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Rank Deficiency and Superstability of Hybrid Systems with Application to Bipedal Robots

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Abstract— **The objectives of this paper are to study the rank properties of flows of hybrid systems, show that they are fundamentally different from those of smooth dynamical systems, and consider applications that emphasize the importance of these differences. In contrast with smooth dynamical systems, the rank of a solution to a hybrid system, a *hybrid execution*, is always less than the dimension of the space on which it evolves and falls within easily-computed and possibly distinct upper and lower bounds. Our main contribution is the derivation of conditions for when an execution fails to have maximal rank, *i.e.*, when it is rank deficient. Given the importance of periodic behavior in many hybrid systems applications, for example in bipedal robots, these rank deficiency conditions are applied to the special case of periodic hybrid executions. Our secondary contribution is the derivation of *superstability* conditions for when a periodic execution has rank equal to 0 and is therefore completely insensitive to perturbations in initial conditions. The results are illustrated in application to a planar kneed biped.**

I. Introduction

Hybrid systems consist of both continuous and discrete components and, as such, are capable of modeling a wide variety of physical systems, *i.e.*, systems that evolve with both continuous and discrete dynamics. Although hybrid systems model a wide variety of applications, we may not in general assume that they share the same fundamental properties as smooth dynamical systems. Moreover, the interaction of the smooth and discrete components of a hybrid system can result in solution behavior that is impossible for smooth dynamical systems to exhibit. For example, the existence and uniqueness properties of solutions of hybrid systems — called *hybrid executions* — are not the same as for smooth systems [1], [2]; therefore, one may not regard the stability of hybrid system equilibria in the same way as the stability of smooth system equilibria [3]. Recent work [4] has also shown that Poincaré maps for hybrid systems are fundamentally different from Poincaré maps for smooth systems.

The first contribution of the present work is the extension of the results in [4] to arbitrary, non-periodic hybrid executions. In particular, we show that the rank of an execution will always fall between possibly distinct upper and lower bounds, and that the upper bound is always less than the dimension of the space on which the execution evolves. This result is in marked contrast with smooth dynamical systems, where the rank of a solution is strictly equal to the dimension of the space. Our main contribution, however, is the derivation of conditions describing when an execution fails to have maximal rank, that is, when it is *rank deficient*. The secondary contribution of this work emerges from application of the main result to periodic solutions of hybrid systems. We show that when an execution is periodic *and* rank deficient it may be possible for the system to be *superstable*. Recall that a discrete dynamical system is said to be superstable when it is completely insensitive to perturbations in initial conditions [5]. This occurs when the linearization of the discrete dynamical system is equal to 0 at a superstable equilibrium point. By considering superstability from within the context of rank deficiency, we obtain a condition describing when a periodic hybrid execution is completely insensitive to perturbations in its initial conditions.

The superstability conditions presented here could enable the design of controllers that reduce the sensitivity of hybrid systems to perturbations. In [6], finite-time controllers and the properties of feedback-linearized systems are used to reduce the stability analysis of a planar biped to an interval of the real line. We foresee that, in analogy to this work, our rank deficiency conditions could be used to enable the design of (feedback-linearizing) controllers that reduce the stability analysis of complex hybrid systems to lower-dimensional spaces.

We begin by briefly reviewing definitions from the standard theory of smooth dynamical systems in Section II. This standard theory applies directly to the smooth components of a hybrid system, leading to straightforward techniques for linearizing executions of hybrid systems, in Section III. This allows us to derive necessary and sufficient conditions for the rank of an arbitrary hybrid execution to fall below its upper bound. These are the rank deficiency conditions. In Section IV we specialize the rank deficiency conditions to periodic hybrid systems and illustrate our results by analyzing the rank deficiency of a planar kneed biped.

II. Smooth dynamical systems

In this section we review standard results [7] on the trajectories of smooth dynamical systems that will be necessary to our analysis of hybrid systems in Section III. In particular we review how to convert the flow, which depends continuously on time, into a discrete map.

A *smooth dynamical system* is a tuple (M, f), where M is a smooth manifold with tangent bundle TM and $f : M \to TM$ is a smooth vector field such that for the canonical projection map $\pi : TM \to M$, $\pi \circ f = \mathrm{Id}$, where Id is

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identity on M. We assume that $M \subset \mathbb{R}^n$, in which case we write the vector field in coordinates as $\dot{x} = f(x)$. If $g : M \to N$ is a smooth map between manifolds, its Jacobian $Dg(x) : T_xM \to T_{g(x)}N$ is a linear map between tangent spaces.

Flows and variational equations. An *integral curve* of the differential equation $\dot{x} = f(x)$ is a trajectory $c : I \subset [0, \infty) \to M$ with initial condition $c(t_0) = x_0$, for $I = [t_0, t_1]$. The *flow* of a smooth vector field $\dot{x} = f(x)$ is a smooth map $\phi : I \times U \to U' \subset M$, where U is some neighborhood of $x_0 = c(t_0)$, satisfying $x_0 = \phi_0(x_0)$, $c(t_0 + t + s) = \phi_{t+s}(x_0) = \phi_t \circ \phi_s(x_0)$, and $\phi_{-r} = (\phi_r)^{-1}$. The flow, with t considered a parameter, is a diffeomorphism $\phi_t : U \to U'$, for all $t \in I$.

