2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Multi-agent Robust Consensus -Part II: Application to Distributed Event-triggered Coordination

Guodong Shi and Karl Henrik Johansson

Abstract— **In the first part of the paper, robust consensus was discussed for continuous-time multi-agent systems with uncertainties in the dynamics. As an application of the robust consensus analysis, this part of the paper further investigates distributed multi-agent coordination via event-triggered strategies, where the control input of each agent is piecewise constant. Each agent chooses the instances to update its control input by checking whether its state error meets a given time-dependent function or not. Proper triggering conditions are given for the system to reach a global consensus using piecewise costant control with directed time-varying communication graphs under neighbor-synchronous and asynchronous updating protocols, respectively.**

Keywords: Multi-agent systems, Joint connection, Event-triggered coordination

I. Introduction

In recent years, there has been tremendous interest on multi-agent coordination problem, due to its broad backgrounds and applications in various fields of science including physics, engineering, biology, ecology and social science [18], [15], [23], [14]. Central to the multi-agent coordination study is distributed control design of a group of autonomous agents using local information only and limited, usually time-varying interconnections over the networks to achieve a consensus or state agreement for the whole group, which requires that all the agents achieve the desired relative position and the same velocity [14], [8], [20].

Efforts have been made for consensus seeking in the literature, and both continuous-time and discrete-time models are investigated [20], [18], [17], [7], [22], [19]. Furthermore, distributed control laws via event-triggered or self-triggered approaches result in new multi-agent dynamics somehow in between, where agents' dynamics are piecewise constant which update the value when certain events are executed [27], [31], [32]. Event-triggered feedback control was shown to be able to preserve desired properties such as stability and convergence with proper design [29], [28]. It has been shown that event-based control needs fewer samples than time-triggered control to achieve the same performance for stochastic systems [26], while up to event-triggered coordination rules, the system also benefits from reducing the communication frequency over the network.

Connectivity of the communication graph plays a key role, and various connectivity conditions have been used to describe frequently switching system topology. The "joint connection", i.e., the union graph over a time interval, and similar concepts are important in the analysis of consensus stability with time-dependent topology. Uniformly joint connectedness, which requests the joint connection is connected for all intervals which are longer than some positive constant, has been employed for different consensus problems from discrete-time to continuous-time agent dynamics, from directed to undirected interconnection topologies [20], [18], [22], [13], [4]. [20] studied the distributed asynchronous iterations, while [18] proved the consensus of a simplified Vicsek model. Furthermore, [13] and [4] investigated the jointly-connected coordination for second-order agent dynamics, while [22] worked on nonlinear continuous-time agent dynamics with directed communications, in which convergence to a consensus is shown to be uniform within bounded initial conditions. [10] and [11] presented the convergence analysis and convergence rate estimations for discrete-time agents' state updating, and furthermore, [21] showed that the convergence time is of order $O(n^2B)$, where n is the number of nodes in the network and B is a lower bound for the time interval in definition of uniformly joint connectedness. $[t,\infty)$-joint connection requires the joint connection is connected for infinitely many disjoint intervals in $[0,+\infty]$, was discussed in [23], in order to achieve the consensus for discrete-time agents. This connectivity concept was then extended in continuous-time distributed control analysis for target set convergence and state agreement in [19].

This part of the paper considers the multi-agent coordination via event-triggered strategies with directed time-varying communication graph. The trigger function time for a agent to decide when it is triggered is defined when the state measurement error equals a given function. Two types of protocols are studied respectively, relying on whether or not each agent will update its control immediately when it receives its neighbor's broadcasting or the communication graph is changing. Based on the robust consensus analysis given in Part I of the paper [16], the results show that a consensus can be achieved with jointly connected, directed interconnections when the class of trigger function is well selected, and Zeno behavior [30] can also be avoided.

The paper is organized as follows. In section II, some preliminary concepts and necessary knowledge are introduced. Neighbor-synchronous updating strategy is studied in section III, in which several conditions are given on the trigger function and connectivity to ensure a consensus for the system. Then in section IV, we turn to asynchronous

This work has been supported in part by the Knut and Alice Wallenberg Foundation and the Swedish Research Council. G. Shi and K. H. Johansson are with ACCESS Linnaeus Centre, School of Electrical Engineering, Royal Institute of Technology, Stockholm 10044, Sweden. Email: guodongs@kth.se, kallej@ee.kth.se

updating strategy. Finally, concluding remarks are given in section V.

II. Problem Formulation and Preliminaries

In this section, we describe the considered consensus problem, and introduce some preliminary knowledge used in the subsequent analysis.

In this paper, we consider a multi-agent system with agent set $\mathcal{V} = \{1, \dots, N\}$, for which the dynamics of each agent is the following first-order integrator:

$$\dot{x}_i = u_i, \quad i = 1, \dots, N \tag{1}$$

where $x_i \in R$ represents the state of agent i, and u_i is the control input which should be designed based on neighborhood information.

A. Communication Graph

In this subsection, we define the communication graph over the network. First we introduce some preliminary knowledge related to directed graph.

