{*\rm T}(k-n_{\rm c}) \tag{46}$$

$$y^{*\rm T}(k-1) \ \cdots \ y^{*\rm T}(k-n_{\rm c})\big]^{\rm T}, \tag{47}$$

and the ideal controller θ^* is assumed to yield the ideal performance

$$z^*(k) \equiv 0. \tag{48}$$

Adding and subtracting $E_1A^r\hat{x}(k-r)$ to and from (44) yields

$$z^*(k) = \mathcal{S}(k) + E_1A^r e(k-r) + \mathcal{H}U^*(k-1), \tag{49}$$

where $\mathcal{S}(k)$ is defined by (10) with $x(k)$ replaced by $\hat{x}(k)$, and $e(k) \triangleq x^*(k) - \hat{x}(k)$.

The *extended ideal system* is given by

$$X^*(k+1) = \tilde{A}X^*(k) + \tilde{B}\tilde{U}^*(k) + \tilde{B}\tilde{U}'(k) + \tilde{D}_1W(k), \tag{50}$$

$$Z^*(k) = \tilde{\mathcal{S}}(k) + \tilde{E}_1\tilde{A}^rE(k-1) + \tilde{\mathcal{H}}\tilde{U}^*(k-1) = 0, \tag{51}$$

where $X^*(k+1)$ and $Z^*(k)$ are defined in the same way as $X(k+1)$ and $Z(k)$, $E(k) \triangleq X^*(k) - \hat{X}(k)$, and

$$\tilde{U}^*(k) \triangleq [I_{l_{\tilde{U}}} \otimes \theta^*]\tilde{\phi}^*(k-1), \tag{52}$$

$$\tilde{\phi}^*(k) \triangleq \big[\ \phi^{*\rm T}(k-q_1) \quad \cdots \quad \phi^{*\rm T}(k-q_{l_{\tilde{U}}})\ \big]^{\rm T}. \tag{53}$$

The goal is to drive the refined controls $\hat{\tilde{U}}(k-1)$ to $\tilde{U}^*(k-1)$ to ensure that $\theta(k) - \theta^* \to 0$ as $k \to \infty$.

Next, subtracting (19) from (51) and solving for $\hat{\tilde{U}}(k-1)$ yields

$$\hat{\tilde{U}}(k-1) = \tilde{\mathcal{H}}^\dagger[\tilde{E}_1\tilde{A}^rE(k-1) + \tilde{\mathcal{H}}\tilde{U}^*(k-1) + \hat{Z}(k)], \tag{54}$$

where $\tilde{\mathcal{H}}^\dagger\tilde{\mathcal{H}} = I_{l_{\tilde{U}}}$ and $\tilde{\mathcal{H}}$ is assumed to have full column rank.

Under the assumptions of Fact 6.1, $\hat{Z}(k) = 0$ and therefore (54) reduces to

$$\hat{\tilde{U}}(k-1) = \tilde{\mathcal{H}}^\dagger\tilde{E}_1\tilde{A}^rE(k-1) + \tilde{U}^*(k-1). \tag{55}$$

Subtracting (37) from (50), and using (55) yields the error dynamics

$$E(k) = (\tilde{A} - \tilde{B}\tilde{\mathcal{H}}^\dagger\tilde{E}_1\tilde{A}^r)E(k-1). \tag{56}$$

Therefore, if $\tilde{A} - \tilde{B}\tilde{\mathcal{H}}^{\dagger}\tilde{E}_1\tilde{A}^r$ is asymptotically stable, then $x(k) - x^*(k) \to 0$ as $k \to \infty$. Furthermore, $z(k) - z^*(k) = E_1x(k) - E_1x^*(k) \to 0$ as $k \to \infty$. Since $z^*(k) = E_1x^*(k) = 0$, it follows that $z(k) \to 0$ as $k \to \infty$.