The *space derivative* or *fundamental matrix* of $\phi_t(x_0)$ is simply the partial derivative of the flow with respect to initial conditions, $D_x\phi_t(x_0) := \partial\phi_t(x_0)/\partial x_0$, (*cf.* [7]–[9]). It satisfies the time-varying, matrix-valued differential equation called the *variational equation*:

$$\dot{\Phi}(t) = A(t)\,\Phi(t), \qquad (1)$$

where $\Phi(t) = D_x\phi_t(x_0)$ and $A(t) := Df\big(\phi_t(x_0)\big)$. As an integral curve, $\Phi(t)$ is nonsingular for all t and has the property that $\dot{\phi}_t(x_0) = \Phi(t)\Phi^{-1}(0)\,\dot{\phi}_0(x_0) = \Phi(t)\,\dot{\phi}_0(x_0)$. That is, with $x_1 = \phi_t(x_0)$, $f(x_1) = \Phi(t)\,f(x_0)$. Note in particular that $\Phi(0) = \mathrm{Id}_n$, the $n \times n$ identity matrix. In general, $\phi_t(x_0)$ and $\Phi(t)$ must be obtained by simultaneous numerical integration, as described in [8], [10].

Flows to sections. We are interested in the properties of integral curves that intersect with particular submanifolds of M called local sections.

A *local section* is a submanifold $S \subset M$ that is *transverse to the flow*, such that $f(x) \notin T_xS$ for all $x \in S$. One can always construct a local section through any point of the flow that is not an equilibrium point [11]. In what follows we consider local sections defined by zero-level sets of smooth functions: $S = \{x \in M \mid h(x) = 0 \text{ and } L_fh(x) \neq 0\}$, where $h : M \to \mathbb{R}$ is smooth and $L_fh(x) = Dh(x)f(x)$ is the Lie derivative.

The time it takes a flow to reach a local section from initial conditions is a well-defined map.

Lemma 1 (Hirsch & Smale, 1974)*: Let S be a local section, $x_0 \in M$ and $x_1 = \phi_t(x_0) \in S$. There exists a unique, C^1 function $\tau : U_0 \to [0, \infty)$ called the* time-to-impact map *such that for U_0 a sufficiently small neighborhood of x_0, $\phi_{\tau(x)}(x) \in S$ for all $x \in U_0$.*

We define the map $\phi_\tau : U_0 \to V$ by $\phi_\tau(x) := \phi_{\tau(x)}(x)$ for all $x \in U_0$, where U_0 is defined as in the Lemma and $V := \phi_\tau(U_0) \cap S$ is the image of ϕ_τ in S.

Rank of flows to sections. The flow $\phi_t : U \to U'$, with t considered a fixed parameter, is a diffeomorphism, so its total derivative, $D\phi_t$, will always have full rank. This is easily confirmed by computing $D\phi_t(x) = D_x\phi_t(x) = \Phi(t)$, which is nonsingular. The total derivative of the flow to a section $\phi_\tau : U_0 \to V \subset S$, on the other hand, is [4], [7], [8]

$$\begin{aligned} D\phi_\tau(x_0) &= \Phi(\tau(x_0)) + \dot{\phi}_\tau(x_0)D\tau(x_0) \\ &= \left(\mathrm{Id}_n - \frac{f(x_1)Dh(x_1)}{L_fh(x_1)}\right)\Phi(\tau(x_0)), \end{aligned} \qquad (2)$$

where $x_0 \in U_0$, $x_1 = \phi_\tau(x_0)$, Id_n is the $n \times n$ identity matrix and h defines the local section S. It was shown in [4] that the rank of (2) is equal to the dimension of the local section S, and that flows to sections satisfy:

- **(S1)** For any local section S of $c(t_0)$ there exists a sufficiently small neighborhood U_0 of $c(t_0)$ such that $\phi_\tau(U_0) \subset S$.
- **(S2)** By Theorem 1 and Corollary 1 of [4], there exists a local section S_0 through $c(t_0)$ such that for $V_0 := U_0 \cap S_0$ and $V := \phi_\tau(U_0) \cap S$, the restricted map $\phi_\tau : V_0 \to V$ is a diffeomorphism with rank $n - 1$.

These properties will be revisited for hybrid systems.

III. Hybrid dynamical systems

Our objective is to understand the rank properties of arbitrary hybrid executions in order to enable the design of controllers that improve the stability properties of hybrid systems. We begin by revisiting the results of the previous section from the perspective of hybrid systems.

Hybrid systems and executions.

