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Approximate Manipulability of Leader-Follower Networks

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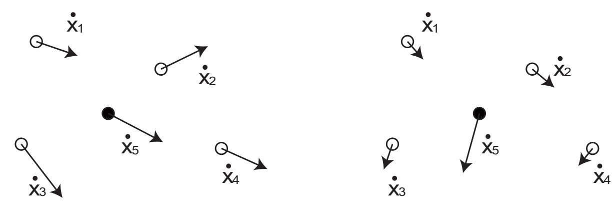
***Abstract*— We introduce the notion of *manipulability* to leader-follower networks as a tool to analyze how effective inputs injected at a leader node are in terms of their impact on the movements of the follower nodes, as a function of the interaction topologies and agent configurations. Classic manipulability is an index used in robotics for analyzing the singularity and efficiency of configurations of robot-arm manipulators. To define similar notions for leader-follower networks, we use a rigid-link approximation of the follower dynamics and under this assumption, we prove that the instantaneous follower velocities can be uniquely determined by that of the leader's, which allows us to define a meaningful manipulability index of the leader-follower networks.**



(a) More effective (b) Less effective

Fig. 1. Effectiveness in terms of the ratio of generated velocity norms of the followers' to the leader's ($N_\ell = 1$ case). The filled circle, x_5, is the leader.

I. Introduction

Consider a system consisting of multiple mobile units, connected together through an information-exchange network, where the agents use the information-exchange network for their coordinations. If the movement of a select agent is viewed as the inputs to the system, one can ask a number of questions pertaining to the inputs effect on the rest of the system, including: (1) What is the set of states reachable under this control input?, (2) How "effective" is the control input in terms of the network's response?, and (3) How we can design or adaptively improve the network topology to render it amenable to "effective" control inputs?

Networked systems where control signals are injected at particular input nodes are referred to as leader-follower networks, and a large body of work has emerged concerning how to control such networks. Examples include optimal control [1], containment control [2], [3], and formation control [4]. And, question (1) above is intimately linked to the controllability properties of such leader-follower networks, which has been investigated, for example, in [5], [6]. In this paper we ignore this question and focus instead on the second question, i.e., the question of how "effective" the control input is. This is not a controllability question but rather it connects instantaneous inputs to instantaneous responses.

In fact, to address the notion of input "effectiveness", we borrow the notion of *manipulability* indices, and transfer it to leader-follower networks as a tool to analyze the instantaneous effectiveness of the leader input to the network under given configurations and network topologies. In robotics, the manipulability indices have been proposed as means for analyzing the singularity and efficiency of particular configurations and controls of robot-arm manipulators [7], [8], [9]. And, while the original manipulability indices are based on taking the Jacobian of the kinematic relation between the input angular velocities of the joints and the generated velocities of the end-effectors, leader-follower network "links" are not rigid in the same way. As such, we are required to approximate the interaction dynamics in order to be able to define manipulability in terms of the relation between the leader's and followers' instantaneous velocities.

The contributions in this paper are twofold. First, we show how the dynamics of leader-follower networks can be approximated as rigid-link networks if the followers move fast enough to maintain given desired distances. Then, we introduce the definition of manipulability of leader-follower networks as the index of how the effort of leader agents effectively affects to the follower velocities (Fig 1).

II. Leader-Follower Networks

We consider a network that consists of N agents divided into two groups: leaders and followers. Let N_ℓ and N_f be the number of leader and follower agents, respectively. Let $x_i(t) \in \mathbb{R}^d$ $(i = 1, ..., N_f, N_f+1, ..., N)$ be the state of agent i at time t, where we, without loss of generality, have assigned the last indices to the leaders. Then, the overall state, which we also refer to as the configuration, of the network is given by $x(t) = [x_1^T(t), ..., x_N^T(t)]^T = [x_f^T(t), x_\ell^T(t)]^T$, where $x_f(t) = [x_1^T(t), ..., x_{N_f}^T(t)]^T \in \mathbb{R}^{N_f d}$ and $x_\ell(t) = [x_{N_f+1}^T(t), ..., x_N^T(t)]^T \in \mathbb{R}^{N_\ell d}$ are follower and leader states, respectively.

We consider the situation where the interaction dynamics are defined through pairwise interactions. We say that when follower agents i and j are *connected*, then they share relative state information, and their pairwise control task is to maintain their distance $||x_i - x_j||$ to a prespecified, positive value d_{ij}. If one of the agents in a connected pair is a leader agent and the other is a follower, then only the follower dynamics is designed to maintain the distance.

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Using a graph representation, the agents are described by nodes $\mathsf{V} = \{v_1, ..., v_N\}$ and the connections between agents become edges $\mathsf{E} \subseteq \mathsf{V} \times \mathsf{V}$. Then, the overall network is described by graph $\mathcal{G} = (\mathsf{V}, \mathsf{E})$. In this paper, we assume networks whose underlying graphs are undirected (the interconnections are symmetric), static, and connected.

