2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Observer-Based Generalized Asymptotic Regulation with Sub-optimal Transient Response

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Abstract— **The generalized asymptotic regulation problem is considered with an observer-based controller. Two different sets of LMI conditions are derived for the optimization of the transient behavior. These conditions are potentially conservative, as would usually be the case for restricted-structure controllers. Nevertheless, it is illustrated by a design example that the performance of the observer-based controller can be quite close to the controller with a general structure.**

I. INTRODUCTION

The asymptotic output regulation problem aims at exact cancelation of infinite-energy disturbances (or tracking of infinite-energy references) in steady-state. It is well worked and even subsumed to multi-objective problems in which additional performance ($\mathscr{H}_\infty$, $\mathscr{H}_2$) indices are minimized (see [9] and the references therein). For sinusoidal disturbances, it is possible to generalize the problem formulation in a meaningful way so that the objective is relaxed to asymptotic attenuation down to a desirable level which need not be zero. In some recent works, solutions are provided for this generalized asymptotic regulation problem under additional performance objectives based on linear matrix inequality (LMI) optimization [3], [4]. The generalized problem formulation becomes especially suitable for uncertain parameter-dependent systems. Synthesis procedures are developed for the control of such systems as well by the use of controllers that are scheduled with the online measurements of the uncertain parameters [5], [6].

The controller used in [3], [4], [6] for generalized asymptotic regulation is formed by a particular combination of the internal model that generates the disturbances and a controller with general structure. In fact, it just imitates the structure of the controller used in exact asymptotic regulation [9]. In spite of this structural restriction on the controller, one can derive convex constraints that enforce the satisfaction of additional performance objectives. It is also possible to solve the generalized asymptotic regulation problem by an observer-based controller whose structure is the same as the one used in the exact asymptotic regulation problem. An observer-based controller is naturally of interest since its state vector can serve as an estimate of the system state vector. Moreover, when the problem is considered for a parameter-dependent plant, an observer-based controller has the advantage of having no dependence on the parameter derivative, in addition to its relatively simpler structure (see and cf. [6]). Nevertheless, when the structure of the controller is fixed to be observer-based, it becomes difficult (if not impossible) to find nonconservative conditions for the satisfaction of additional performance objectives.

In this paper, we consider the synthesis of observer-based controllers for generalized asymptotic regulation with sub-optimal transient response. After formulating the basic generalized asymptotic regulation problem in the next section, we summarize its observer-based solution in Section III. The main contributions of the paper are presented in Section IV, where we consider optimizing the transient response when the controller is restricted to be observer-based. With the Riccati equation approach being not suitable due to the extended plant eigenvalues on the imaginary axis, we derive two alternative sets of LMI conditions for a (potentially) conservative solution of the problem. The proposed solutions are evaluated in Section V by an example synthesis for rudder roll stabilization and sinusoidal wave disturbance cancelation.

II. GENERALIZED ASYMPTOTIC REGULATION

We start by describing the problem setup and formulating the generalized asymptotic regulation problem. Consider a linear time-invariant (LTI) plant

$$\Sigma_\mathrm{p}: \left[\begin{array}{c} \dot{x} \\ \hline e \\ y \end{array}\right] = \left[\begin{array}{c|cc} A & B_\mathrm{r} & B \\ \hline C_\mathrm{r} & D_\mathrm{r} & D_\mathrm{rc} \\ C & D_\mathrm{cr} & 0 \end{array}\right] \left[\begin{array}{c} x \\ \hline v \\ u \end{array}\right], \tag{1}$$

where $x(t) \in \mathbb{R}^k$ denotes the state and $u(t) \in \mathbb{R}^n$ represents the control input. The basic control goal is to use the measurement vector $y(t) \in \mathbb{R}^m$ to regulate the error output $e(t) \in \mathbb{R}^r$ against an infinite-energy disturbance $v(t) \in \mathbb{R}^l$. The disturbance is generated by the exogenous system

$$\Sigma_\mathrm{e}: \ \dot{v} = A_\mathrm{e} v. \tag{2}$$

By appending the dynamics of the exo-system to the dynamics of the plant, we express the dynamics of the extended plant as

$$\begin{aligned} \dot{\tilde{x}} &= \underbrace{\begin{bmatrix} A & B_\mathrm{r} \\ 0 & A_\mathrm{e} \end{bmatrix}}_{\tilde{A}} \underbrace{\begin{bmatrix} x \\ v \end{bmatrix}}_{\tilde{x}} + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{\tilde{B}} u, \\ y &= \underbrace{\begin{bmatrix} C & D_\mathrm{cr} \end{bmatrix}}_{\tilde{C}} \tilde{x}. \end{aligned} \tag{3}$$

*The author acknowledges the support of King Fahd University of Petroleum and Minerals (KFUPM).