Definition 1: A hybrid system is a tuple $\mathcal{H} = (\Gamma, D, G, R, F)$, where

- $\Gamma = (Q, E)$ is a graph such that $Q = \{q_1, \ldots, q_k\}$ is a set of k vertices and $E = \{e_1 = (q_1, q_2), e_2 = (q_2, q_3), \ldots\} \subset Q \times Q$. With the set E we define maps $\mathrm{sor} : E \to Q$ which returns the source of an edge (the first element in the edge tuple), and $\mathrm{tar} : E \to Q$, which returns the target of an edge (second element in the edge tuple).
- $D = \{D_q\}_{q \in Q}$ is a collection of smooth manifolds called *domains*, where D_q is assumed to be embedded submanifolds of $\mathbb{R}^{n_q}$ with $\dim(D_q) = n_q \geq 1$.
- $G = \{G_e\}_{e \in E}$ is a collection of *guards*, where G_e is assumed to be an embedded submanifold of $D_{\mathrm{sor}(e)}$.
- $R = \{R_e\}$ is a collection of *reset maps* which are smooth maps $R_e : G_e \to D_{\mathrm{tar}(e)}$.
- $F = \{f_q\}_{q \in Q}$ is a collection of Lipschitz vector fields on D_q, such that $\dot{x} = f_q(x)$.

The continuous and discrete dynamics of a hybrid system are described using a notion of solution called a hybrid execution.

Definition 2: A *hybrid execution* is a tuple $\chi = (\Lambda, I, \rho, C)$, where

- $\Lambda = \{0, 1, 2, 3, \ldots\} \subseteq \mathbb{N}$ is an indexing set.
- $I = \{I_i\}_{i \in \Lambda}$ such that with $|\Lambda| = N$, $I_i = [t_i, t_{i+1}] \subset \mathbb{R}$ and $t_i \leq t_{i+1}$ for $0 \leq i < N - 1$. If N is finite then $I_{N-1} = [t_{N-1}, t_N]$ or $[t_{N-1}, t_N)$ or $[t_{N-1}, \infty)$, with $t_{N-1} \leq t_N$.
- $\rho : \Lambda \to Q$ is a map such that $e_{\rho(i)} := (\rho(i), \rho(i+1)) \in E$.

- $C = \{c_i\}_{i\in\Lambda}$ is a set of continuous trajectories where each c_i is the integral curve of the vector field $f_{\rho(i)}$ on $D_{\rho(i)}$. Specifically, $c_i(t) = \phi^{\rho(i)}_{t-t_i}(c_i(t_i))$, where $\phi^{\rho(i)}_t$ is the flow associated with $f_{\rho(i)}$.

We require the consistency conditions:

- For $i < |\Lambda|$ and for all $t \in I_i$, $c_i(t_i) = \phi^i_0(c_i(t_i))$, $c_i(t) \in D_{\rho(i)}$ and $c_i(t_{i+1}) \in G_{e_{\rho(i)}}$.
- For $i < |\Lambda| - 1$, $R_{e_{\rho(i)}}(c_i(t_{i+1})) = c_{i+1}(t_{i+1})$.

Assumptions. We only consider executions that are *deterministic* and *non-blocking* [1] and are sufficiently "well-behaved," described as follows. Let $i < |\Lambda| - 1$ and $e = (\rho(i), \rho(i+1))$.

- (**A1**) The execution does not have any equilibria, *i.e.*, $f_{\rho(i)}(c_i(t)) \neq 0$, for all $t \in I_i$.
- (**A2**) R_e has constant rank r_e and $R_e(G_e)$ is a submanifold of $D_{\mathrm{tar}(e)}$.
- (**A3**) G_e is a codimension-1 local section of $f_{\mathrm{sor}(e)}$.
- (**A4**) $R_e(G_e)$ is transverse to $f_{\mathrm{tar}(e)}$ whenever $\dim(D_{\mathrm{sor}(e)}) \leq \dim(D_{\mathrm{tar}(e)})$, that is, $f_{\mathrm{tar}(e)}(y) \notin T_y R_e(G_e)$ for all $y \in R_e(G_e)$.

Assumption (**A4**) allows us to tighten the lower bound on the rank of our executions.

Properties. We may extend properties (**S1-2**) of flows to sections from Section II to every integral curve $c_i \in C$, $i < |\Lambda|$, satisfying (**A1-4**).

- (**H1**) For any local section $S^i \subset G_e$ of $c_i(t_{i+1})$ there exists a sufficiently small neighborhood U^i_0 of $c_i(t_i)$ such that $\phi^{\rho(i)}_\tau(U^i_0) \subset S^i$.
- (**H2**) There exists a local section S^i_0 through $c_i(t_i)$ such that for $V^i_0 := U^i_0 \cap S^i_0$ and $V^i := \phi^{\rho(i)}_\tau(U^i_0) \cap S^i$, the restricted map $\phi^{\rho(i)}_\tau : V^i_0 \to V^i$ is a diffeomorphism with rank equal to $\dim(D_{\rho(i)}) - 1$.

These properties are generically satisfied by any flow that reaches a guard, and will be necessary to our results on rank deficiency in the following subsections.