VII. Regularized Retrospective Cost

We now let $\eta(k) > 0$. In this case, choosing $\hat{\tilde{U}}(k-1)$ as in (27) yields

$$\hat{Z}(k) = Z(k) - \tilde{\mathcal{H}}\tilde{U}(k-1) + \tilde{\mathcal{H}}(\tilde{\mathcal{H}}^{\mathrm{T}}R(k)\tilde{\mathcal{H}} + \eta(k)I_{l_{\tilde{U}}})^{-1} \cdot \tilde{\mathcal{H}}^{\mathrm{T}}R(k)[-Z(k) + \tilde{\mathcal{H}}\tilde{U}(k-1)]). \tag{57}$$

The following result is an extension of Fact 6.1, where we no longer assume that $\eta(k) = 0$.

Fact 7.1: Assume that $\tilde{\mathcal{H}}$ has full column rank, $Z(k)$ is in the range of $\tilde{\mathcal{H}}$ for all k, $u(k) - \hat{u}(k) \to 0$ as $k \to \infty$, and let $\hat{\tilde{U}}(k-1)$ be given by (27). Then $Z(k) - \hat{Z}(k) \to 0$ as $k \to \infty$.

Proof. Since $u(k) - \hat{u}(k) \to 0$ as $k \to \infty$, it follows that $\hat{\tilde{U}}(k) - \tilde{U}(k) \to 0$ as $k \to \infty$. Next, the retrospective cost function is

$$\hat{Z}(k) = Z(k) - \tilde{\mathcal{H}}(\hat{\tilde{U}}(k) - \tilde{U}(k)), \tag{58}$$

therefore, $\hat{Z}(k) - Z(k) \to 0$ as $k \to \infty$. □

In view of Fact 7.1, we assume henceforth that k is sufficiently large that the difference between $\hat{x}(k)$, $\hat{u}(k)$, $\hat{y}(k)$, and $\hat{z}(k)$ and $x(k)$, $u(k)$, $y(k)$, and $z(k)$, respectively, is negligible. For convenience we set $d = r$. The following analysis focuses on the subsequent behavior of $\hat{x}(k)$ and $\hat{z}(k)$, when $\eta(k) > 0$.

Substituting (27) into (37) yields

$$\begin{aligned}\hat{X}(k) = {}& \tilde{A}\hat{X}(k) + \tilde{B}(\tilde{\mathcal{H}}^{\mathrm{T}}R(k)\tilde{\mathcal{H}} + \eta(k-1)I_{l_{\tilde{U}}})^{-1} \\ & \cdot \tilde{\mathcal{H}}^{\mathrm{T}}R(k)[-\hat{Z}(k) + \tilde{\mathcal{H}}\hat{\tilde{U}}(k-1)] + \tilde{B}'\hat{\tilde{U}}'(k-1) \\ & + \tilde{D}_1W(k-1), \end{aligned}\tag{59}$$

$$\hat{Z}(k) = \tilde{E}_1\hat{X}(k) + \tilde{E}_0W(k). \tag{60}$$

Next, we write the performance as

$$\begin{aligned}\hat{Z}(k) = {}& \tilde{E}_1\tilde{A}^r\hat{X}(k-1) + \tilde{\mathcal{H}}\hat{\tilde{U}}(k-1) + \tilde{\mathcal{H}}'\hat{\tilde{U}}'(k-1) \\ & + \tilde{D}\tilde{A}^rW(k-1). \end{aligned}\tag{61}$$

Substituting (61) into (59) yields

$$\begin{aligned}\hat{X}(k) = {}& [\tilde{A} - \tilde{B}(\tilde{\mathcal{H}}^{\mathrm{T}}R(k)\tilde{\mathcal{H}} + \eta(k-1)I_{l_{\tilde{U}}})^{-1} \\ & \cdot \tilde{\mathcal{H}}^{\mathrm{T}}R(k)\tilde{E}_1\tilde{A}^r]\hat{X}(k-1), + [\tilde{D}_1 - \tilde{B}(\tilde{\mathcal{H}}^{\mathrm{T}}R(k)\tilde{\mathcal{H}} \\ & + \eta(k-1)I_{l_{\tilde{U}}})^{-1}\tilde{\mathcal{H}}^{\mathrm{T}}R(k)\tilde{D}\tilde{A}^r]W(k-1) \\ & + [\tilde{B} - \tilde{B}(\tilde{\mathcal{H}}^{\mathrm{T}}R(k)\tilde{\mathcal{H}} \\ & \cdot + \eta(k-1)I_{l_{\tilde{U}}})^{-1}\tilde{\mathcal{H}}^{\mathrm{T}}R(k)\tilde{\mathcal{H}}']\hat{\tilde{U}}'(k-1). \end{aligned}\tag{62}$$