A directed graph (digraph) $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a finite set $\mathcal{V}$ of nodes and an arc set $\mathcal{E}$, in which an arc is an ordered pair of distinct nodes of $\mathcal{V}$ [2]. An element $e = (i, j)$ in $\mathcal{E}$ is called an *arc* leaving from node $i \in \mathcal{V}$ and entering node $j \in \mathcal{V}$. If the e_j's are pairwise distinct in an alternating sequence $v_0 e_1 v_1 e_2 v_2 \dots e_n v_n$ of nodes v_i and arcs $e_i = (v_{i-1}, v_i) \in \mathcal{E}$ for $i = 1, 2, \dots, n$, the sequence is called a (directed) *path* with *length* n, and for $v_0 = v_n$ a (directed) *cycle*. A path from i to j is denoted as $i \to j$, and the length of $i \to j$ is denoted as $|i \to j|$. A digraph without cycles is said to be *acyclic*. $\mathcal{G}$ is said to be *strongly connected* if it contains path $i \to j$ and $j \to i$ for every pair of nodes i and j. The length If there exists a path from node i to node j, then node j is said to be reachable from node i. In particular, each node is thought to be reachable by itself. A node v from which any other node is reachable is called a *center* (or a *root*) of $\mathcal{G}$. $\mathcal{G}$ is said to be *quasi-strongly connected* (QSC) if $\mathcal{G}$ has a center [3].

The communication in the network is modeled as a time-varying graph $\mathcal{G}_{\sigma(t)} = (\mathcal{V}, \mathcal{E}_{\sigma(t)})$ with $\sigma : [0, +\infty) \to \mathcal{Q}$ as a piecewise constant function, where $\mathcal{Q}$ is a finite set with all the possible graphs with node set $\mathcal{V}$. Moreover, node j is said to be a *neighbor* of i at time t when there is an arc $(i, j) \in \mathcal{E}_{\sigma(t)}$, and $\mathcal{N}_i(\sigma(t))$ represents the set of agent i's neighbors at time t.

An assumption is given to the time-varying topology.

A1. (Dwell Time) There is a $\tau_D > 0$ for $\sigma(t)$, as a lower bound between two switching time instants.

Denote the joint graph of $\mathcal{G}_{\sigma(t)}$ in time interval $[t_1, t_2)$ with $t_1 < t_2 \le +\infty$ as $\mathcal{G}([t_1, t_2)) = \cup_{t\in[t_1,t_2)} \mathcal{G}(t) = (\mathcal{V}, \cup_{t\in[t_1,t_2)} \mathcal{E}_{\sigma(t)})$. Then we have the following definition.

Definition 2.1: (i) $\mathcal{G}_{\sigma(t)}$ is said to be *uniformly (jointly) quasi-strongly connected* (UQSC) if there exists a constant $T > 0$ such that $\mathcal{G}([t, t+T))$ is quasi-strongly connected for any $t \ge 0$.

(ii) $\mathcal{G}_{\sigma(t)}$ is said to be *uniformly (jointly) strongly connected* (USC) if there exists a constant $T > 0$ such that $\mathcal{G}([t, t+T))$ is strongly connected for any $t \ge 0$.

B. Continuous-time Dynamics

Suppose the state of agent i is $x_i \in R (i = 1, \dots, n)$. Denote $x = (x_1, \dots, x_N)^T \in R^N$ and let continuous function $a_{ij}(x, t) > 0$ be the weight of arc (j, i), if any, for $i, j \in \mathcal{V}$. The control input for each agent is presented in the following.

$$u_i = \sum_{j\in\mathcal{N}_i(\sigma(t))} a_{ij}(x, t)(x_j - x_i) + w_i(t), \ i = 1, \dots, N \tag{2}$$

where $w_i(t)$ is a function to describe the disturbances in communication links and individual dynamics to agent i.

An assumption is given to each $a_{ij}(x, t)$.

A2. (Weights Rule) There are two constants $0 < a_* \le a^*$ such that $a_* \le a_{ij}(x, t) \le a^*$, $x \in R^N, t \in R^+$.

Remark 2.1: In practice, the weights for a multi-agent network, a_{ij}, may not be constant because of the complex communication and environment uncertainties, and then the multi-agent system become time-varying or nonlinear (referring to [22], [19], [23]). Here $a_{ij}(x, t)$ is written in a general form simply for convenience, and global information is not required in the study. For example, a_{ij} can depend only on the state of x_i, time t and x_j $(j \in N_i)$, which is certainly a special form of $a_{ij}(x, t)$. In this case, the control laws of form (2) are still decentralized.

Denote $\|z\|_\infty < \infty\}$ with $\|z\|_\infty \triangleq \sup\{|z(t)|, t \ge 0\}$. Here we take $|z(t)| \triangleq \max_i |z_i(t)|$ as the maximum norm of $z(t)$. Then define $\mathcal{F} \triangleq \{z : \mathbb{R}_{\ge 0} \to \mathbb{R}^N | z(t)$ is continuous except for a set with measure zero with $\|z\|_\infty < \infty\}$. Then denoting $w(t) \triangleq (w_1(t), \dots, w_n(t))^T$, another assumption is given to the regularity of the noise functions in order to ensure the existence of the solutions of system (2).

A3. (Noise Regularity) $w(t) \in \mathcal{F}$.

In this paper, we assume that assumptions A1, A2 and A3 always hold as the standing assumptions. With assumptions A1 and A3, the set of discontinuous points for the right hand side of equation (2) has measure 0. Therefore, the Caratheodory solutions [1] for (2) are existent for arbitrary initial conditions, which are absolutely continuous function such that satisfies (2) for almost all t on the maximum interval of existence. Furthermore, it is not hard to see that assumption A2 ensures each Caratheodory solution of (2) exists on $[t_0, \infty]$ without finite time escape. In the following of the paper, the trajectories of system (2) mentioned are Caratheodory solutions.