A. Edge-Tension Energy

To formulate the follower dynamics, we use a general, energy-based definition (e.g., [1]), rather than tying the results to any specific set of interaction dynamics. In other words, we introduce the following edge-tension energy

$$\mathcal{E}(x_f(t), x_\ell(t)) = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\mathcal{E}_{ij}(x_i(t), x_j(t)), \quad (1)$$

where

$$\mathcal{E}_{ij}(x_i, x_j) = \begin{cases} \frac{1}{2}\{e_{ij}(||x_i - x_j||)\}^2 & (v_i, v_j) \in \mathsf{E} \\ 0 & (v_i, v_j) \notin \mathsf{E}, \end{cases} \quad (2)$$

where $e_{ij} : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ is a strictly increasing twice differentiable function such that $e_{ij}(d_{ij}) = 0 \quad (d_{ij} > 0)$, i.e., the edge-tension energy is zero when the desired distance between agent i and j is realized.

An example for e_{ij} is (see [1] and the references therein)

$$e_{ij}(||x_i - x_j||) = ||x_i - x_j|| - d_{ij}. \quad (3)$$

B. Agent Dynamics

Given the velocity of the leaders $\dot{x}_\ell(t)$, we assume that each of the followers tries to maintain (locally) the desired distances between connected agent pairs by minimizing the related parts of the edge-tension energy (1) through a gradient descent direction:

$$\dot{x}_i(t) = -\sum_{j\in\mathcal{N}(i)} \frac{\partial \mathcal{E}_{ij}(x_i(t), x_j(t))}{\partial x_i}^T \; (i = 1, ..., N_f) \quad (4)$$

where $\mathcal{N}(i) = \{j \in \{1, ..., N\} \,|\, (v_i, v_j) \in \mathsf{E}\}$ is the neighbor set of agent i. Using the facts that $\mathcal{E}_{ij} = \mathcal{E}_{ji}$ and $\frac{\partial \mathcal{E}}{\partial x_i} = \frac{1}{2}\sum_{j\in\mathcal{N}(i)}\left(\frac{\partial \mathcal{E}_{ij}}{\partial x_i} + \frac{\partial \mathcal{E}_{ji}}{\partial x_i}\right)$, the dynamics of overall followers in the network can be described by

$$\dot{x}_f(t) = -\frac{\partial \mathcal{E}(x_f, x_\ell)}{\partial x_f}^T. \quad (5)$$

Therefore, using this dynamics, the followers try to decrease (locally) the overall energy (1) since $\dot{\mathcal{E}} = \frac{\partial \mathcal{E}}{\partial x_f}\dot{x}_f + \frac{\partial \mathcal{E}}{\partial x_\ell}\dot{x}_\ell = -||\frac{\partial \mathcal{E}}{\partial x_f}||^2 + \frac{\partial \mathcal{E}}{\partial x_\ell}\dot{x}_\ell$.

III. Manipulability of Leader-Follower Networks

To introduce the effectiveness of the input to the network, we define the manipulability of a leader-follower network based on the ratio between the norm of follower velocities and the norm of leader velocities similar to the definition used by Bicchi, et al. in robot arms [8], [9]. In other words, this ratio is given by

$$R(x, \mathsf{E}, \dot{x}_\ell) = \frac{\dot{x}_f^T Q_f \dot{x}_f}{\dot{x}_\ell^T Q_\ell \dot{x}_\ell}, \quad (6)$$

where $Q_f = Q_f^T \succ 0$ and $Q_\ell = Q_\ell^T \succ 0$ are positive definite weight matrices.

Once we successfully define this kind of indices under a given configuration and topology, it becomes possible to estimate the most effective inputs to the network by maximizing (6) with $\dot{x}_\ell$:

$$\dot{x}_{\ell,\max}(x, \mathsf{E}) = \arg\max_{\dot{x}_\ell} R(x, \mathsf{E}, \dot{x}_\ell), \quad (7)$$

$$R_{\max}(x, \mathsf{E}) = \max_{\dot{x}_\ell} R(x, \mathsf{E}, \dot{x}_\ell). \quad (8)$$

Another possible application, albeit beyond the scope of this paper, is to find an effective, adaptive topology process when the configuration and input velocities are given.

While the manipulability is an intuitively clear notion, it needs to be connected to the agent dynamics in the previous section in a meaningful way, which presents some difficulty. The reason is that since $\dot{x}_f = -\frac{\partial \mathcal{E}}{\partial x_f}^T$ is a function of x_f and x_ℓ but not $\dot{x}_\ell$, we need to introduce an integral action to see the influence of $\dot{x}_\ell$. But, the input velocity $\dot{x}_\ell$ can change on the time interval of the integration. Thus, it is impossible to calculate an instantaneous measure given by (6). Two choices present themselves. The first is to change the agent dynamics. But, we do not want to follow that route since edge-tension functions (and weighted consensus equations) are used quite frequently. As such, to define a notion that is practically relevant, we choose to go with the second option instead, namely to introduce an approximate notion of manipulability instead, i.e., to assume that the followers move fast enough to always maintain the desired distances.