The standing assumptions of the paper are as follows:

A.1. A_{e} is skew-symmetric:

$$\mathfrak{He}A_{\mathrm{e}} \triangleq A_{\mathrm{e}} + A_{\mathrm{e}}^T = 0. \quad (4)$$

This implies that $\|v(t)\| = \|v(0)\|, \forall t \geq 0$.

A.2. (A,B) is stabilizable: There exists a matrix $F \in \mathbb{R}^{n \times k}$ with which $A+BF$ is Hurwitz (i.e. all of its eigenvalues have strictly negative real parts).

A.3. $(\tilde{A},\tilde{C})$ is detectable: There exists a matrix $G \in \mathbb{R}^{(k+l)\times m}$ with which $\tilde{A}+G\tilde{C}$ is Hurwitz.

A.1 restricts our attention to the generators of multi-sinusoidal disturbances. A.2 is necessary for the existence of a stabilizing controller, while A.3 can be replaced by the detectability of (A,C). Nevertheless, a reduction technique can be employed to reformulate the problem with a new plant-exo-system pair that satisfies A.3 when (A,C) is detectable (see [9]). This is to be applied when the plant has common eigenvalues with the exo-system.

Within this setting, the basic generalized asymptotic regulation problem is formulated as follows:

Problem 1: Given a plant Σ_{p} and an exogenous system Σ_{e} that fulfill the assumptions A.1-A.3, design a controller

$$\Sigma_{\mathrm{c}}: \left[\begin{array}{c} \dot{\xi} \\ \hline u \end{array}\right] = \left[\begin{array}{c|c} A_{\mathrm{c}} & B_{\mathrm{c}} \\ \hline C_{\mathrm{c}} & D_{\mathrm{c}} \end{array}\right] \left[\begin{array}{c} \xi \\ \hline y \end{array}\right], \quad (5)$$

such that:

C.1. (**Internal Stability**) The feedback system formed by Σ_{p} and Σ_{c} is asymptotically stable, i.e. when $v(0)=0$

$$\lim_{t\to\infty} \|e(t)\| = 0, \text{ for any } x(0) \in \mathbb{R}^k, \xi(0) \in \mathbb{R}^{k+l}. \quad (6)$$

C.2. (**Generalized Asymptotic Regulation**) When $v(0) \neq 0$, the closed-loop system satisfies

$$\limsup_{t\to\infty} \|e(t)\| \leq \kappa \|\Phi^{1/2} v(0)\|, \quad (7)$$

where Φ is a given symmetric positive-semi-definite attenuation profile and $\kappa \geq 0$ represents the desired level of steady-state attenuation.

We first note that this is a generalized asymptotic regulation problem since κ is allowed to be greater than zero. The attenuation profile Φ is introduced to add extra flexibility into the problem formulation. The simplest choice would clearly be $\Phi = I$. When A_{e} has a block-diagonal structure, with which multi-sinusoidal disturbances can be generated, it might be preferable to emphasize the attenuation level at certain frequencies. Recall that a sinusoid of frequency ω_i can be generated by

$$\dot{v}_i = \underbrace{\begin{bmatrix} 0 & -\omega_i \\ \omega_i & 0 \end{bmatrix}}_{A_{\mathrm{e}}^i} \underbrace{\begin{bmatrix} v_{i1} \\ v_{i2} \end{bmatrix}}_{v_i}, \; v_i(0) = \begin{bmatrix} v_{i1}^0 \\ v_{i2}^0 \end{bmatrix}. \quad (8)$$

The state of this exo-system can be obtained explicitly as

$$v_i(t) = \begin{bmatrix} \cos(\omega_i t) & -\sin(\omega_i t) \\ \sin(\omega_i t) & \cos(\omega_i t) \end{bmatrix} \begin{bmatrix} v_{i1}^0 \\ v_{i2}^0 \end{bmatrix}. \quad (9)$$

For $A_{\mathrm{e}} = \mathsf{blockdiag}\,(A_{\mathrm{e}}^i)_{i=1}^p$, one might use $\Phi = \mathsf{blockdiag}\,(\alpha_i I)_{i=1}^p$, where α_i's represent nonnegative scalars. For purposes of normalization, it would be convenient to set one of the α_i's to 1. The others should then be chosen to reflect the emphasis given to the attenuation of the i'th frequency component in the error. Note that smaller α_i values will lead to relatively more attenuation of the i'th frequency component. When exact asymptotic regulation is required for a particular frequency, the associated α_i should be chosen as zero.

III. Observer-Based Control for Generalized Asymptotic Regulation

In this section we summarize the observer-based solution to the generalized asymptotic regulation problem. The controller structure is actually the same as those employed for solving the exact asymptotic regulation problem for linear time-invariant as well as time-varying systems [9], [2]. An observer-based controller is to be constructed by integrating an observer into the solution of the full-information problem.