Fundamental hybrid executions. The rank of a hybrid execution is determined by the rank of its linearization, or total derivative, at every point. This motivates the following definition.

Definition 3: The *fundamental hybrid execution* associated with a given execution χ is a tuple $\mathcal{F}\chi = (\Lambda, I, \rho, C, W)$ where Λ, I, ρ, and C are given in Definition 2 and $W = \{\Phi_i\}_{i\in\Lambda}$ is a set of continuous trajectories, where each Φ_i is an integral curve of the variational equation:

$$\dot{\Phi}_i(t - t_i) = Df_{\rho(i)}(c_i(t))\,\Phi_i(t - t_i). \tag{3}$$

Furthermore, every $\Phi_i \in W$ has the property $\dot{\phi}^i_t(c_i(t_i)) = \Phi_i(t - t_i)\,\dot{\phi}^i_0(c_i(t_i))$.

The fundamental execution allows us to compute the total derivative of the flow on each domain. Given an execution χ, we integrate (3) on $D_{\rho(i)}$, $i < |\Lambda|$, to obtain $\mathcal{F}\chi$ and then use (2) to compute the total derivative of $\phi^{\rho(i)}_\tau$,

$$\begin{aligned} D\phi^{\rho(i)}_\tau(x_0) &= \Phi_i(\tau(x_0)) + f_{\rho(i)}(x_0)\,D\tau(x_0) \\ &= \left(\mathrm{Id}_{n_i} - \frac{f_{\rho(i)}(x_1)\,Dh_i(x_1)}{Dh_i(x_1)\,f_{\rho(i)}(x_1)}\right)\Phi_i(\tau(x_0)), \end{aligned} \tag{4}$$

where, for ease of notation, we set $x_0 = c_i(t_i)$, $x_1 = c_i(t_{i+1})$, $h_j : D_{\rho(i)} \to \mathbb{R}$ defines the local section $S^i \subset G_{e_{\rho(i)}}$, $\tau(x_0) = t_{i+1} - t_i$ is the time it takes the flow to reach the guard and $n_i = \dim(D_{\rho(i)})$.

Rank of edge maps. Let $\mathcal{H}$ be a hybrid system and χ its hybrid execution with initial condition in the guard, $c_0(t_0) \in G_{e_{\rho(0)}}$. This initial condition is related to a point $c_i(t_{i+1})$ in the guard $G_{e_{\rho(i)}}$, for some $i < |\Lambda|$, by the *partial* function $\psi_{\rho(i)} : V^0 \to V^i$ defined by

$$\psi_{\rho(i)} = \phi^{\rho(i)}_\tau \circ R_{e_{\rho(i-1)}} \circ \cdots \circ \phi^{\rho(1)}_\tau \circ R_{e_{\rho(0)}}. \tag{5}$$

The neighborhoods V^0 of $c_0(t_0)$ and V^i of $c_i(t_{i+1})$ are defined as in (**H2**). We may think of the partial function as describing the progress of the execution through the hybrid system $\mathcal{H}$. Our analysis of the rank properties of χ is aided by identifying the terms in (5) that can be associated with each edge in the graph Γ of $\mathcal{H}$.

Definition 4: Let $i < |\Lambda| - 1$. For every edge $e = (\rho(i), \rho(i+1)) \in E$, the *edge map* $\psi_e : V^i \to V^{i+1}$ takes the guard of one domain to the next and is defined $\psi_e = \phi^{\mathrm{tar}(e)}_\tau \circ R_e$. Using the edge map, (5) becomes

$$\psi_{\rho(i)} = \psi_{e_{\rho(i-1)}} \circ \cdots \circ \psi_{e_{\rho(0)}}, \tag{6}$$

$$D\psi_{\rho(i)} = D\psi_{e_{\rho(i-1)}} \circ \cdots \circ D\psi_{e_{\rho(0)}}. \tag{7}$$

Let $\{A_i\}^k_{i=1}$ be a collection of matrices, with $A_i \in \mathbb{R}^{n_{i+1}\times n_i}$. Repeated application of Sylvester's inequality shows that the composition $\prod^k_{i=1} A_i = A_1 \circ A_2 \circ \cdots \circ A_k$ is bounded above and below:

$$\mathrm{rank}\left(\prod^k_{i=1} A_i\right) \leq \min_{i\in\{1,\ldots,k\}}\{\mathrm{rank}(A_i)\}, \tag{8}$$

$$\mathrm{rank}\left(\prod^k_{i=1} A_i\right) \geq \sum^k_{i=1}\mathrm{rank}(A_i) - \sum^{k-1}_{i=1} n_i. \tag{9}$$

Recall the rank-nullity theorem [12]: for every linear map $A : \mathbb{R}^n \to \mathbb{R}^m$, $\mathrm{rank}(A) + \mathrm{nty}(A) = n$, where $\mathrm{nty}(A)$ is the dimension of the nullspace of A. The following Lemma is a consequence of rank-nullity. We omit proofs of the following two results for the sake of brevity.