IV. Rigid-Link Approximation

A. Approximation

Definition 4.1: The *rigid-link approximation* of the dynamics in a given leader-follower network is the ideal situation when all the given desired distances $\{d_{ij}\}_{(v_i,v_j)\in\mathsf{E}}$ are perfectly maintained by the followers (i.e., $||x_i - x_j|| = d_{ij} \; \forall(v_i, v_j) \in \mathsf{E}$).

Note that this approximation is valid if the scale of edge-tension energy $\mathcal{E}(t)$ is large enough compared to that of the leader velocities $\dot{x}_\ell(t)$. Note also that, in real situations, $\mathcal{E}(t)$ needs to be greater than zero in order for the followers to move, while this approximation implies $\mathcal{E}(t) = 0 \; \forall t$. Therefore, the situation of Definition 4.1 is never realized perfectly in actual dynamics as long as leaders are moving. Nevertheless, this approximation gives us a good estimation of actual network responses to injected leader inputs unless leaders move much faster than followers. We will show in simulation that the approximation is reasonable.

Now, to analyze the approximated dynamics, we first introduce the method of using the rigidity matrix [10], [11]. If the connections in agent pairs associated with the edges can be viewed as rigid links, the distances between connected agents do not change in time. Assume that the trajectories of $x_i(t)$ are smooth and differentiable, then

$$\frac{d}{dt}||x_i - x_j||^2 = 0 \quad \forall (v_i, v_j) \in \mathsf{E},$$

and therefore

$$(x_i - x_j)^T(\dot{x}_i - \dot{x}_j) = 0 \quad \forall (v_i, v_j) \in \mathsf{E}. \tag{9}$$

Here, (9) can be written in the following matrix form

$$C(x)\begin{bmatrix}\dot{x}_f \\ \dot{x}_\ell\end{bmatrix} = [C_f(x)|C_\ell(x)]\begin{bmatrix}\dot{x}_f \\ \dot{x}_\ell\end{bmatrix} = 0, \tag{10}$$

where $C(x) \in \mathbb{R}^{M\times Nd}$, $C_f(x) \in \mathbb{R}^{M\times N_f d}$, $C_\ell(x) \in \mathbb{R}^{M\times N_\ell d}$, and M is the number of edges (i.e., $M = |\mathsf{E}|$). The matrix C is known as the rigidity matrix, which is a function of the current configuration x and also of the network topology E in the underlying graph $\mathcal{G}$. Specifically, considering that C consists of $M \times N$ blocks of $1 \times d$ row vectors, its (k, i_k) and (k, j_k) blocks are either $(x_{i_k} - x_{j_k})^T$ and $-(x_{i_k} - x_{j_k})^T$, respectively, or $-(x_{i_k} - x_{j_k})^T$ and $(x_{i_k} - x_{j_k})^T$, respectively, where i_k and j_k are the agents connected by edge k.

Assume that the leaders move in a feasible manner so that the approximation in Definition 4.1 stays valid. (We will discuss this point in Section V.) From the constraint equation (10) and the property of C_f that will be shown in (17) or (26), the possible set of $\dot{x}_f$ associated with $\dot{x}_\ell$ can be obtained as the following general solution:

$$\dot{x}_f = -C_f^\dagger C_\ell \dot{x}_\ell + [\text{null}(C_f)]p, \tag{11}$$

where $C_f^\dagger$ is the Moore-Penrose pseudo inverse of C_f, $[\text{null}(C_f)]$ is a matrix whose columns span $\text{null}(C_f)$, and $p \in \mathbb{R}^{\text{nullity}(C_f)}$ is arbitrary. This means that there may exist infinite possibilities of $\dot{x}_f$ (i.e., rotational freedom and/or formation flexibility) once an input $\dot{x}_\ell$ is given.

In this indeterminate case, the definition of the manipulability (6) cannot be determined uniquely, and it seems that we need to modify the definition of manipulability, for example, by using the "worst-case approach" [9] that assumes the least object velocity (follower velocity, in our case). However, when we approximate the follower dynamics (5) based on Definition 4.1, it can be proven that $\dot{x}_f$ is uniquely determined by given $\dot{x}_\ell$. This is the key for introducing the notion of manipulability (6) in leader-follower networks. In the following paragraphs, we prepare some facts and then show how $\dot{x}_f$ is determined uniquely.

Lemma 4.1: Let $A \in \mathbb{R}^{n\times n}$ be a negative semidefinite matrix, which can be decomposed into $A = -V\Lambda V^T \preceq 0$, where the i-th column vector of $V \in \mathbb{R}^{n\times r}$ is an eigenvector corresponding to eigenvalue $\lambda_i > 0 \quad (i = 1, ..., r)$, $r = \text{rank}(A)$, $\Lambda = \text{diag}([\lambda_1, ..., \lambda_r])$, and $V^T V = I_r$. Then, the following equation is satisfied:

$$\left(\lim_{s\to\infty}\int_0^s e^{A(s-\tau)}d\tau\right)V = V\Lambda^{-1}. \tag{12}$$