Therefore, it follows from (62) that if $\tilde{A} - \tilde{B}(\tilde{\mathcal{H}}^{\mathrm{T}}R(k)\tilde{\mathcal{H}} + \eta(k-1)I_{l_{\tilde{U}}})^{-1}\tilde{\mathcal{H}}^{\mathrm{T}}R(k)\tilde{E}_1\tilde{A}^r$ is asymptotically stable, then $\hat{X}(k)$ and $Z(k)$ are bounded. Furthermore, note that $\tilde{A} - \tilde{B}(\tilde{\mathcal{H}}^{\mathrm{T}}R(k)\tilde{\mathcal{H}} + \eta(k-1)I_{l_{\tilde{U}}})^{-1}\tilde{\mathcal{H}}^{\mathrm{T}}R(k)\tilde{E}_1\tilde{A}^r \to \tilde{A}$ as $\eta(k) \to \infty$.

VIII. Frequency-Domain Convergence Analysis

Let $G_{\mathrm{FIR}}(\mathbf{q})$ be an FIR transfer function whose numerator coefficients are the Markov parameters of G_{zu} that comprise $\tilde{\mathcal{H}}$. Furthermore, let the external signal $w(k)$ be a sinusoid whose frequency is Θ.

Next, assume that A is asymptotically stable, and assume that the system is turned on at $k = 1$ and allowed to reach harmonic steady state, which occurs at $k_0 > k$. Then for $0 \le k_i < k_0$, $\beta(k_i) = 0$, and $\beta(k_0) = 1$. Furthermore, $\beta(k_0+1) = 0$, where $\beta(k) = 1$, once the system has again reached harmonic steady state.

Assume that $\tilde{\mathcal{H}}$ has full column rank, $\eta(k) \to 0$ as $z(k) \to 0$, $R(k) = I$, $Z(k)$ is in the range of $\tilde{\mathcal{H}}$, and let $\hat{\tilde{U}}(k-1)$ be given by (27). Furthermore, assume that $u(k) - \hat{u}(k) \to 0$ as $k \to \infty$ and

$$\left|1 - \frac{G_{zu}(e^{\jmath\Theta})}{G_{\mathrm{FIR}}(e^{\jmath\Theta})}\right| < 1. \tag{63}$$

Then $z(k) \to 0$ as $k \to \infty$. To show this consider the performance in harmonic steady state we have

$$z_\nu = G_{zw}(e^{\jmath\Theta})w + G_{zu}(e^{\jmath\Theta})\hat{u}_\nu + G_{zu}(e^{\jmath\Theta})g_\nu, \tag{64}$$

where z_ν,w, g_ν are phasors, and $\nu = \beta(0) + \cdots + \beta(k)$, that is, the number of times the controller $\theta(k)$ has been allowed to adapt, and $g_\nu \stackrel{\triangle}{=} u_\nu - \hat{u}_\nu$.

Next, the retrospective cost in harmonic steady state is

$$\hat{z}_\nu \stackrel{\triangle}{=} z_{\nu-1} - G_{\mathrm{FIR}}(e^{\jmath\Theta})u_{\nu-1} + G_{\mathrm{FIR}}(e^{\jmath\Theta})\hat{u}_\nu, \tag{65}$$

$$\begin{aligned}\hat{z}_\nu = {}& G_{zw}(e^{\jmath\Theta})w + [G_{zu}(e^{\jmath\Theta}) - G_{\mathrm{FIR}}(e^{\jmath\Theta})]u_{\nu-1} \\ & + G_{\mathrm{FIR}}(e^{\jmath\Theta})\hat{u}_\nu. \end{aligned}\tag{66}$$