Suppose $x(t) = (x_1, \dots, x_N)^T \in R^N$ is the trajectory of system (2) with initial condition $x(t_0) = x^0 = (x_1(t_0), \dots, x_N(t_0))^T \in R^N$, where $t_0 \ge 0$ is the initial time. Furthermore, let

$$\hbar(t) = \max_{i\in\mathcal{V}}\{x_i(t)\}, \quad \ell(t) = \min_{i\in\mathcal{V}}\{x_i(t)\}$$

be the maximum and minimum within all the agents at time t along $x(t)$. Moreover, denote

$$\mathcal{H}(x(t)) \triangleq \hbar(t) - \ell(t).$$

Remark 2.2: In this paper, $|\cdot|$ denotes the maximum norm for a vector or the absolute value of a scalar. All the results obtained in this paper will also hold if $|\cdot|$ takes the Euclidean norm.

Furthermore, we define global consensus and global asymptotic consensus in the following way.

Definition 2.2: (i) A global consensus (GC) is achieved for system (2) if

$$\lim_{t\to\infty}\mathcal{H}(x(t))=0$$

for any initial condition $x(t_0)=x^0$;

(ii) Assume that $\mathcal{F}_0\subseteq\mathcal{F}$. Then a global asymptotic consensus (GAC) with respect to $\mathcal{F}_0$ is achieved for system (2) if $\forall w\in\mathcal{F}_0$, $\forall\varepsilon>0$, $\forall c>0$, $\exists T>0$ such that $\forall t_0\geq 0$,

$$\mathcal{H}(x^0)\leq c\Rightarrow\mathcal{H}(x(t))\leq\varepsilon,\forall t\geq t_0+T.$$

Remark 2.3: GAC is to say, with bounded initial condition $\mathcal{H}(x^0)$, $\mathcal{H}(x(t))$ not only converges to 0, but also converge uniformly in t for all $w\in\mathcal{F}_0$ along trajectories of system (2).

C. Preliminary Results

The following results were proved in Part I of the paper [16].

Proposition 2.1: (i) System (1) with control rule (2) achieves a GC for any $w\in\mathcal{F}_1$ if $\mathcal{G}_{\sigma(t)}$ is UQSC, where

$$\mathcal{F}_1\triangleq\{z(t)\in\mathcal{F}_\infty:\lim_{t\to\infty}z(t)=0\}.$$

(ii) System (1) with control rule (2) achieves a GAC with respect to $\mathcal{F}_1^0$ if and only if $\mathcal{G}_{\sigma(t)}$ is UQSC, where $\mathcal{F}_1^0\subseteq\mathcal{F}_1$ is a subset with $\lim_{t\to\infty}\sup_{z\in\mathcal{F}_1^0}|z(t)|=0$.

Proposition 2.2: (i) Assume that either $\mathcal{G}_{\sigma(t)}$ being undirected for any $t\geq 0$, or $\mathcal{G}([0,+\infty))$ being acyclic. Then system (1) with control rule (2) achieves a GC for all $w\in\mathcal{F}_2$ if and only if $\mathcal{G}([t,\infty))$ is QSC for any $t\geq 0$, where

$$\mathcal{F}_2\triangleq\{z\in\mathcal{F}|\int_0^\infty|z(t)|dt<\infty\}.$$

(ii) Let $\mathcal{F}_2^0\subseteq\mathcal{F}_2$ be a subset with $\int_0^\infty\sup_{z\in\mathcal{F}_2^0}|z(t)|dt<\infty$. Then system (1) with control rule (2) achieves a GAC with respect to $\mathcal{F}_2^0$ if $\mathcal{G}_{\sigma(t)}$ is UQSC.

Moreover, the following estimation was also obtained on the convergence rates in Part I of the paper [16]. When $\mathcal{G}_{\sigma(t)}$ is UQSC, we have

$$\mathcal{H}(x(t))\leq(1-\alpha_{N-1})^{\lfloor\frac{t}{K_0}\rfloor}\mathcal{H}(x^0)+(4N-3)\int_0^t(1-\alpha_{N-1})^{\lfloor\frac{t}{K_0}\rfloor-g(\tau)}|w(\tau)|d\tau \quad (3)$$

where $K_0=(N-1)^2\hat{T}$ with $\hat{T}=T_0+2\tau_D$, $0<\alpha_{N-1}<1$ are two constants, and

$$g(\tau)=\begin{cases}i+1, & \tau\in[iK_0,(i+1)K_0), i=0,\dots,\lfloor\frac{t}{K_0}\rfloor-1\\ \lfloor\frac{t}{K_0}\rfloor, & \tau\in[\lfloor\frac{t}{K_0}\rfloor\cdot K_0,t]\end{cases} \quad (4)$$

On the other hand, when $\mathcal{G}_{\sigma(t)}$ is USC, similar estimation can also be given by

$$\mathcal{H}(x(t))\leq(1-\alpha_{N-1}^*)^{\lfloor\frac{t}{K_*}\rfloor}\mathcal{H}(x^0)+(4N-3)\int_0^t(1-\alpha_{N-1}^*)^{\lfloor\frac{t}{K_*}\rfloor-g(\tau)}|w(\tau)|d\tau \quad (5)$$

where $K_*=(N-1)\hat{T}$, $0<\alpha_{N-1}<\alpha_{N-1}^*<1$ are constants, and

$$g(\tau)=\begin{cases}i+1, & \tau\in[iK_*,(i+1)K_*), i=0,\dots,\lfloor\frac{t}{K_*}\rfloor-1\\ \lfloor\frac{t}{K_*}\rfloor, & \tau\in[\lfloor\frac{t}{K_*}\rfloor\cdot K_*,t]\end{cases} \quad (6)$$

D. Event-Triggered Coordination

Control laws via event-triggered or self-triggered approaches are piecewise constant inputs which update the value when certain events are executed [31], [32].