Proof: Using the fact that $e^{-V\Lambda V^T t} - I_n = \sum_{k=1}^\infty \frac{t^k}{k!}V(-\Lambda)^k V^T = V(e^{-\Lambda t} - I_r)V^T$ with $V^T V = I_r$,

$$\begin{aligned} LHS &= \lim_{s\to\infty}\int_0^s \{V(e^{-\Lambda(s-\tau)} - I_r) + V\}d\tau \\ &= V\lim_{s\to\infty}\int_0^s e^{-\Lambda(s-\tau)}d\tau \\ &= V\lim_{s\to\infty}\text{diag}\left(\left[\frac{1-e^{-\lambda_1 s}}{\lambda_1}, \cdots, \frac{1-e^{-\lambda_r s}}{\lambda_r}\right]\right) \\ &= RHS. \end{aligned}$$

■

Corollary 4.1: Given a linear system $\dot{x}(s) = Ax(s) + Bu$ with $x(0) = 0$ and constant input $u \in \mathbb{R}^m$, where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are time-invariant matrices that can be decomposed into $A = -G^T G$ and $B = G^T H$, respectively, where $G \in \mathbb{R}^{M\times n}$, $H \in \mathbb{R}^{M\times m}$, and $M \in \mathbb{N} (= \{1, 2, ..\})$, the state converges to $\lim_{s\to\infty} x(s) = G^\dagger Hu$.

Proof: Let $G = U\Sigma V^T$ be the singular value decomposition of G, where $U \in \mathbb{R}^{M\times r}$ and $V \in \mathbb{R}^{n\times r}$ are column-orthogonal matrices (i.e., $V^T V = I_r$ and $U^T U = I_r$), $\Sigma \in \mathbb{R}^{r\times r}$ is a diagonal matrix, and $r = \text{rank}(A) \leq \min\{n, M\}$. Then, the zero-state response of the system converges to

$$\begin{aligned} \lim_{s\to\infty} x(s) &= \lim_{s\to\infty}\int_0^s e^{A(s-\tau)}d\tau Bu \\ &= \left(\lim_{s\to\infty}\int_0^s e^{-V\Sigma^2 V^T(s-\tau)}d\tau V\right)\Sigma U^T Hu \\ &= (V\Sigma^{-2})\Sigma U^T Hu = (V\Sigma^{-1}U^T)Hu. \end{aligned}$$

Note that all the diagonal elements in Σ are non-zero (strictly positive); hence, Σ^{-1} exists and $G^\dagger = V\Sigma^{-1}U^T$. ■

Lemma 4.2: The second-order partial derivatives of the edge-tension energy (1) with respect to x_f and x_ℓ have the following form if all the connected agents satisfy their desired distances (i.e., $||x_i - x_j|| = d_{ij} \; \forall (v_i, v_j) \in \mathsf{E}$):

$$\frac{\partial^2 \mathcal{E}}{\partial x_f^2} = D_f^T D_f, \quad \frac{\partial^2 \mathcal{E}}{\partial x_f \partial x_\ell} = D_f^T D_\ell, \tag{13}$$

where $\frac{\partial^2 \mathcal{E}}{\partial x_f^2} \in \mathbb{R}^{N_f d\times N_f d}$, $\frac{\partial^2 \mathcal{E}}{\partial x_f \partial x_\ell} \in \mathbb{R}^{N_f d\times N_\ell d}$, and

$$D_f = WC_f, \quad D_\ell = WC_\ell. \tag{14}$$

$W \in \mathbb{R}^{M\times M}$ is a diagonal matrix whose elements are

$$[W]_{kk} = \frac{e'_{i_k j_k}(d_{i_k j_k})}{d_{i_k j_k}} \quad (k = 1, ..., M), \tag{15}$$

where $e'_{ij}(z) \triangleq \frac{de_{ij}(z)}{dz}$; i_k and j_k are the two agents connected by edge k.

Recall that we assumed $d_{ij} \in (0, \infty)$ and that $e_{ij}(z)$ is a strictly increasing twice differentiable function for all $(v_i, v_j) \in \mathsf{E}$. Therefore, in (15), $[W]_{kk} \in (0, \infty)$ exists for all $k \in \{1, ..., M\}$.

Proof: Let $(v_i, v_j) \in \mathsf{E}$. The first-order and second-order derivatives of $\mathcal{E}_{ij}(x_i, x_j)$ in (2) with respect to x_i in a general configuration of x_i and x_j (i.e., without assuming

$||x_i - x_j|| = d_{ij}$) become the followings.

$$\frac{\partial \mathcal{E}_{ij}(x_i, x_j)}{\partial x_i} = w_{ij}(||x_i - x_j||)(x_i - x_j)^T,$$

$$\frac{\partial^2 \mathcal{E}_{ij}(x_i, x_j)}{\partial x_i^2} \left(= - \frac{\partial^2 \mathcal{E}_{ij}(x_i, x_j)}{\partial x_i \partial x_j} \right)$$

$$= \frac{w'_{ij}(||x_i - x_j||)}{||x_i - x_j||}(x_i - x_j)(x_i - x_j)^T + w_{ij}(||x_i - x_j||) I_d,$$

where the equality in the bracket follows from the fact that $\frac{\partial \mathcal{E}_{ij}}{\partial x_i}$ is a function of $x_i - x_j$; and, let $e''_{ij}(z) \triangleq \frac{d^2 e_{ij}(z)}{dz^2}$,

$$w_{ij}(z) \triangleq \frac{e_{ij}(z) e'_{ij}(z)}{z},$$

$$w'_{ij}(z) \triangleq \frac{dw_{ij}}{dz} = \frac{\{e'_{ij}(z)^2 + e_{ij}(z) e''_{ij}(z)\} z - e_{ij}(z) e'_{ij}(z)}{z^2}.$$