Solving (66) for $\hat{u}_\nu$ yields

$$\begin{aligned}\hat{u}_\nu = {}& G_{\mathrm{FIR}}^{-1}(e^{\jmath\Theta})\left[\hat{z}_\nu - G_{zw}(e^{\jmath\Theta})w - [G_{zu}(e^{\jmath\Theta}) \right. \\ & \left. -G_{\mathrm{FIR}}(e^{\jmath\Theta})]u_\nu\right]. \end{aligned}\tag{67}$$

Substituting (67) into (64) yields

$$\begin{aligned}z_\nu = {}& [1 - G_{zu}(e^{\jmath\Theta})G_{\mathrm{FIR}}^{-1}(e^{\jmath\Theta})][G_{zw}(e^{\jmath\Theta})w \\ & - G_{zu}(e^{\jmath\Theta})u_{\nu-1}] + G_{zu}(e^{\jmath\Theta})G_{\mathrm{FIR}}^{-1}\hat{z}_\nu + G_{zu}(e^{\jmath\Theta})g_\nu. \end{aligned}$$

Using this process we write z_ν in terms of u_0 as

$$\begin{aligned}z_\nu = {}& [1 - G_{zu}(e^{\jmath\Theta})G_{\mathrm{FIR}}^{-1}(e^{\jmath\Theta})]^\nu[G_{zw}(e^{\jmath\Theta})w \\ & - G_{zu}(e^{\jmath\Theta})u_0] + [G_{zu}(e^{\jmath\Theta})G_{\mathrm{FIR}}^{-1}]^\nu\hat{z}_1 + G_{zu}(e^{\jmath\Theta})g_\nu. \end{aligned}\tag{68}$$

It follows from (68) that

$$\begin{aligned}|z_\nu| \le {}& \left|[1 - G_{zu}(e^{\jmath\Theta})G_{\mathrm{FIR}}^{-1}(e^{\jmath\Theta})]^\nu\right| \\ & \cdot \left|G_{zw}(e^{\jmath\Theta})w - G_{zu}(e^{\jmath\Theta})u_0\right| \\ & + \left|G_{zu}(e^{\jmath\Theta})G_{\mathrm{FIR}}^{-1}\right|^\nu|\hat{z}_1| + \left|G_{zu}(e^{\jmath\Theta})g_\nu\right|. \end{aligned}\tag{69}$$

Therefore, since $\left|1 - \frac{G_{zu}(e^{\jmath\Theta})}{G_{\mathrm{FIR}}(e^{\jmath\Theta})}\right| < 1$, it follows that $\left|1 - \frac{G_{zu}(e^{\jmath\Theta})}{G_{\mathrm{FIR}}(e^{\jmath\Theta})}\right|^\nu \to 0$ as $\nu \to \infty$, then $|z_\nu| \to 0$ as $h \to \infty$.

Condition (63) has a simple geometric interpretation, namely, $G_{\mathrm{FIR}}(e^{\jmath\Theta})$ must lie in a half plane that contains $G_{zu}(e^{\jmath\Theta})$ and whose boundary is perpendicular to $|G_{zu}(e^{\jmath\Theta})|$ and passes through $\frac{1}{2}|G_{zu}(e^{\jmath\Theta})|$. Figure 2 illustrates the region of admissible $G_{\rm FIR}(e^{\jmath\Theta})$ for a given $|G_{zu}(e^{\jmath\Theta})|$ and frequency Θ.

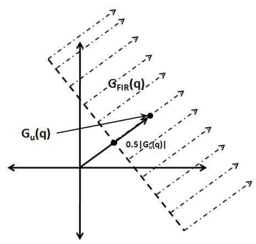

Fig. 2. The dashed region on the complex plane illustrates the region of admissible $G_{\rm FIR}(e^{\jmath\Theta})$ for a given $|G_{zu}(e^{\jmath\Theta})|$ and frequency Θ as determined by (63). The admissible region is a half plane.