The full-information problem assumes that the state vectors of the plant and exo-system are both available for synthesis. In order to derive a solution for this case, one introduces a design matrix $\Pi \in \mathbb{R}^{k\times l}$, in terms of which a new state variable is defined as

$$\varsigma \triangleq \underbrace{\begin{bmatrix} I & \Pi \end{bmatrix}}_{\tilde{\Pi}} \underbrace{\begin{bmatrix} x \\ v \end{bmatrix}}_{\tilde{x}}. \quad (10)$$

The evolution of the plant state and the error to be regulated can be expressed equivalently in terms of this new state vector as

$$\begin{aligned} \dot{\varsigma} &= A\varsigma + (B_{\mathrm{r}} + \Pi A_{\mathrm{e}} - A\Pi)v + Bu, && (11) \\ e &= C_{\mathrm{r}}\varsigma + (D_{\mathrm{r}} - C_{\mathrm{r}}\Pi)v + D_{\mathrm{rc}}u. && (12) \end{aligned}$$

When the state of the extended plant, i.e. $\tilde{x}$, and thus x and v are available, one can construct the control input as

$$u = F\varsigma - \Gamma v = \underbrace{\begin{bmatrix} F & F\Pi - \Gamma \end{bmatrix}}_{\tilde{F}} \underbrace{\begin{bmatrix} x \\ v \end{bmatrix}}_{\tilde{x}} \quad (13)$$

where F and Γ represent the matrices to be designed. With this control input, the dynamics of the plant are modified as

$$\begin{aligned} \dot{\varsigma} &= (A+BF)\varsigma + \underbrace{(B_{\mathrm{r}} + \Pi A_{\mathrm{e}} - A\Pi - B\Gamma)}_{Z} v, && (14) \\ e &= (C_{\mathrm{r}} + D_{\mathrm{rc}}F)\varsigma + \underbrace{(D_{\mathrm{r}} - C_{\mathrm{r}}\Pi - D_{\mathrm{rc}}\Gamma)}_{\Lambda} v. && (15) \end{aligned}$$

In order stabilize the plant, one clearly needs to choose F such that $A+BF$ is Hurwitz. This can simply be realized by solving the LMI

$$\mathfrak{He}(AY + BN) \preccurlyeq -2\rho Y \prec 0, \quad (16)$$

over $Y \in \mathbb{S}_+^k$ and $N \in \mathbb{R}^{n\times k}$, by setting $\rho \in \mathbb{R}_+$ to a sufficiently small value. A state-feedback can then be constructed as

$$F = NY^{-1}. \quad (17)$$

The basic idea behind the choices of Π and Γ is to render $Z=0$. This will then ensure $\lim_{t\to\infty}\|\varsigma(t)\|=0$, which means that e will approach Λv in the steady-state. In order to guarantee the generalized asymptotic regulation constraint on e, one then only needs to impose $\|\Lambda v(0)\|\leq\kappa\|\Phi^{1/2}v(0)\|$. In conclusion, the generalized asymptotic regulation objective will be achieved, if Π and Γ are chosen to satisfy

$$B_{\mathrm{r}}+\Pi A_{\mathrm{e}}-A\Pi-B\Gamma = 0, \quad (18)$$

$$\begin{bmatrix} \kappa\Phi & (D_{\mathrm{r}}-C_{\mathrm{r}}\Pi-D_{\mathrm{rc}}\Gamma)^T \\ D_{\mathrm{r}}-C_{\mathrm{r}}\Pi-D_{\mathrm{rc}}\Gamma & \kappa I \end{bmatrix} \succcurlyeq 0. \quad (19)$$

The synthesis of the required observer is standard, except that it should provide the estimates of the plant as well as the exo-system states. It is hence constructed by imitating the evolution of the extended plant state $\tilde{x}$ in (3) with the addition of a correction term as

$$\dot{\xi}=\tilde{A}\xi+\tilde{B}u+G(\tilde{C}\xi-y), \quad (20)$$

where $G\in\mathbb{R}^{(k+l)\times m}$ is the output injection gain matrix to be designed. The estimation error defined as

$$\zeta\triangleq\xi-\tilde{x} \quad (21)$$

evolves according to

$$\dot{\zeta}=(\tilde{A}+G\tilde{C})\zeta. \quad (22)$$

In order to ensure the convergence of the error to zero, one needs to choose G in such a way that $\tilde{A}+G\tilde{C}$ is Hurwitz. This can be done by solving the LMI

$$\mathfrak{He}\left(X\tilde{A}+M\tilde{C}\right)\preccurlyeq-2\eta X\prec 0, \quad (23)$$

over $X\in\mathbb{S}_+^{k+l}$ and $M\in\mathbb{R}^{(k+l)\times m}$, by using a sufficiently small value for $\eta\in\mathbb{R}_+$. An observer gain matrix is then constructed as

$$G=X^{-1}M. \quad (24)$$

When the control input is constructed by using ξ in place of $\tilde{x}$ in (13), one obtains an output feedback controller as

$$\left[\begin{array}{c}\dot{\xi}\\ \hline u\end{array}\right]=\left[\begin{array}{c|c}\tilde{A}+\tilde{B}\tilde{F}+G\tilde{C} & -G\\ \hline \tilde{F} & 0\end{array}\right]\left[\begin{array}{c}\xi\\ \hline y\end{array}\right]. \quad (25)$$