If $||x_i - x_j|| = d_{ij}$, then

$$\frac{\partial \mathcal{E}_{ij}}{\partial x_i} = 0,$$

$$\frac{\partial^2 \mathcal{E}_{ij}}{\partial x_i^2} = - \frac{\partial^2 \mathcal{E}_{ij}}{\partial x_i \partial x_j} = \left(\frac{e'_{ij}(d_{ij})}{d_{ij}} \right)^2 (x_i - x_j)(x_i - x_j)^T.$$

Hence, we get $\frac{\partial^2 \mathcal{E}}{\partial x_f^2} = C_f^T W^2 C_f$ and $\frac{\partial^2 \mathcal{E}}{\partial x_f \partial x_\ell} = C_f^T W^2 C_\ell$. ■

Example 4.1: If the edge-tension energy is given by (3), $e'_{ij}(z) = 1$ and $[W]_{kk} = (d_{i_k j_k})^{-1} \quad (k = 1, ..., M)$.

In the following, we assume single-leader networks (i.e., $N_\ell = 1$), and assume that the leader can move arbitrarily. In cases of $N_\ell > 1$, we need to restrict the freedom of the leaders. While we now focus on single-leader cases, we will later extend the result being derived here to multi-leader cases in Section V.

Lemma 4.3: If $N_\ell = 1$, then $C_f^\dagger C_\ell = D_f^\dagger D_\ell$.

Proof: Since all diagonal elements in W are non-zero, C_f and $D_f (= W C_f)$ have the same row space. Therefore, their projection matrices onto the row space are identical:

$$C_f^\dagger C_f = D_f^\dagger D_f. \tag{16}$$

Now, since we assume that $N_\ell = 1$, the matrices C_f and D_f have the following properties, respectively:

$$C_f [\underbrace{I_d \cdots I_d}_{N_f \text{matrices}}]^T = -C_\ell, \quad D_f [\underbrace{I_d \cdots I_d}_{N_f \text{matrices}}]^T = -D_\ell. \tag{17}$$

Using (17) with (16), we get $C_f^\dagger C_\ell = -C_f^\dagger C_f [I_d \cdots I_d]^T = -D_f^\dagger D_f [I_d \cdots I_d]^T = D_f^\dagger D_\ell$. ■

Theorem 4.1: If $N_\ell = 1$ (i.e., single-leader cases), the rigid-link approximation of dynamics (5) is given by

$$\dot{x}_f(t) = -C_f^\dagger C_\ell \dot{x}_\ell(t). \tag{18}$$

Note that (18) does not depend on a specific choice of function e_{ij} in (2).

Proof: We here see the details of the approximation described in Definition 4.1. The most part of this proof can also be applied to the cases of $N_\ell > 1$. Consider that the velocity of leaders gives a small displacement, $\delta x_\ell(t)$, of their configuration from time t to $t + \delta t$. Here, $\dot{x}_\ell(t) = \lim_{\delta t \to 0} \frac{\delta x_\ell(t)}{\delta t}$. Since we assume that the desired distances are perfectly maintained by the followers, we introduce another time axis s and track the configuration of followers, $\tilde{x}_f(t, s) \triangleq x_f(t) + \delta \tilde{x}_f(t, s)$, to see its convergence in $s \to \infty$, where the leader configuration $\tilde{x}_\ell(t, s) \triangleq x_\ell(t) + \delta x_\ell(t)$ is constant on the axis of s. We can think of s describing the time evolution when the system is executing the actual, as opposed to the approximate, dynamics. Then, we consider the approximation in Definition 4.1 as $\dot{x}_f(t) = \lim_{\delta t \to 0} \lim_{s \to \infty} \frac{\delta \tilde{x}_f(t,s)}{\delta t}$. We also assume that $\tilde{x}_f(t, 0) = x_f(t)$ and all the desired distances are satisfied at $s = 0$.