In this paper, we study the trigger condition in the following. Let $t_1^i<t_2^i<\cdots<t_k^i<\dots$ be the time sequence when agent i is triggered. Denote $e_i(t)\triangleq x_i(t)-x_i(t_k^i)$ as the state measurement error for node i.

Let $t_1^i=t_0$. Having got t_k^i, t_{k+1}^i is determined by the solution of the following equation:

$$|e_i(t)|=\delta(t), \quad (7)$$

where $\delta(t):\mathbb{R}_{\geq 0}\to\mathbb{R}_{>0}$ is a given function.

In the next two sections, we will discuss neighbor-synchronous and asynchronous updating rule respectively, which study event-triggered coordination under two different communication protocols.

III. Neighbor-Synchronous Coordination

In this section, we study a class of self-triggered coordination rule in which each agent will update its control input when it is triggered or its neighbor updates the control.

Denote $\hat{a}_{ij}(k)=a_{ij}(x(t_k^i),t_k^i)$. Then the control input for agent $i, i=1,\dots,N$ is defined in the following:

$$u_i(t)=\sum_{j\in\mathcal{N}_i(\sigma(t))}\hat{a}_{ij}(k)(x_j(t^j_{\mathcal{T}_j(t)}))-x_i(t_k^i)),\ t\in[t_k^i,t_{k+1}^i) \quad (8)$$

where $\mathcal{T}_j(t)\triangleq\arg\max_l\{t_l^j|t_l^j\leq t\}$ for $j=1,\dots,N$.

Remark 3.1: With (8), agent i will update the control input on the time instants when it is triggered, or its neighbors are changing or triggered. However, it is not hard to see that $u_i(t)$ is still piece-wise constant.

The communications of protocol (8) over the network can be described in the following way: (i) Each agent i broadcasts its state $x_i(t_k^i)$ during $[t_k^i,t_{k+1}^i)$ until it is triggered another time. (ii) Agent i's neighbors update the parts related to i's state in their control inputs once they receive the broadcasting states. (iii) The control inputs also updates synchronously as the communication graph switches.

Denote $\hat{w}_i(t)=\sum_{j\in\mathcal{N}_i(\sigma(t))}\hat{a}_{ij}(k)(e_i(t)-e_j(t))$. Then (8) can be transformed into the following form:

$$u_i(t)=\sum_{j\in\mathcal{N}_i(\sigma(t))}\hat{a}_{ij}(k)(x_j(t)-x_i(t))+\hat{w}_i(t). \quad (9)$$

Noting the fact that

$$|\hat{w}_i(t)|\leq\sum_{j\in\mathcal{N}_i(\sigma(t))}\hat{a}_{ij}(k)|e_i(t)+e_j(t)|\leq 2(N-1)a^*|\delta(t)|. \quad (10)$$

we obtain following conclusions immediately based on Propositions 2.1 and 2.2.

Theorem 3.1: (i) System (1) with control law (8) achieves a GC for any $\delta \in \mathcal{F}_1 \cup \mathcal{F}_2$ if $\mathcal{G}_{\sigma(t)}$ is UQSC.

(ii) System (1) with control law (8) achieves a GAC with respect to $\delta \in \mathcal{F}_1^0 \cup \mathcal{F}_2^0$ if $\mathcal{G}_{\sigma(t)}$ is UQSC.

Theorem 3.2: Assume that either $\mathcal{G}_{\sigma(t)}$ being undirected for any $t \geq 0$, or $\mathcal{G}([0, +\infty))$ being acyclic. Then System (1) with control law (8) achieves a GC for any $\delta \in \mathcal{F}_2$ if $\mathcal{G}([t, \infty))$ is QSC for any $t \geq 0$.

Furthermore, denote $\tau_{k+1}^i \triangleq t_{k+1}^i - t_k^i,\ k = 0, 1, \ldots, i = 1, \ldots, N$ as the spaces between two trigged time instants for each agent, and denote $\tau_0 \triangleq \min_i \inf_k \{\tau_{k+1}^i\}$ as their lower bound. Then *Zeno* behavior [30] is avoided if $\tau_0 > 0$.

The following conclusion holds.

Theorem 3.3: (i) Assume that $\mathcal{G}_{\sigma(t)}$ is UQSC and $\delta(t) = c_0 e^{-\lambda t}$ with $c_0 > 0$. Then System (1) with control law (8) achieves a GAC with $\tau_0 > 0$ if $0 < \lambda < -\frac{\ln(1-\alpha_{N-1})}{K_0}$.