This is clearly a structured controller since its system matrices (identified according to (5)) will necessarily satisfy $A_{\mathrm{c}}=\tilde{A}+\tilde{B}C_{\mathrm{c}}-B_{\mathrm{c}}\tilde{C}$ with $D_{\mathrm{c}}=0$. With this controller, the evolution of ς will be influenced by the estimation error ζ as well. In order to derive the state equation, we set $u=\tilde{F}(\zeta+\tilde{x})$ and use the relation $\tilde{F}\tilde{x}=F\varsigma-\Gamma v$ in (11)-(12). In this fashion, we obtain

$$\dot{\varsigma} = (A+BF)\varsigma+B\tilde{F}\zeta+Zv, \quad (26)$$

$$e = \underbrace{(C_{\mathrm{r}}+D_{\mathrm{rc}}F)\varsigma+D_{\mathrm{rc}}\tilde{F}\zeta}_{e_{\mathrm{tr}}}+\underbrace{\Lambda v}_{e_{\mathrm{ss}}}. \quad (27)$$

Choosing Π and Γ in a way to render $Z=0$ eliminates the effect of v on the state evolution in this case as well. Hence, when the observer is designed to have a finite-energy estimation error ζ and the state-feedback is synthesized to stabilize the evolution of ς, we will have $\lim_{t\to\infty}\varsigma(t)=0$. This means that the generalized asymptotic regulation constraint can be guaranteed via (18) and (19) in the case of output feedback as well. It can be shown that these conditions are also necessary [3], [4].

The generalized asymptotic regulation constraint clearly relates to the steady-state behavior. In order to shape the transient behavior of the closed-loop system, one can use large values for ρ and η when solving (16) and (23). Since any scaled solution of these LMIs are also solutions, it is necessary to introduce some extra constraints for numerical conditioning. For this purpose, one might might ensure an upper bound on the condition numbers of Y and X. Expressed for Y, this can be realized via the LMI conditions

$$Y\preccurlyeq\sigma I \text{ and } \begin{bmatrix} Y & I\\ I & \sigma I\end{bmatrix}\succcurlyeq 0. \quad (28)$$

The state-feedback design ingredients can then be obtained by minimizing σ subject to the constraints (16) and (28). Although keeping the condition numbers of Y and X small might be advantageous, it is hard to ensure a desirable transient behavior in this fashion. As a matter of fact, the main contribution of the paper is on how to synthesize F and G such that the transient behavior is sub-optimal.

IV. Generalized Asymptotic Regulation with Suboptimal Transient Response

There is clearly much freedom in the choice of the design variables for constructing the controller of (25). We hence consider in this section the problem of optimizing the transient response by proper choices of the design parameters. Although it is e_{tr} that constitutes the regulation transients, we consider a more general problem in which the energy of

$$z=\left(C_{\mathrm{p}}+D_{\mathrm{pc}}F\right)\varsigma+D_{\mathrm{pc}}\tilde{F}\zeta \quad (29)$$

is to be kept small. As can be justified by the decomposition of e in (27), $z(t)\in\mathbb{R}^q$ can be viewed as the transient component of a new output signal obtained from the plant of (1) as $C_{\mathrm{p}}x+D_{\mathrm{pc}}u$. A typical choice of z would hence be $z=[e_{\mathrm{tr}}^T\ \ \lambda u_{\mathrm{tr}}^T]^T$, where $u_{\mathrm{tr}}=F\varsigma+\tilde{F}\zeta=u+\Gamma v$ and λ is a nonnegative scalar, which should be chosen large enough to avoid undesirably large transients for the control input. This means choosing the performance channel matrices as

$$\left[\begin{array}{c|c} C_{\mathrm{p}} & D_{\mathrm{pc}}\end{array}\right]=\left[\begin{array}{c|c} C_{\mathrm{r}} & D_{\mathrm{rc}}\\ 0 & \lambda I\end{array}\right]. \quad (30)$$

Based on these ingredients, we formulate the synthesis problem for suboptimal transient response as follows (see and cf. [1], [4]):

Problem 2: Design a controller of the form (25) with $\xi(0)=0$ such that C.1 and C.2 are satisfied as well as

$$\|z\|_2^2\triangleq\int_0^\infty\|z(t)\|^2dt<\gamma^2\|\tilde{x}(0)\|^2,\ \forall\tilde{x}(0)\in\mathbb{R}^{k+l}\setminus\{0\}, \quad (31)$$

where $\gamma\in\mathbb{R}_+$ represents the level of desired performance.