Lemma 2: *Let A and B be linear maps. Then*

$$\mathrm{nty}(B \circ A) - \mathrm{nty}(A) = \dim(\mathrm{ns}(B) \cap \mathrm{im}(A)).$$

Lemma 2 and the rank-nullity theorem allow us to compute the rank of the execution by determining the rank of every edge map in $\psi_{\rho(i)}$.

Lemma 3: *Let $i < |\Lambda| - 1$. For every edge $e = (\rho(i), \rho(i+1))$, the rank of the edge map $\psi_e : V^i \to V^{i+1}$ is bounded from below by* $\mathrm{rank}(\psi_e) \geq \mathrm{rank}(R_e) - 1$.

However, if $R_e(G_e)$ is transverse to $f_{\mathrm{tar}(e)}$ then $\mathrm{rank}(\psi_e) = \mathrm{rank}(R_e)$.

Lemma 3 indicates that the rank of an edge map is only known *a priori* whenever transversality of the image of the reset map with the vector field in the target domain is known. As we will see, transversality in the target domain allows us to tighten the lower bound on the rank of the execution.

Rank of hybrid executions. The following definitions allow us to track the progress of the execution through the graph Γ of $\mathcal{H}$.

Definition 5: Given $i < |\Lambda| - 1$, the set of *traversed edges* is $E_i = \{e_{\rho(0)}, \ldots, e_{\rho(i-1)}\}$, and the set of *visited vertices* is the set of all source and target vertices of E_i,

$$Q_i = \mathrm{sor}(E_i) \cup \mathrm{tar}(E_i) = \{\rho(0), \ldots, \rho(i)\}.$$

Definition 6: Let m be the number of *non-transverse edges* for which we do not assume $R_e(G_e)$ is transverse to $f_{\mathrm{tar}(e)}$. Then m is given by

$$m = \left|\{e \in E_i : \dim(D_{\mathrm{sor}(e)}) > \dim(D_{\mathrm{tar}(e)})\}\right|.$$

We now show that the rank of an execution falls between possibly distinct upper and lower bounds. The following result is the extension of Theorem 4 in [4] to arbitrary, non-periodic hybrid systems and executions, and so we omit the proof for reasons of space.

Theorem 4*: Let $\mathcal{H}$ be a hybrid system with execution χ satisfying assumptions (**A1-4**). For any $i < |\Lambda| - 1$,*

$$\mathrm{rank}(\psi_{\rho(i)}) \leq \min_{e \in E_i}\{\mathrm{rank}(R_e)\} \leq \min_{q \in Q_i}\{\dim(D_q) - 1\},$$
$$\mathrm{rank}(\psi_{\rho(i)}) \geq \sum_{e \in E_i} \mathrm{rank}(R_e) - m - \sum_{q \in \mathrm{sor}(E_i) - \{\rho(0)\}} (\dim(D_q) - 1),$$

and where m, E_i and Q_i are given in Definitions 5 and 6.

If the upper and lower bounds on rank in Theorem 4 are distinct then there must be a mechanism that causes an execution to fail to have maximal rank. We determine this mechanism in the next section.

Rank deficiency of hybrid executions. Our objective is to understand the causes of rank deficiency. As we will see, rank deficiency can result in superstable hybrid systems that are completely insensitive to perturbations in initial conditions. We begin by formally defining the rank deficiency of a hybrid execution.

Definition 7: Let $\mathcal{H}$ be a hybrid system with execution χ satisfying assumptions (**A1-4**). We say the execution is *rank deficient* at a point $c_i(t_{i+1})$, $i < |\Lambda| - 1$, if $\psi_{\rho(i)}(c_i(t_{i+1}))$ does not have maximal rank, that is, if $\mathrm{rank}\left(\psi_{\rho(i)}(c_0(t_0))\right) < r$, where r is the upper bound on $\mathrm{rank}(\psi_{\rho(i)})$ from Theorem 4.

The following Theorem is the main result of this paper.

Theorem 5*: Let $\mathcal{H}$ be a hybrid system with execution χ satisfying* (**A1-4**)*, initial condition $x_0 = c_0(t_0)$ and $i < |\Lambda| - 1$. Then $\psi_{\rho(i)}$ is rank deficient if and only if*

$$\sum_{e \in E_i - \{e_{\rho(0)}\}} \dim(\mathrm{ns}(D\psi_e) \cap \mathrm{im}(D\psi_{\mathrm{sor}(e)})) > \mathrm{rank}(\psi_{e_{\rho(0)}}) - r,$$

where r is the upper bound on $\psi_{\rho(i)}$ from Theorem 4.

Proof: The proof will follow from recursively applying the rank-nullity theorem and Lemma 2 to the sequence of linear maps (7).