(ii) Assume that $\mathcal{G}_{\sigma(t)}$ is USC and $\delta(t) = c_0 e^{-\lambda t}$ with $c_0 > 0$. Then System (1) with control law (8) achieves a GAC with $\tau_0 > 0$ if $0 < \lambda < -\frac{\ln(1-\alpha_{N-1}^*)}{K_*}$.

Proof: The proof of (i) results from (3), and (ii) can be obtained in the same way based on (5). Therefore, we just focus on part (i) of the conclusion. We just have to prove $\tau_0 > 0$. With (18), we have

$$|u_i(t)| \leq (N-1)a^*\mathcal{H}(x(t)) + 2(N-1)a^*\delta(t), \quad i = 1, \ldots, N. \tag{11}$$

Then according to the event condition (7), (11) will lead to

$$\begin{aligned}\tau_{k+1}^i &= \frac{\delta(t_{k+1}^i)}{|u_i(t_k^i)|} \\ &\geq \frac{\delta(t_k^i)}{(N-1)a^*\mathcal{H}(x(t_k^i)) + 2(N-1)a^*\delta(t_k^i)} \cdot e^{-\lambda \tau_{k+1}^i}.\end{aligned} \tag{12}$$

Furthermore, combing (3) and (10), one has

$$\begin{aligned}\mathcal{H}(x(t)) \leq\ & (1-\alpha_{N-1})^{\lfloor \frac{t}{K_0} \rfloor}\mathcal{H}(x^0) + 2(N-1) \\ & \cdot (4N-3)a^* \int_0^t (1-\alpha_{N-1})^{\lfloor \frac{t}{K_0} \rfloor - g(\tau)}\delta(\tau)d\tau.\end{aligned}$$

Therefore, denoting $F(t) \triangleq \frac{\delta(t)}{(N-1)a^*\mathcal{H}(x(t)) + 2(N-1)a^*\delta(t)}$, we obtain

$$\begin{aligned}F(t) \geq\ & \frac{\delta(t)}{(1-\alpha_{N-1})^{\lfloor \frac{t}{K_0} \rfloor}\mathcal{H}(x^0) + g(t) + 2\delta(t)} \\ & \cdot \frac{1}{(N-1)a^*},\end{aligned} \tag{13}$$

where

$$g(t) = 2(N-1)(4N-3)a^* \int_0^t (1-\alpha_{N-1})^{\lfloor \frac{t}{K_0} \rfloor - g(\tau)}\delta(\tau)d\tau.$$

Furthermore, recalling that $(1-\alpha_{N-1})^{\lfloor \frac{t}{K_0} \rfloor} \leq c_* e^{-\lambda_* t}$, where $c_* = \frac{1}{1-\alpha_{N-1}}$ and $\lambda_* = -\frac{\ln(1-\alpha_{N-1})}{K_0}$, and also noticing the fact that $(1-\alpha_{N-1})^{-g(\tau)} \leq c_* e^{\lambda_* \tau}$, we have

$$\begin{aligned}F(t) &\geq \frac{\delta(t)}{c_* e^{-\lambda_* t}\mathcal{H}(x^0) + g_0 e^{-\lambda_* t}\int_0^t e^{\lambda_* \tau}\delta(\tau)d\tau + 2\delta(t)} \\ &\quad \cdot \frac{1}{(N-1)a^*} \\ &= \frac{c_0}{c_* e^{(\lambda-\lambda_*)t}\mathcal{H}(x^0) + \frac{g_0 c_0}{\lambda-\lambda_*}(e^{(\lambda-\lambda_*)t} - 1) + 2c_0} \\ &\quad \cdot \frac{1}{(N-1)a^*} \\ &\geq \frac{c_0}{c_*\mathcal{H}(x^0) + \frac{g_0 c_0}{\lambda_*-\lambda} + 2c_0} \cdot \frac{1}{(N-1)a^*} \\ &\triangleq M_*.\end{aligned} \tag{14}$$

where $g_0 = 2(N-1)(4N-3)a^* c_*^2$. As a result, (12) leads to

$$\tau_{k+1}^i \geq M_* e^{-\lambda \tau_{k+1}^i}, \tag{15}$$

which implies

$$\tau_0 \geq M_* e^{-\lambda \tau_0} \tag{16}$$

immediately since τ_{k+1}^i is arbitrarily chosen in (15). Then it is not hard to find $\tau_0 \geq m_*$, where $m_* > 0$ is the unique solution of equation $y = M_* e^{-\lambda y}$. This completes the proof. □

IV. Asynchronous Event-Triggered Coordination

In this subsection, we consider asynchronous self-triggered coordination when the agents' control updating no longer synchronize with the neighbors.

To be precise, an asynchronous self-triggered coordination rule should include the following properties.

(i) (Broadcasting) Each agent i broadcasts its state $x_i(t_k^i)$ during $[t_k^i, t_{k+1}^i)$ until it is triggered another time at t_{k+1}^i.

(ii) (Receiving) Agent j can receive $x_i(t_k^i)$ if and only if there exists a time $t_1 \in [t_k^i, t_{k+1}^i)$ such that i is a neighbor of j at time t_1. Moreover, agent j can store this message until another message from i is received.

(iii) (Updating) Each agent i updates its control input at time $x_i(t_k^i)$ with $k \geq 2$ once it is triggered, based on the messages it receives from the neighbor set $\hat{\mathcal{N}}_i(k) \doteq \cup_{t \in [t_{k-1}^i, t_k^i)}\mathcal{N}_i(\sigma(t))$.