Since the dynamics of the followers is given by (5), the system equation of $\delta \tilde{x}_f(t, s)$ becomes

$$\frac{d}{ds} \delta \tilde{x}_f(t, s) = \frac{d}{ds} \tilde{x}_f(t, s) = - \frac{\partial \mathcal{E}(\tilde{x}_f(t, s), \tilde{x}_\ell(t, s))}{\partial x_f}^T$$

$$= - \frac{\partial \mathcal{E}(x_f(t) + \delta \tilde{x}_f(t, s), x_\ell(t) + \delta x_\ell(t))}{\partial x_f}^T$$

$$= - \frac{\partial^2 \mathcal{E}(x_f(t), x_\ell(t))}{\partial x_f^2} \delta \tilde{x}_f(t, s) - \frac{\partial^2 \mathcal{E}(x_f(t), x_\ell(t))}{\partial x_f \partial x_\ell} \delta x_\ell(t),$$

where we assumed that $\delta x_\ell(t)$ and $\delta \tilde{x}_f(t, s)$ are small enough to use the first-order approximation. We also used $\frac{\partial \mathcal{E}(x_f(t), x_\ell(t))}{\partial x_f} = 0$. Note that $\frac{\partial^2 \mathcal{E}(x_f(t), x_\ell(t))}{\partial x_f^2}$ and $\frac{\partial^2 \mathcal{E}(x_f(t), x_\ell(t))}{\partial x_f \partial x_\ell}$ are constant on the time axis of s.

Using Lemma 4.2, we can rewrite the above system equation as

$$\frac{d}{ds} \delta \tilde{x}_f(t, s) = -(D_f^T D_f) \delta \tilde{x}_f(t, s) - (D_f^T D_\ell) \delta x_\ell(t). \tag{19}$$

Recall that the initial condition is $\delta \tilde{x}_f(t, 0) = 0$. Therefore, using corollary 4.1, we know that (19) converges and its convergence point is given by

$$\delta x_f(t) \triangleq \lim_{s \to \infty} \delta \tilde{x}_f(t, s) = -D_f^\dagger D_\ell \delta x_\ell(t). \tag{20}$$

Here, $\delta x_f(t)$ gives the displacement of the followers caused by the displacement $\delta x_\ell(t)$. Thus, dividing (20) by δt and taking $\delta t \to 0$, we obtain

$$\dot{x}_f(t) = -D_f^\dagger D_\ell \dot{x}_\ell(t). \tag{21}$$

Finally, if $N_\ell = 1$, (21) and Lemma 4.3 yield (18). ■

B. Manipulability with Rigid-Link Approximations

As a corollary to Theorem 4.1, the manipulability (6) of a leader-follower network under the rigid-link approximation of the follower dynamics is given by the Rayleigh quotient

$$R(x, \mathsf{E}, \dot{x}_\ell) = \frac{\dot{x}_\ell^T J^T Q_f J \dot{x}_\ell}{\dot{x}_\ell^T Q_\ell \dot{x}_\ell}, \tag{22}$$

where $J(x, \mathsf{E}) \triangleq -C_f^\dagger C_\ell$. Hence, similar to the manipulability indices in robot-arm manipulators, the maximum/minimum values of the manipulability index can be obtained by the eigenvalue analysis. That is, $R_{\max}$ is the maximum eigenvalue, $\lambda_{\max}$, of the generalized eigenvalue problem $J^T Q_f J v = \lambda Q_\ell v$, and $\dot{x}_{\ell,\max}$ is obtained from its corresponding eigenvector, $v_{\max}$, as $\dot{x}_{\ell,\max} = \alpha v_{\max}$ ($\alpha \neq$

0). Similarly, the minimum value and its corresponding inputs can be obtained from the minimum eigenvalue, $\lambda_{\min}$, and its corresponding eigenvector, respectively.

Now, we introduce a tool to depict effective input directions (axes) in case of $Q_\ell \propto I_{N_\ell d}$. Let us first consider a robot-arm manipulability index defined by $\frac{\dot{r}^T\dot{r}}{\dot{\theta}^T\dot{\theta}}$, where θ and r are the states of joint angles and the end-effector, respectively. Given a kinematic relation $r = f(\theta)$, thus $\dot{r} = \frac{\partial f}{\partial \theta}\dot{\theta}$, the manipulability ellipsoid can be defined as $\dot{r}^T(\frac{\partial f}{\partial \theta}\frac{\partial f}{\partial \theta}^T)^\dagger \dot{r} = 1$, which depicts the range of end-effector velocities under inputs $\dot{\theta}$ with norm $||\dot{\theta}|| \leq 1$. In contrast, since what we are interested in is the effective direction (axis) of inputs, we define a similar ellipsoid not in the space of follower velocities but in the space of leader velocities:

$$\dot{x}_\ell^T(J^TQ_fJ)^\dagger \dot{x}_\ell = \text{const.}, \tag{23}$$

which we refer to as the *leader-side manipulability ellipsoid*. Here, the longest axis of the ellipsoid corresponds to the eigenvector that gives the maximum eigenvalue of J^TQ_fJ.

V. MULTIPLE LEADERS

In case that multiple leaders exist (i.e., $N_\ell > 1$), it is obvious that the leaders cannot take arbitrary velocities one another under the rigid-link approximation of Definition 4.1. For instance, when two leaders take opposite directions for a while, then it becomes impossible to maintain some of the desired distances since those desired distances have finite constant lengths. We here show a method to take multiple leaders into account by preserving the assumption of Definition 4.1.