First, realize that any two linear maps defined on the same domain are related by the rank-nullity theorem. In particular, it is an immediate consequence of rank-nullity that for all j such that $i \geq j \geq 2$,

$$\begin{aligned}\dim(T_{c_0(t_0)}V^0) &= \mathrm{rank}(D\psi_{e_{\rho(0)}}) + \mathrm{nty}(D\psi_{e_{\rho(0)}}) \\ &= \mathrm{rank}(D\psi_{\rho(j)}) + \mathrm{nty}(D\psi_{\rho(j)}),\end{aligned}$$

where the statement is obvious for $j = 1$ since $\psi_{\rho(1)} = \phi_\tau^{\rho(1)} \circ R_{e_{\rho(0)}} = \psi_{e_{\rho(0)}}$. Thus, the rank-nullity of $\psi_{\rho(i)}$ is certainly equal to the rank-nullity of $\psi_{\rho(i-1)}$:

$$\mathrm{rank}(\psi_{\rho(i)}) + \mathrm{nty}(\psi_{\rho(i)}) = \mathrm{rank}(\psi_{\rho(i-1)}) + \mathrm{nty}(\psi_{\rho(i-1)}).$$

Applying Lemma 2 to the above equation while noting that $\psi_{\rho(i)} = \psi_{e_{\rho(i-1)}} \circ \psi_{\rho(i-1)}$ yields

$$\begin{aligned}\mathrm{rank}(\psi_{\rho(i)}) &= \mathrm{rank}(\psi_{\rho(i-1)}) \\ &\quad - \dim(\mathrm{ns}(D\psi_{e_{\rho(i-1)}}) \cap \mathrm{im}(D\psi_{\rho(i-1)})).\end{aligned}$$

If we continue in this vein by relating the rank-nullity of $\psi_{\rho(j)}$ with $\psi_{\rho(j-1)}$ for $j = i-1, \ldots, 2$, we obtain

$$\begin{aligned}\mathrm{rank}(\psi_{\rho(i)}) &= \mathrm{rank}(\psi_{e_{\rho(0)}}) \\ &\quad - \sum_{e \in E_i - \{e_{\rho(0)}\}} \dim(\mathrm{ns}(D\psi_e) \cap \mathrm{im}(D\psi_{\mathrm{sor}(e)})).\end{aligned}$$

The result follows by observing that the execution is rank deficient if and only if $r - \mathrm{rank}(\psi_{\rho(i)}) > 0$, where r is the upper bound on rank from Theorem 4. ■

Remark 1*:* The left-hand side of the inequality in the statement of Theorem 5 shows that rank deficiency is primarily affected by the intersection of the nullspace of every reset map with the tangent space over the execution. To see this, realize that for any given $e \in E$, the nullspace of the edge map ψ_e is the union of the tangent spaces

$$\mathrm{ns}(D\psi_e) = \left(\mathrm{ns}(D\phi_\tau^{\mathrm{tar}(e)}) \cap \mathrm{im}(DR_e)\right) \cup \mathrm{ns}(DR_e).$$

Therefore, because the nullspace of the flow to the guard on the target is small, *i.e.*, $\mathrm{nty}(D\phi_\tau^{\mathrm{tar}(e)}) = \dim(\mathrm{span}\{f_{\mathrm{tar}(e)}\}) = 1$, in general the nullspace of every edge map is primarily determined by $\mathrm{ns}(DR_e)$.

Remark 2*:* The right-hand side of Theorem 5 shows that the rank of the first edge map in the execution significantly affects rank deficiency. Since $r = \min_{e \in E_i}\{\mathrm{rank}(R_e)\}$, $\mathrm{rank}(\psi_{e_{\rho(0)}}) \geq r$ and the inequality will not be satisfied unless enough intersections occur in the left-hand side, or the intersections are large enough. This is a consequence of the fact that perturbations to initial conditions propagate differently through the execution depending on the starting domain $D_{\rho(0)}$.

IV. Application to Periodic Hybrid Systems

We are interested in applying the general results obtained thus far to periodic solutions of hybrid systems. To this end, we restrict our attention to hybrid systems with cyclic graphs and consider the rank properties of hybrid periodic orbits.

Definition 8: A *hybrid system on a cycle* is a hybrid system $\mathcal{H} = (\Gamma, D, G, R, F)$ where $\Gamma = (Q, E)$ is a *directed cycle* such that $Q = \{q_1, \ldots, q_k\}$ is a set of k vertices and $E = \{e_1 = (q_1, q_2), e_2 = (q_2, q_3), \ldots, e_k = (q_k, q_1)\} \subset Q \times Q$.

Definition 9: A *hybrid periodic orbit* $\mathcal{O} = (\Lambda, I, \rho, C)$ with period T is an execution of the hybrid system on a cycle $\mathcal{H}$ such that for all $n \in \Lambda$,

- $\rho(n) = \rho(n + k)$,
- $I_n + T = I_{n+k}$,
- $c_n(t) = c_{n+k}(t + T)$.

Remark 3*:* Since $\mathcal{O}$ is periodic we may index the elements S_0^n, S^n, U_0^n, V_0^n and V^n defined in (**H1-2**) using the vertex set Q of the graph Γ of $\mathcal{H}$ rather than the indexing set Λ (*e.g.*, one can take $S^n = S^{n+k}$).