Note that, when the upper restrictions are satisfied, the control input of an agent i may equals 0 at some time t_k^i, and then it will never be triggered again according to the trigger condition (7). Consequently, a global consensus will not be achieved. In order to avoid this, we have to modify the definition of the event condition. Having got t_k^i, we redefine the solution of (7) as $\hat{t}_{k+1}^i$, and another assumption is given as follows for the definition of t_{k+1}^i.

A4. (Forcing Waking Up) There is a constant L^* such that

$$(i)\ t_{k+1}^i = \hat{t}_{k+1}^i \ if\ \hat{t}_{k+1}^i - t_k^i \leq L^*;$$

$$(ii)\ t_{k+1}^i = t_k^i + L^* \ if\ \hat{t}_{k+1}^i - t_k^i > L^*.$$

We present the following distributed asynchronous self-triggered coordination rule:

$$u_i(t) = \sum_{j\in\hat{\mathcal{N}}_i(k)} \hat{a}_{ij}(k)(x_j(t^j_{\mathcal{T}_i^j(k)}) - x_i(t^i_k)),\ t\in[t^i_k, t^i_{k+1}), \tag{17}$$

where $\mathcal{T}_i^j(k);\ i,j = 1,\dots,N$ are defined by $\mathcal{T}_i^j(k) \triangleq \max_l\{l|t^j_l \le T^*_{ij}(k)\}$ with $T^*_{ij}(k) \triangleq \max_t\{t\in[t^i_{k-1}, t^i_k)|j\in \mathcal{N}_i(\sigma(t))\}$. It is not hard to see (17) satisfies properties $(i)-(iii)$.

Remark 4.1: In [12], an asynchronous consensus protocol is studied, where each node independently updates its state at times determined by its own clock and each node's position between two event times is formulated as a given piece-wise continuous signal. In (17), each node also independently updates its control by its own clock. From this point, we have the same meaning by saying "asynchronous" as in [12]. Here using piece-wise constant control, the trajectory of each node is a linear function between two event times.

Denote

$$\begin{aligned}\tilde{w}_i(t) = &\sum_{j\in\hat{\mathcal{N}}_i(k)} \hat{a}_{ij}(k)(e_i(t) - e_j(t))\\ &+ \sum_{j\in\hat{\mathcal{N}}_i(k)} \hat{a}_{ij}(k)(x_j(t^j_{\mathcal{T}_i^j(k)}) - x_j(t^j_{\mathcal{T}_j(t)})).\end{aligned}$$

Then (17) can be transformed into the following form:

$$u_i(t) = \sum_{j\in\hat{\mathcal{N}}_i(k)} \hat{a}_{ij}(k)(x_j(t) - x_i(t)) + \tilde{w}_i(t). \tag{18}$$

Then we propose our main result on asynchronous self-triggered coordination.

Theorem 4.1: Assume that $\delta(t) = c_0e^{-\lambda t}$ with $0<\lambda< -\frac{\ln(1-\alpha_{N-1})}{K_0}$, and L^* is chosen to satisfy the following inequality:

$$2L^*e^{2\lambda L^*}[\frac{(N-1)(4N-3)a^*c_*^2}{\lambda_*-\lambda}+1](N-1)a^* < 1. \tag{19}$$

Then System (1) with control law (17) achieves a GAC with $\tau_0>0$ if $\mathcal{G}_{\sigma(t)}$ is UQSC.

Proof: We define a function

$$M(t) \triangleq \inf\{\tau^i_{k+1}|t^i_{k+1}<t, i=1,\dots,N; k=0,\dots\}$$

as the lower bound for the inter-event times before time t. Then $M(t)$ is obviously non-increasing. Thus, based on the definition of $\mathcal{T}_i^j(k)$, it is not hard to find that every agent $j\in\hat{\mathcal{N}}_i(k)$ is triggered as many as $\frac{L^*}{M(t)}$ times during time interval $[t^i_{k-1}, t^i_{k+1})$ for $t^i_{k+1}\le t$.

Suppose $\delta(t) = c_0e^{-\lambda t}$, and therefore we have

$$|x_j(t^j_{\mathcal{T}_i^j(k)}) - x_j(t^j_{\mathcal{T}_j(t)})| \le \frac{2L^*}{M(t)}\delta(t^i_{k-1}) \le \frac{2L^*}{M(t)}\delta(t)e^{\lambda 2L^*}$$

for any $j\in\hat{\mathcal{N}}_i(k)$ and $t\in[t^i_{k-1}, t^i_{k+1})$, which leads to

$$|\tilde{w}_i(t)| \le (N-1)a^*[2+\frac{2L^*e^{2\lambda L^*}}{M(t)}]\delta(t). \tag{20}$$

Noting the fact that in the communication graph defined by $\hat{\mathcal{N}}_i(k), k=1,\dots$ for every agent i, every arc exists longer than in the graph $\mathcal{G}_{\sigma(t)}$, we can also obtain

$$\begin{aligned}\mathcal{H}(x(t)) \le\ &(1-\alpha_{N-1})^{\lfloor\frac{t}{K_0}\rfloor}\mathcal{H}(x^0) + (N-1)(4N-3)a^*\\ &\int_0^t (1-\alpha_{N-1})^{\lfloor\frac{t}{K_0}\rfloor - g(\tau)}[2+\frac{2e^{2\lambda L^*}L^*}{M(\tau)}]\delta(\tau)d\tau.\end{aligned} \tag{21}$$