To extend the discussion in the previous section, we exploit the notion of *motion feasibility* of multi-agent networks [12]. Let us consider the following matrices:

$$\begin{bmatrix} K_f \\ K_\ell \end{bmatrix} \triangleq [\text{null}(C)], \tag{24}$$

where $K_f \in \mathbb{R}^{N_f d\times n_c}$, $K_\ell \in \mathbb{R}^{N_\ell d\times n_c}$, $n_c = \text{nullity}(C)$, and $[\text{null}(C)]$ is a matrix whose columns span $\text{null}(C)$. Then, the set of a feasible motion of the agents can be represented by

$$\begin{bmatrix} \dot{x}_f \\ \dot{x}_\ell \end{bmatrix} = \begin{bmatrix} K_f \\ K_\ell \end{bmatrix} q, \tag{25}$$

where $q(t) \in \mathbb{R}^{n_c}$ is an arbitrary (time-varying) vector.

Definition 5.1: Given an agent configuration x and their topology E in the underlying graph $\mathcal{G} = (\mathsf{V}, \mathsf{E})$, a *feasible leader motion* is an instantaneous velocity given by $\dot{x}_\ell = \tilde{K}_\ell\tilde{q}$, where $\tilde{q} \in \mathbb{R}^{\text{rank}(K_\ell)}$ is arbitrary and the columns of $\tilde{K}_\ell$ span the column space of K_ℓ defined in (24). If $\text{rank}(K_\ell) = N_\ell d$, the leaders can take arbitrary instantaneous velocities, which we refer to as an *arbitrary motion*.

Example 5.1: In the configurations of two-leader single-follower networks shown in Fig. 2, $\text{rank}(K_\ell)$ of (a), (b), and (c) are 4, 3, and 3, respectively, where $N_\ell = 2$ and $d = 2$. Therefore, only the leaders in (a) can take arbitrary motion under the given configuration.

Once a feasible leader motion is given, we show that the result in the previous section, i.e. (18), is true even if $N_\ell > 1$.

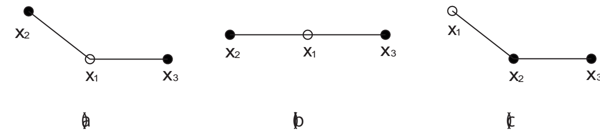


Fig. 2. 2-leader 1-follower networks. (a) and (b) are the same leader-follower networks with different configurations; (a) and (c) are different leader-follower networks (same configuration with different assignments).

To generalize Theorem 4.1 for multiple-leader cases, we first extend (17).

Lemma 5.1: Given a rigidity matrix $C = [C_f|C_\ell]$,

$$C_fK_f = -C_\ell K_\ell, \quad D_fK_f = -D_\ell K_\ell \tag{26}$$

is always satisfied by arbitrary choices of K_f and K_ℓ, where $\text{null}(C)$ is spanned by the column vectors of $[K_f^T|K_\ell^T]^T$.

Proof: This is directly obtained from the fact that $C = [C_f|C_\ell]$ and $WC = [D_f|D_\ell]$ have the same null space; thus, $[C_f|C_\ell][K_f^T|K_\ell^T]^T = [D_f|D_\ell][K_f^T|K_\ell^T]^T = 0$. ■

Theorem 5.1: If the velocities of leaders are given by a feasible leader motion, then (18) is true even if $N_\ell > 1$.

Proof: From (25), a feasible leader motion can also be written as a redundant form $\dot{x}_\ell = K_\ell q$; there exists a set of q, $\{q|K_\ell q = \tilde{K}_\ell\tilde{q}\}$, corresponding to given $\tilde{q}$. Let us pick one q. Using Lemma 5.1 with (16) and (21), we get

$$\begin{aligned}\dot{x}_f(t) &= -D_f^\dagger D_\ell\dot{x}_\ell(t) = -D_f^\dagger D_\ell K_\ell q = D_f^\dagger D_f K_f q \\ &= C_f^\dagger C_f K_f q = -C_f^\dagger C_\ell K_\ell q = -C_f^\dagger C_\ell \dot{x}_\ell(t). \quad ■\end{aligned}$$

Note that (8) needs to be solved [1] with respect to $\tilde{q}$ instead of $\dot{x}_\ell$ to obtain $R_{\max}$ in case of $\text{rank}(K_\ell) < N_\ell d$.

VI. EXAMPLES

In order to verify the approximation of dynamics, we first compare original dynamics (5) with rigid-link approximated dynamics (18) using $d = 2$ (i.e., the state of each agent corresponds to its two dimensional position in the 2-d plane). Then, we show how the defined manipulability can be used to analyze the effectiveness of leader inputs. In the following examples, $d_{ij} = 1$ is used for all the desired distances. For simplicity, we used $Q_f = I_{N_fd}$ and $Q_\ell = I_{N_\ell d}$ for the weight matrices in (6).

A. Rigid-Link Approximation

Fig. 3 shows an example of agent motion generated by (5) and (18), where a single-leader network with $N = 7$ ($N_f = 6$ and $N_\ell = 1$) was used. Uniformly-accelerated motion $\dot{x}_\ell(t) = t[\cos(\pi/4), \sin(\pi/4)]^T$ was used for the leader input. For the follower dynamics (5), the edge-tension energy (3) multiplied by 200 was used, which ensures the connected agents almost satisfy the desired distances.