The solution of Problem 2 with a general controller has been obtained in [4] based on LMI optimization. In this paper, we are interested in the derivation of LMI conditions that can be used to shape the transient behavior of the observer-based controller in (25). When the structure of the controller is fixed, it becomes difficult (if not impossible) to derive convex problems for obtaining the design ingredients in a nonconservative way. In the sequel, we derive two alternative sets of conditions which are potentially conservative.

A. The First Set of Conditions

The first set of conditions are derived based on an alternative expression of the closed-loop dynamics with

$$\theta \triangleq \varsigma + \tilde{\Pi}\zeta = \tilde{\Pi}\xi. \tag{32}$$

When Π and Γ are chosen to render $Z = 0$, the equations for the evolution of θ and z can be derived as

$$\left[\begin{array}{c} \dot{\theta} \\ \hline z \end{array}\right] = \left[\begin{array}{c|c} A+BF & \tilde{\Pi}G\tilde{C} \\ \hline C_{\rm p}+D_{\rm pc}F & -C_{\rm p}\tilde{\Pi}-D_{\rm pc}\tilde{\Gamma} \end{array}\right] \left[\begin{array}{c} \theta \\ \hline \zeta \end{array}\right]. \tag{33}$$

The idea behind our approach is to assume a choice for G as in (24) and introduce a new signal as

$$\vartheta = \begin{bmatrix} X^{-1/2} & 0 \\ 0 & -S^{-1/2} \end{bmatrix} \begin{bmatrix} M\tilde{C} \\ C_{\rm p}\tilde{\Pi}+D_{\rm pc}\tilde{\Gamma} \end{bmatrix} \zeta, \tag{34}$$

where $\tilde{\Gamma} \triangleq \begin{bmatrix} 0_{n\times k} & \Gamma \end{bmatrix}$ and $S \in \mathbb{S}_+^q$ is a positive-definite matrix introduced as a slack variable to reduce conservatism. We view this signal as the output of the unexcited observer system (22) and as the input to a second system with state θ (which has a zero initial condition) and output z. In this fashion, we represent the closed-loop as a series combination of the following systems:

$$\left[\begin{array}{c} \dot{\zeta} \\ \hline \vartheta \end{array}\right] = \left[\begin{array}{c} \tilde{A}+G\tilde{C} \\ \hline X^{-1/2}M\tilde{C} \\ -S^{-1/2}\left(C_{\rm p}\tilde{\Pi}+D_{\rm pc}\tilde{\Gamma}\right) \end{array}\right]\zeta, \tag{35}$$

$$\left[\begin{array}{c} \dot{\theta} \\ \hline z \end{array}\right] = \left[\begin{array}{c|cc} A+BF & \tilde{\Pi}X^{-1/2} & 0 \\ \hline C_{\rm p}+D_{\rm pc}F & 0 & S^{1/2} \end{array}\right] \left[\begin{array}{c} \theta \\ \hline \vartheta \end{array}\right]. \tag{36}$$

Sufficient conditions for the solvability of Problem 2 are then derived as follows:

(i) Bound, from above, the energy of ϑ obtained in response to a nonzero initial condition $\zeta(0) = -\tilde{x}_0$ as

$$\|\vartheta\|_2^2 < \phi\tilde{x}_0^T X\tilde{x}_0, \tag{37}$$

where ϕ is an arbitrary positive scalar.

(ii) Bound the energy gain from ϑ to z from above as

$$\|z\|_2^2 < \phi^{-1}\gamma\|\vartheta\|_2^2. \tag{38}$$

Conditions (37) and (38) clearly imply

$$\|z\|_2^2 < \gamma\tilde{x}_0^T X\tilde{x}_0. \tag{39}$$

In order ensure (31), we hence need to have

$$X \prec \gamma I. \tag{40}$$

By following the approach sketched above, we arrive at the following first main result of the paper:

Theorem 1: There exists a controller with a realization as in (25) that solves Problem 2, if there exist $Y \in \mathbb{S}_+^k$, $N \in \mathbb{R}^{n\times k}$, $\Pi \in \mathbb{R}^{k\times l}$, $\Gamma \in \mathbb{R}^{n\times l}$, $X \in \mathbb{S}_+^{k+l}$, $M \in \mathbb{S}^{(k+l)\times m}$, $S :\in \mathbb{S}_+^q$ such that (18), (19) and (40) are satisfied as well as

$$\begin{bmatrix} \mathfrak{He}\left(X\tilde{A}+M\tilde{C}\right) & * & * \\ M\tilde{C} & -\phi X & * \\ C_{\rm p}\tilde{\Pi}+D_{\rm pc}\tilde{\Gamma} & 0 & -S \end{bmatrix} \prec 0, \tag{41}$$

$$\begin{bmatrix} \mathfrak{He}\left(AY+BN\right) & * & * \\ \tilde{\Pi}^T & -\phi^{-1}X & * \\ C_{\rm p}Y+D_{\rm pc}N & 0 & S-\gamma I \end{bmatrix} \prec 0, \tag{42}$$

where $\phi \in \mathbb{R}_+$ is a fixed arbitrary positive scalar and $*$'s represent the entries that are identifiable from symmetry. The controller can then be constructed with F and G obtained as in (17) and (24) respectively.

Proof: We complete the proof simply by deriving the conditions for energy and energy-gain bounding. The energy of ϑ resulting from a nonzero initial condition $\zeta(0) = -\tilde{x}_0$ is bounded from above based on the Lyapunov function

$$\mathscr{V}_{\rm d}(\zeta) \triangleq \zeta^T X\zeta. \tag{43}$$

This function can be ensured to decay along the trajectories of the closed-loop in a required way by the $\mathscr{H}_2$-type matrix inequality

$$\begin{bmatrix} \mathfrak{He}\left(X\tilde{A}+M\tilde{C}\right) & * & * \\ X^{-1/2}M\tilde{C} & -\phi I & * \\ -S^{-1/2}\left(C_{\rm p}\tilde{\Pi}+D_{\rm pc}\tilde{\Gamma}\right) & 0 & -\phi I \end{bmatrix} \prec 0. \tag{44}$$

This can be established by multiplying (44) from the left with $\tilde{\zeta}^T = \begin{bmatrix} \zeta^T & \phi^{-1}\vartheta^T \end{bmatrix}$ and from the right with $\tilde{\zeta}$. In this fashion, we are able to show (for nonzero $\zeta(0)$) that

$$\frac{d\mathscr{V}_{\rm d}(\zeta(t))}{dt} + \phi^{-1}\|\vartheta(t)\|^2 < 0. \tag{45}$$

Integrating this inequality from zero to infinity and using $\lim_{t\to\infty}\zeta(t) = 0$ as follows from internal stability that is assured by $\mathfrak{He}(X\tilde{A}+M\tilde{C}) \prec 0$, we conclude that (37) is satisfied. By applying a congruence transformation to (44) with $\mathsf{blockdiag}(I, X^{1/2}, -S^{1/2})$ and making the replacement $\phi S \to S$, we can express it equivalently as in (41).

The worst-case energy gain from ϑ to z is bounded from above by using the Lyapunov function

$$\mathscr{V}_{\rm s}(\theta) = \theta^T Y^{-1}\theta. \tag{46}$$

The required decay condition along the trajectories of the closed-loop can be imposed on this function by the $\mathscr{H}_\infty$-type matrix inequality

$$\begin{bmatrix} \mathfrak{He}(AY+BN) & * & * & * \\ X^{-1/2}\tilde{\Pi}^T & -\phi^{-1}I & * & * \\ 0 & 0 & -\phi^{-1}I & * \\ C_{\rm p}Y+D_{\rm pc}N & 0 & S^{1/2} & -\gamma I \end{bmatrix} \prec 0. \tag{47}$$

By multiplying this inequality from the left with $\tilde{\theta}^T = \begin{bmatrix} \theta^T Y^{-1} & \vartheta^T & \gamma^{-1}z^T \end{bmatrix}$ and from the right with $\tilde{\theta}$, we obtain

$$\frac{d\mathscr{V}_{\rm s}(\theta(t))}{dt} + \gamma^{-1}\|z(t)\|^2 - \phi^{-1}\|\vartheta(t)\|^2 < 0, \tag{48}$$

for nonzero ϑ and θ along the trajectories of the closed-loop. Integrating from zero to infinity and using $\lim_{t\to\infty}\theta(t) = 0$

(as follows from internal stability assured by $\mathfrak{He}(AY+BN) \prec 0$ and the fact that ϑ is an exponentially decaying input), we obtain

$$\|z\|_2^2 < \gamma\phi\|\vartheta\|_2^2 + \gamma\theta(0)^T Y^{-1}\theta(0). \tag{49}$$

Recalling $\theta(0) = 0$, we conclude that (38) is satisfied. $X^{-1/2}$ can be eliminated from the $(2,1)$ and $(1,2)$ blocks by a congruence transformation with $\mathsf{blockdiag}(I, X^{1/2}, I, I)$, which changes the $(2,2)$ block to $-\phi^{-1}X$. An application of the Schur complement with respect to the third row/column blocks then allows us to remove them simply by adding ϕS to the $(4,4)$ block. By again making the replacement $\phi S \to S$, we arrive at the condition in (42). ■