Moreover, since $M(t)$ is non-increasing, (21) leads to

$$\begin{aligned}\mathcal{H}(x(t)) \le\ &(1-\alpha_{N-1})^{\lfloor\frac{t}{K_0}\rfloor}\mathcal{H}(x^0) + (N-1)(4N-3)a^*\\ &[2+\frac{2e^{2\lambda L^*}L^*}{M(t)}]\int_0^t (1-\alpha_{N-1})^{\lfloor\frac{t}{K_0}\rfloor - g(\tau)}\delta(\tau)d\tau.\end{aligned}$$

Therefore, based on similar analysis by which we obtain (15), it is not hard to obtain that for $i=1,\dots,N$ and $k=0,1,\dots,$

$$\begin{aligned}&\tau^i_{k+1}\\ &\ge \frac{c_0M(t^i_k)}{[c_*\mathcal{H}(x^0)+\frac{g_0c_0}{\lambda_*-\lambda}+2c_0]M(t^i_k) + [\frac{g_0c_0}{\lambda_*-\lambda}+2c_0]L^*e^{2\lambda L^*}}\\ &\cdot\frac{1}{(N-1)a^*}e^{-\lambda\tau^i_{k+1}},\end{aligned} \tag{22}$$

where $g_0 = (N-1)(4N-3)a^*c_*^2$. Next, we prove $\tau_0>0$ by contradiction. Assume that $\tau_0=0$. Then we have $\lim_{t\to\infty}M(t)=0$. Therefore, for any fixed number $0<\mu<1$, there exists $N_1>0$ such that when $k>N_1$, one has

$$\tau^i_{k+1} \ge \frac{M(t^i_k)}{[\frac{g_0}{\lambda_*-\lambda}+2](N-1)a^*L^*e^{2\lambda L^*}}\cdot\mu e^{-\lambda\tau^i_{k+1}}. \tag{23}$$

It is not hard to see that if $\tau^i_{k+1}\ge M(t^i_k)$ for all i and $k>N_1$, then $M(t^i_k)$ is nondecreasing when $k>N_1$, and therefore trivially we have $\tau_0>0$. Otherwise, there has to be $\tau^{i_0}_{k_0+1}\to 0$ as k_0 tends to infinity such that $\tau^{i_0}_{k_0+1} = M(t^{i_0}_{k_0}+\tau^{i_0}_{k_0+1})$. According to (19), choosing k_0 sufficiently large to enforce

$$\frac{1}{[\frac{g_0}{\lambda_*-\lambda}+2](N-1)a^*L^*e^{2\lambda L^*}}\cdot\mu e^{-\lambda\tau^{i_0}_{k_0+1}} > 1,$$

(23) will lead to

$$M(t^{i_0}_{k_0}+\tau^{i_0}_{k_0+1}) > M(t^{i_0}_{k_0}), \tag{24}$$

which contradicts the fact that $M(t)$ is non-increasing. Therefore, we have proved that $\tau_0>0$.

As a result, we finally obtain

$$|\hat{w}_i(t)| \le (N-1)a^*[2+\frac{2L^*e^{2\lambda L^*}}{\tau_0}]\delta(t), \tag{25}$$

which guarantees GAC for system (1) immediately according to Proposition 2.2. This completes the proof. □

Similarly, we also have the following conclusion for the USC case, whose proof is omitted.

Theorem 4.2: Assume that $\delta(t) = c_0e^{-\lambda t}$ with $0<\lambda<\hat{\lambda}_*$, where $\hat{\lambda}_* = -\frac{\ln(1-\alpha^*_{N-1})}{K_*}$, and L^* is chosen to satisfy the following inequality:

$$2L^*e^{2\lambda L^*}[\frac{(N-1)(4N-3)a^*\hat{c}_*^2}{\hat{\lambda}_*-\lambda}+](N-1)a^* < 1, \tag{26}$$

where $\hat{c}_* = \frac{1}{1-\alpha^*_{N-1}}$. Then System (1) with control law (17) achieves a GAC with $\tau_0 > 0$ if $\mathcal{G}_{\sigma(t)}$ is USC.

V. Conclusions

The paper studied event-triggered coordination for multi-agent systems with directed switching communication graphs. Both neighbor-synchronous and asynchronous updating rules are investigated, and proper conditions on trigger function and connectivity were proposed for the system to each a consensus. In practice, multi-agent systems reaching a consensus with event-triggered control under communication constraints deserves more attention since in many cases the communication costs can be reduced using event-triggered feedback over the networks.

References

[1] A.F. Filippov, *Differential Equations with Discontinuous Righthand Sides*. Norwell, MA: Kluwer, 1988.

[2] C. Godsil and G. Royle. *Algebraic Graph Theory.* New York: Springer-Verlag, 2001.

[3] C. Berge and A. Ghouila-Houri. *Programming, Games, and Transportation Networks*, John Wiley and Sons, New York, 1965.

[4] D. Cheng, J. Wang, and X. Hu, An extension of LaSalle's invariance principle and its applciation to multi-agents consensus, *IEEE Trans. Automatic Control*, 53, 1765-1770, 2008.