We see that the follower motion is almost identical between the original and approximated dynamics. When we used the edge-tension energy with smaller scale, the distances between connected agents vary more. This is prominent when the leader takes large velocity (e.g., the last part of the

[1] Eigenvalue problem $(\tilde{K}_\ell^TJ^TQ_fJ\tilde{K}_\ell)\tilde{q} = \lambda(\tilde{K}_\ell^TQ_\ell\tilde{K}_\ell)\tilde{q}$ can be used.

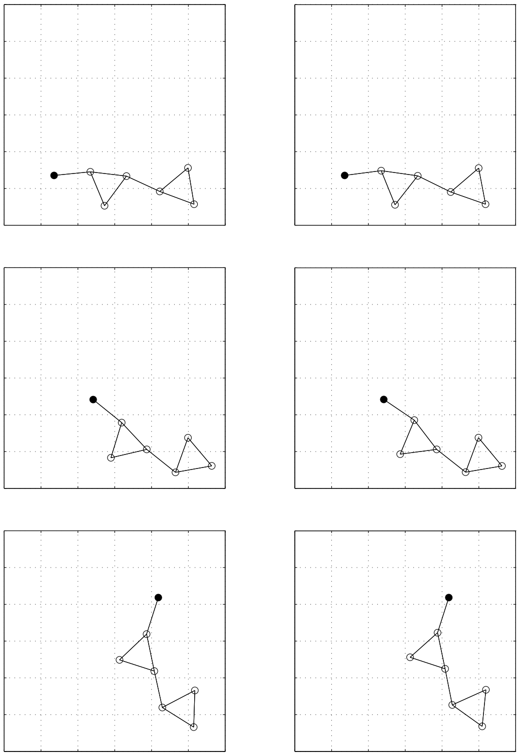
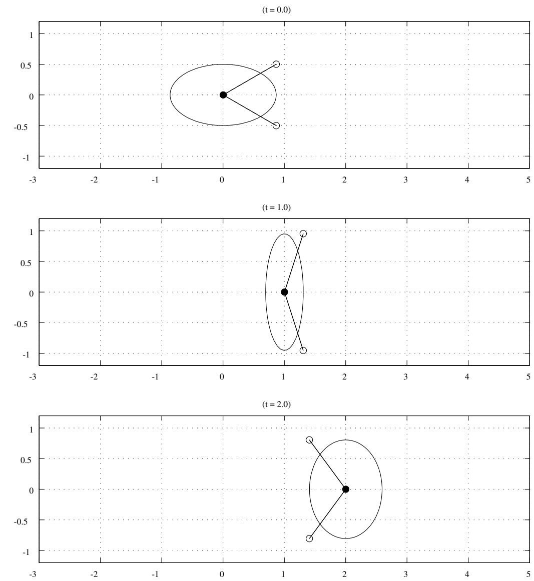
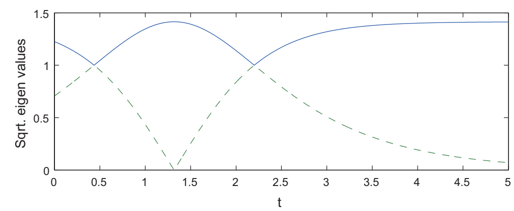


Fig. 3. Comparison of agent motion between original dynamics Eq. (5) (left) and rigid-link approximated dynamics Eq. (18) (right). The filled circle in each figure is the leader agent.



(a) Agent motion and leader-side manipulability ellipsoids



(b) Temporal change of the square root of eigenvalues

Fig. 4. Temporal change of the manipulability: (a) leader-side manipulability ellipsoids and (b) the maximum and minimum manipulability indices with respect to possible $\dot{x}_\ell(t)$ at each t, i.e., $\lambda_{\max}^{1/2}(t)$ and $\lambda_{\min}^{1/2}(t)$.

example). However, the rough characteristics of the agent motion are still preserved in many cases even if the distances between connected agents vary.

B. Manipulability

Fig. 4 shows an example of the temporal change of the manipulability index during a single leader moves with $\dot{x}_\ell(t) = [1, 0]^T$, where $N = 3$ ($N_f = 2$ and $N_\ell = 1$) and $|\mathsf{E}| = 2$. From the leader-side manipulability ellipsoids depicted in Fig. 4 (a), we see that the effective direction was the horizontal direction in the first and the last parts of the motion. Fig. 4 (b) shows the maximum and minimum square-root eigenvalues of $J^T J$. From these figures, we see that the vertical direction was once the most effective around $t = 1.3$, when the three agents were lined in the vertical direction.

VII. Conclusions

In this paper, we defined the notion of manipulability in leader-follower networks by using rigid-link approximation of network dynamics, where every connected agent pairs keep their desired distances. This enables us to find the relation between instantaneous velocities of leaders and followers, which is crucial to define the approximate manipulability indices in leader-follower networks.

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