B. The Second Set of Conditions

The second set of conditions are derived based on the expression of the closed-loop dynamics with the equation in (22) and

$$\left[\begin{array}{c}\dot{\varsigma}\\ \hline z\end{array}\right] = \left[\begin{array}{c|c} A+BF & B\tilde{F} \\ \hline C_{\mathrm{p}}+D_{\mathrm{pc}}F & D_{\mathrm{pc}}\tilde{F}\end{array}\right]\left[\begin{array}{c}\varsigma\\ \hline \zeta\end{array}\right]. \tag{50}$$

In this case, the choice of F is fixed as in (17) and the new signal is introduced as

$$\varphi = \begin{bmatrix} Y^{-1/2} & 0 \\ 0 & -R^{1/2}\end{bmatrix}\begin{bmatrix}\tilde{\Pi} \\ E\end{bmatrix}\zeta, \tag{51}$$

where $E \triangleq \begin{bmatrix} 0_{l\times k} & I\end{bmatrix}$ and $R \in \mathbb{S}_+^l$ is a positive-definite matrix introduced again as a slack variable. In this case, the unexcited observer system and the second system are identified as

$$\left[\begin{array}{c}\dot{\zeta}\\ \hline \varphi\end{array}\right] = \left[\begin{array}{c}\tilde{A}+G\tilde{C}\\ \hline Y^{-1/2}\tilde{\Pi}\\ -R^{1/2}E\end{array}\right]\zeta, \tag{52}$$

$$\left[\begin{array}{c}\dot{\varsigma}\\ \hline z\end{array}\right] = \left[\begin{array}{c|c} A+BF & B\tilde{N}\tilde{Y}^{-1/2} \\ \hline C_{\mathrm{p}}+D_{\mathrm{pc}}F & D_{\mathrm{pc}}\tilde{N}\tilde{Y}^{-1/2}\end{array}\right]\left[\begin{array}{c}\varsigma\\ \hline \varphi\end{array}\right], \tag{53}$$

where $\tilde{Y}$ and $\tilde{N}$ are defined as

$$\tilde{Y} \triangleq \begin{bmatrix} Y & 0 \\ 0 & R\end{bmatrix}, \tag{54}$$

$$\tilde{N} \triangleq \begin{bmatrix} N & \Gamma\end{bmatrix}. \tag{55}$$

We emphasize at this point that the initial state of the second system is given by $\varsigma(0) = \tilde{\Pi}\tilde{x}(0)$. The steps of the derivation are quite similar, except for the difference that arises from nonzero initial state of the second system. In this case, sufficient conditions for solvability are derived as follows:

(i) Bound, from above, the energy of φ obtained in response to a nonzero initial condition $\zeta(0) = -\tilde{x}_0$ as

$$\|\varphi\|_2^2 < \psi^{-1}\tilde{x}_0^T H\tilde{x}_0, \tag{56}$$

where $H \in \mathbb{S}_+^{k+l}$ is a positive-definite matrix and ψ is an arbitrary positive scalar.

(ii) Bound the energy of z from above as

$$\|z\|_2^2 < \psi\gamma\|\varphi\|_2^2 + \gamma\varsigma(0)^T Y^{-1}\varsigma(0). \tag{57}$$

Conditions (56) and (57) will now guarantee

$$\|z\|_2^2 < \gamma\tilde{x}_0^T \underbrace{\left(H+\tilde{\Pi}^T Y^{-1}\tilde{\Pi}\right)}_{X}\tilde{x}_0, \tag{58}$$

where we used the fact that $\varsigma(0) = \tilde{\Pi}\tilde{x}(0)$. Thanks to the introduction of X as in (58), we can express the condition under which (31) is satisfied again as in (40). With X chosen as the design variable, H can be obtained from X simply as

$$H = X - \tilde{\Pi}^T Y^{-1}\tilde{\Pi}. \tag{59}$$

The positive-definiteness constraint on H introduces a coupling condition on X and Y as

$$\begin{bmatrix} Y & \tilde{\Pi} \\ \tilde{\Pi}^T & X\end{bmatrix} \succ 0, \tag{60}$$

which can be obtained by a standard application of the Schur complement.