[5] S. Martinez, J. Cortés, and F. Bullo. Motion coordination with distributed information, *IEEE Control Systems Magazine*, vol. 27, no. 4, 75-88, 2007.

[6] W. Ren and R. Beard. *Distributed Consensus in Multi-vehicle Cooperative Control*, Springer-Verlag, London, 2008.

[7] W. Ren and R. Beard. Consensus seeking in multiagent systems under dynamically changing interaction topologies. *IEEE Transactions on Automatic Control*, 50(5), 655-661, 2005.

[8] R. Olfati-Saber, Flocking for multi-agent dynamic systems: algorithms and theory, *IEEE Trans. Automatic Control*, 51(3): 401-420, 2006.

[9] M. Cao, D. A. Spielman and A. S. Morse. A lower bound on convergence of a distributed network consensus algorithm. *IEEE CDC*, 2356-2361, 2005.

[10] M. Cao, A. S. Morse and B. D. O. Anderson. Reaching a consensus in a dynamically changing environment: a graphical approach. *SIAM J. Control Optim.*, vol. 47, no. 2, 575-600, 2008.

[11] M. Cao, A. S. Morse and B. D. O. Anderson. Reaching a consensus in a dynamically changing environment: convergence rates, measurement delays, and asynchronous events. *SIAM J. Control Optim.*, vol. 47, no. 2, 601-623, 2008.

[12] M. Cao, A. S. Morse and B. D. O. Anderson. Agreeing Asynchronously. *IEEE Trans. Automatic Control*, vol. 53, no. 8, 1826-1838, 2008.

[13] Y. Hong, L. Gao, D. Cheng, and J. Hu. Lyapuov-based approach to multi-agent systems with switching jointly connected interconnection. *IEEE Trans. Automatic Control*, vol. 52, 943-948, 2007.

[14] J. Fax and R. Murray. Information flow and cooperative control of vehicle formations. *IEEE Trans. Automatic Control*, vol. 49, no. 9, 1465-1476, 2004.

[15] R. Olfati-Saber and R. Murray. Consensus problems in the networks of agents with switching topology and time dealys, *IEEE Trans. Automatic Control*, vol. 49, no. 9, 1520-1533, 2004.

[16] G. Shi and K. H. Johansson, Multi-agent Robust Consensus-Part I: Convergence Analysis, submitted to the 50th IEEE Conference on Decision and Control and European Control Conference.

[17] H. G. Tanner, A. Jadbabaie, G. J. Pappas, Flocking in fixed and switching networks, *IEEE Trans. Automatic Control*, 52(5): 863-868, 2007.

[18] A. Jadbabaie, J. Lin, and A. S. Morse. Coordination of groups of mobile agents using nearest neighbor rules. *IEEE Trans. Automatic Control*, vol. 48, no. 6, 988-1001, 2003.

[19] G. Shi and Y. Hong, Global target aggregation and state agreement of nonlinear multi-agent systems with switching topologies, *Automatica*, vol. 45, 1165-1175, 2009.

[20] J. N. Tsitsiklis, D. Bertsekas, and M. Athans. Distributed asynchronous deterministic and stochastic gradient optimization algorithms, *IEEE Trans. Automatic Control*, 31, 803-812, 1986.

[21] A. Nedić, A. Olshevsky, A. Ozdaglar, and J. N. Tsitsiklis. On distributed averaging algorithms and quantization effects. *IEEE Trans. Automatic Control*, 54, 11, 25062517, 2009.

[22] Z. Lin, B. Francis, and M. Maggiore. State agreement for continuous-time coupled nonlinear systems. *SIAM J. Control Optim.*, vol. 46, no. 1, 288-307, 2007.

[23] L. Moreau, Stability of multiagent systems with time-dependent communication links, *IEEE Trans. Automatic Control*, 50, 169-182, 2005.

[24] L. Wang and L. Guo. Robust consensus and soft control of multi-agent systems with noises. *Journal of Systems Science and Complexity*, vol.21, no.3, 406-415, 2008.

[25] L. Wang and Z. Liu. Robust consensus of multi-agent systems with noise. *Science in China, Series F: Information Sciences*, vol. 52, no. 5, 824-834, 2009.

[26] K.J. Åström and B. Bernhardsson. Comparison of Riemann and Lebesgue sampling for first order stochastic systems. *The 41st IEEE Conference on Decision and Control*, 2011-2016, 2002.

[27] X. Wang and M.D. Lemmon. Event-triggered broadcasting across distributed networked control systems. American Control Conference, 2008.

[28] X. Wang and M. Lemmon. Self-triggered feedback control systems with finite-gain L_2 stability. *IEEE Transactions on Automatic Control*, 54(3):452467, 2009.

[29] P. Tabuada. Event-triggered real-time scheduling of stabilizing control tasks. *IEEE Transactions on Automatic Control*, 52(9):1680-1685, 2007.

[30] K.H. Johansson, M. Egerstedt, J. Lygeros, and S.S. Sastry. On the regularization of Zeno hybrid automata. *Systems and Control Letters*, 38:141150, 1999.

[31] D. Dimarogonas and K. Johansson. Event-triggered control for multi-agent systems. The 48th IEEE Conference on Decision and Control, Shanghai, China, 7131-7136, 2009.

[32] G. Seyboth, D. Dimarogonas and K. Johansson. Control of multi-agent systems via event-based communication. To appear. IFAC World Congress, Milan, 2011.