Definition 10: The *fundamental hybrid periodic orbit* associated with $\mathcal{O}$ is the fundamental execution $\mathcal{FO} = (\Lambda, I, \rho, C, W)$, with the fundamental matrix solutions $\Phi_n \in W$ such that $\Phi_n(t - t_n) = \Phi_{n+k}(t + T - t_{n+k})$.

Extending equations (5) and (6) to periodic orbits yields the following definition for a Poincaré map of a hybrid system.

Definition 11: Let $\mathcal{O}$ be a given hybrid periodic orbit of $\mathcal{H}$ with initial condition $x^* = c_0(t_0) \in D_{\rho(0)}$, where $\rho(0) = q = \rho(k)$ and so $c_0(t_0) = \phi^q_\tau(c_k(t_k))$. The *hybrid Poincaré map* $P_q : V^q \to S^q$ is given by

$$P_q(x^*) = \psi_{\rho(k)} = \psi_{e_{\rho(k-1)}} \circ \cdots \circ \psi_{e_q} \tag{10}$$

It is well-known that the stability of hybrid periodic orbits is related to the stability of the hybrid Poincaré map. In particular, the following result is a corollary to Theorem 1 of [13] and the results of [4].

Corollary 6*: Let $\mathcal{H}$ be a hybrid system with hybrid periodic orbit $\mathcal{O}$ satisfying* (**A1-4**)*. Then $x^* = P_q(x^*)$ is an exponentially stable fixed point of the hybrid Poincaré map $P_q : V^q \to S^q$ if and only if $\mathcal{O}$ is exponentially stable.*

As a discrete dynamical system, the stability of the Poincaré map is determined by the eigenvalues of its derivative evaluated at a fixed point. The following is a corollary to Theorem 4.

Corollary 7*: The hybrid Poincaré map $P_q : V^q \to S^q$ is exponentially stable if and only if all eigenvalues of $DP_q(x^*)$ fall within the unit circle. In particular, $P_q(x^*)$ has only $r_q = \operatorname{rank}(DP_q(x^*))$ many nontrivial eigenvalues, where*

$$r_q \le \min_{e \in E}\{\operatorname{rank}(R_e)\} \le \min_{q \in Q}\{\dim(D_q) - 1\},$$
$$r_q \ge \sum_{e \in E} \operatorname{rank}(R_e) - m - \sum_{q \in \operatorname{sor}(E) - \{q\}} (\dim(D_q) - 1)\,,$$

m is the number of non-transverse edges in the cycle, and E and Q are the edge and vertex sets of Γ.

It follows that the stability of a rank deficient Poincaré map is determined by fewer eigenvalues than a Poincaré map with maximal rank.

We now consider the case when a Poincaré map is completely rank deficient. Recall that a *superstable* discrete dynamical system [5] is characterized by the derivative of the system equal to 0. When this occurs, the discrete dynamical system is said to be completely insensitive to perturbations in initial conditions. This notion adapts to periodic hybrid systems as follows.

Definition 12: The hybrid periodic orbit $\mathcal{O}$ with initial condition x^* and its associated Poincaré map P_q are said to be *superstable* at x^* if $\operatorname{rank}(DP_q(x^*)) = 0$.

All eigenvalues of a superstable Poincaré map are equal to 0, implying that not only is it exponentially stable, it is completely insensitive to perturbations in initial conditions. We obtain the following Corollary to Theorem 5.

Corollary 8*: The Poincaré map P_q is superstable if and only if the lower bound on rank in Corollary 7 is equal to 0 and*

$$\sum_{e \in E - \{e_q\}} \dim(\operatorname{ns}(D\psi_e) \cap \operatorname{im}(D\psi_{\operatorname{sor}(e)})) = \operatorname{rank}(\psi_{e_q}).$$

From the Corollary it is easy to see that single-domain hybrid systems are rank deficient if and only if the reset map has rank equal to 0, in which case it is superstable. Thus, the planar compass biped, a single-domain hybrid system studied in [14], [15], will never be rank deficient nor superstable. On the other hand, it is certainly possible to construct simple multi-domain hybrid systems that exhibit superstability, like the application first presented in [4].

In the following example we illustrate how one might use our results to achieve rank deficiency of a two-domain planar kneed biped.

A. Planar kneed biped

The planar kneed biped walks on flat ground with locking knees using controlled symmetries [15], and was studied in [4], [16]. It may be considered the augmentation of the planar compass biped with an additional domain where the stance leg is locked and the non-stance leg is unlocked at the knee. See Figure 1.