The above analysis is based on the assumption that the state of the system reaches harmonic steady state after each period of adaptation. This assumption is an approximation invoked to facilitate the analysis. In fact, RCAC adapts at each step, and thus the state does not reach harmonic state. The examples in the next section show that this condition is sufficient but not necessary, and thus provides a conservative estimate of the allowable uncertainty that can be tolerated in the FIR approximation error.

IX. Numerical Examples

For the following numerical examples we use the recursive least squares update (33) and (34). Furthermore, we consider only the disturbance rejection problem, where $D_1 \neq 0$, $D_2 = 0$, and $E_0 = 0$. We also choose $\eta(k) = \bar\eta(k) Z^{\rm T}(k-1) Z(k-1)$, where $\bar\eta(k)$ is a nonnegative number for all $k \geq 1$.

Example 9.1: (SISO NMP) Consider the asymptotically stable, nonminimum-phase system

$$A = \begin{bmatrix} 1.7 & -1.2 & 0.7 \\ 1 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \tag{70}$$

$$D_1 = \begin{bmatrix} 0.9794 \\ -0.2656 \\ -0.5484 \end{bmatrix}, C = E_1 = \begin{bmatrix} 0.5 \\ -1.25 \\ 1 \end{bmatrix}^{\rm T}. \tag{71}$$

The goal is to reject the disturbance $w(k) = \sin(\frac{\pi}{5}k)$. We choose $\tilde{\mathcal{H}} = H_1 = 1$, $n_c = 5$, $\bar\eta(k) = 2$, and $\gamma = 1$. Figure 3 shows the adaptive filter in closed loop with the nonminimum-phase system. Note that the controller does not have any knowledge of the nonminimum-phase zero.

Example 9.2: (SISO NMP) We consider the same plant and disturbance as in Example 9.1. Furthermore we choose the controller parameters as in Example 9.1. However, we now assume that the $2^{\rm nd}$ and $6^{\rm th}$ Markov parameters are known, and thus $\tilde{\mathcal{H}} = [-0.1076 \;\; -0.8]^{\rm T}$. Figure 4 shows the resulting closed-loop performance.

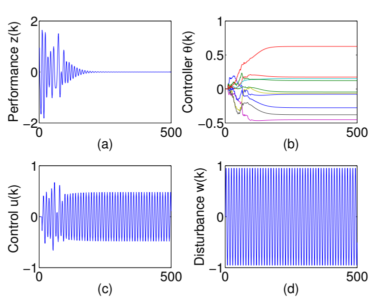

Fig. 3. For this example, the plant is SISO and nonminimum phase. We choose $\tilde{\mathcal{H}} = H_1 = 1$, and $\bar\eta(k) = 2$. (a) shows the performance $z(k)$, (b) shows the controller parameters $\theta(k)$, (c) shows the control signal $u(k)$, and (d) shows the disturbance $w(k)$.

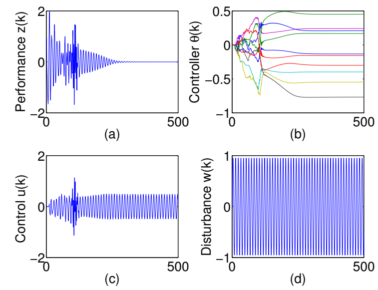

Fig. 4. For this example, the plant is SISO and nonminimum phase. We choose $\tilde{\mathcal{H}} = [-0.1076 \;\; -0.8]^{\rm T}$ and $\bar\eta(k) = 2$. (a) shows the performance $z(k)$, (b) shows the controller parameters $\theta(k)$, (c) shows the control signal $u(k)$, and (d) shows the disturbance $w(k)$.

X. Conclusions

In this paper we extended the RCAC adaptive control algorithm and investigated its ability to adaptively control systems without knowledge of the nonminimum-phase zeros, if any. A frequency-domain conditions that ensures stability of the error system was derived. Furthermore, the algorithm was demonstrated on several SISO examples. In all cases, the number of Markov parameters that are used is not sufficient to determine the nonminimum-phase zeros of the system. Numerical examples showed that the frequency-domain convergence analysis, which is based on a harmonic steady-state assumption, is conservative. Future analysis will refine this analysis to better reflect the robustness of RCAC observed in the numerical examples.

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