As a two-domain hybrid system on a cycle, $\mathcal{H} = (\Gamma, D, G, R, FG)$ has graph structure $\Gamma = \{Q = \{u, l\}, E = \{e_u = (u, l), e_l = (l, u)\}\}$. In the *unlocked domain* D_u, the non-stance calf rotates at the knee and the biped is a 3-link mechanism, so the dynamics evolve on the tangent bundle to the configuration space $\Theta_u := \mathbb{T}^3$ with chosen coordinates $\theta_u = (\theta_s, \theta_{ns}, \theta_k)^T$, where the stance leg angle is θ_s, non-stance thigh angle is θ_{ns}, and non-stance calf angle is θ_k. Since the non-stance thigh and calf are locked together in the *locked domain* D_l, the biped is a 2-link mechanism with dynamics on the tangent bundle to $\Theta_l := \mathbb{T}^2$, coordinates given by $\theta_l = (\theta_s, \theta_{ns})^T$. The dynamics on D_u are specified

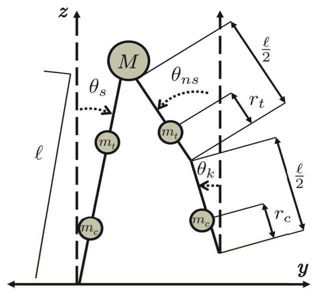

Fig. 1. Diagram of a planar kneed biped. The annotations indicate dimensions, point-mass locations and measuring conventions for stance, calf and thigh angles from the vertical.

by the controlled vector field f^u and on D_l by f^l, with $FG = \{f^u, f^l\}$. We transition from D_u to D_l when the knee locks, and from D_l to D_u when the foot strikes the ground. The reset maps $R = \{R_{e_u}, R_{e_l}\}$ model these transitions as perfectly plastic impacts; see [6], [16], [17] for details. Note that since $\dim(D_u) = 4$ and $\dim(D_l) = 6$, the transition to the unlocked domain has full rank, $\mathrm{rank}(DR_{e_l}) = 3$, since it is a map to a higher-dimensional domain. On the other hand, we find that $\mathrm{rank}(DR_{e_u}) = 4$ and $\mathrm{nty}(DR_{e_u}) = 1$.

Define the Poincaré map for initial conditions in the locked domain by $P_l = \phi^l_\tau \circ R_{e_u} \circ \phi^u_\tau \circ R_{e_l} = \psi_{e_u} \circ \psi_{e_l}$, and in the unlocked domain by $P_u = \phi^u_\tau \circ R_{e_l} \circ \phi^l_\tau \circ R_{e_u} = \psi_{e_l} \circ \psi_{e_u}$. Corollary 7 implies that $2 \leq \mathrm{rank}(P_u) \leq 3$ and $0 \leq \mathrm{rank}(P_l) \leq 3$, so the stability of the biped is determined by at most 3 eigenvalues. Immediately we see that superstability might be possible in D_l, but sensitivity to perturbations in initial conditions cannot be removed from D_u.

In [4], exactly 3 stable eigenvalues were found for a given execution χ of this hybrid system by computing the eigenvalues of DP_u and DP_l using $\mathcal{F}\chi$, as described after Definition 3 in Section III.

We use Theorem 5 to assess the rank deficiency of P_u for *any* given execution χ. First, realize that since transversality of $R_{e_u}(G_{e_u})$ is not guaranteed, Lemma 3 implies $\mathrm{rank}(\psi_{e_u}) \geq \mathrm{rank}(R_{e_u}) - 1 = 3$. However, since $\mathrm{rank}(D\phi^l_\tau) = 3$, it follows that P_u is rank deficient if and only if $\dim(\mathrm{ns}(D\phi^u_\tau \circ DR_{e_l}) \cap \mathrm{im}(D\phi^l_\tau \circ DR_{e_u})) > 0$. Since DR_{e_l} is full rank, we conclude that rank-deficiency of P_u is achieved when $\mathrm{im}(D\phi^l_\tau)$ intersects with the 1-dimensional subspace $\mathrm{ns}(D\phi^u_\tau \circ DR_{e_l})$. On the other hand, since $\mathrm{rank}(\psi_{e_l}) = \mathrm{rank}(R_{e_l})$ by Lemma 3 and (**A4**), rank deficiency of P_l requires $\dim(\mathrm{ns}(D\phi^l_\tau \circ DR_{e_u}) \cap \mathrm{im}(D\phi^u_\tau \circ DR_{e_l})) > 0$ by Theorem 5. Rank deficiency occurs when $\mathrm{im}(D\phi^u_\tau)$ aligns with either of the 1-dimensional subspaces $\mathrm{ns}(D\phi^l_\tau \circ DR_{e_u})$ or $\mathrm{ns}(DR_{e_u})$. We have thus determined that our objective in each case is to align the linearization of the flow with a 1-dimensional nullspace.

V. CONCLUSION

Rank deficiency emphasizes fundamental differences between hybrid systems and smooth systems and implies a depth to hybrid systems that is not yet fully understood. We have shown that the rank of a hybrid execution is always less than the dimension of the space on which solutions evolve, and that the upper and lower bounds on rank are known *a priori*. The rank deficiency condition is determined by the alignment of the tangent space to the execution with the nullspace of a reset map. We applied our results to a planar kneed biped and determined which tangent spaces needed to align to induce rank deficiency.

A future research direction is to employ existing techniques, such as those in [18], to design hybrid systems controllers that directly induce rank deficiency and superstability, and hence improve robustness to perturbations.

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