The solvability conditions obtained in the way described above constitute the second main result of the paper expressed as follows:

Theorem 2: There exists a controller with a realization as in (25) that solves Problem 2, if there exist $Y \in \mathbb{S}_+^k$, $N \in \mathbb{R}^{n\times k}$, $\Pi \in \mathbb{R}^{k\times l}$, $\Gamma \in \mathbb{R}^{n\times l}$, $X \in \mathbb{S}_+^{k+l}$, $M \in \mathbb{R}^{(k+l)\times m}$, $R \in \mathbb{S}_+^l$ such that (18), (19), (40) and (60) are satisfied as well as

$$\begin{bmatrix} \mathfrak{He}\left(X\tilde{A}+M\tilde{C}\right)+E^T RE & * \\ \tilde{\Pi} & -\psi^{-1}Y\end{bmatrix} \prec 0, \tag{61}$$

$$\begin{bmatrix} \mathfrak{He}\left(AY+BN\right) & * & * & * \\ N^T B^T & -\psi Y & * & * \\ \Gamma^T B^T & 0 & -R & * \\ C_{\mathrm{p}}Y+D_{\mathrm{pc}}N & D_{\mathrm{pc}}N & D_{\mathrm{pc}}\Gamma & -\gamma I\end{bmatrix} \prec 0, \tag{62}$$

where $\psi \in \mathbb{R}_+$ is a fixed arbitrary positive scalar. The controller can then be constructed with F obtained as in (17) and G computed as

$$G = H^{-1}M = \left(X - \tilde{\Pi}^T Y^{-1}\tilde{\Pi}\right)^{-1}M. \tag{63}$$

Proof: The matrix inequality constraint that ensures the bound on the energy of φ can be expressed as

$$\begin{bmatrix} \mathfrak{He}\left(H\tilde{A}+M\tilde{C}\right) & * & * \\ Y^{-1/2}\tilde{\Pi} & -\psi^{-1}I & * \\ -R^{1/2}E & 0 & -\psi^{-1}I\end{bmatrix} \prec 0. \tag{64}$$

We first observe that the third row and column blocks can be removed by an application of the Schur complement, which results in the addition of $\psi^{-1}E^T RE$ in the $(1,1)$ block. By then applying a congruence transformation with

$$\begin{bmatrix} I & 0 \\ Y^{-1/2}\tilde{\Pi}^T\tilde{A} & Y^{1/2}\end{bmatrix}. \tag{65}$$

we can replace H by X, remove the term $Y^{-1/2}$ from blocks (1,2) as well as (2,1) and change the (2,2) block to $-\phi Y$. After making the replacement $\psi^{-1}R \to R$, we obtain the the condition in (61). The constraint that ensures an energy bound on z can easily be expressed in this case as in (62). ■

V. Illustrative Example

In this section, we consider the course control problem in ship steering based on a model from [8], Section 4.2:

$$\left[\begin{array}{c|c|c} A & B_r & B \end{array}\right] = \left[\begin{array}{ccccc|cc|c} -0.13 & 0 & 0 & 0 & 0 & 0 & 0 & 0.01 \\ -0.35 & -0.77 & 0 & 0 & 0 & 0.77 & 0 & -0.02 \\ 0 & 1.00 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.53 & 0 & 0 & -0.10 & -0.25 & 0 & 0.25 & -0.04 \\ 0 & 0 & 0 & 1.00 & 0 & 0 & 0 & 0 \end{array}\right].$$

The states are identified as follows: x_1 is the sway velocity (in m/s), x_3 is the yaw angle (in degrees) and x_2 is its rate, x_5 is the roll angle and x_4 is its rate. We assume that the measurements of the yaw and roll angles are available without any noise or disturbance effect. The control problem that we consider is the adjustment of the rudder deflection u (in degrees) for the stabilization of the ship dynamics and the regulation of the roll angle in the face of a single-frequency sinusoidal wave disturbance (i.e. $e = x_5$). The disturbance is assumed to be generated by (8) with a wave frequency of $\omega_1 = 0.3382\,\mathrm{rad/s}$, as felt by the ship with a constant velocity.

We have designed four controllers for exact cancellation of the sinusoidal disturbance effect in the roll angle (i.e. $\kappa = 0$). The optimization problems are coded in Matlab by the Yalmip parser [7] and solved by SeDuMi [10]. Σ_1 is designed with the basic LMI conditions presented in Section II by using $\rho = \eta = 0.2$. The design ingredients are obtained by minimizing the bounds imposed on the condition numbers of Y and X. The remaining three controllers are designed to shape the transient behavior of $z = [e \;\; \lambda u]^T$ with $\lambda = 0.37$. Σ_2 and Σ_3 are observer-based and are synthesized based on the LMI conditions of Theorem 1 and Theorem 2 respectively. Recall that these conditions have dependence on the scalars ϕ and ψ, over which we had to perform line searches. The values of these scalars that provided the minimum γ values are obtained as $\phi = 0.14$ and $\psi = 0.28$. With these values, we were able to obtain the minimum γ values associated with Σ_2 and Σ_3 as 16.08 and 6.89. Σ_4 has a general structure and is hence synthesized based on the LMI conditions in [4]. The optimum γ value obtained with this controller is 5.75. In all the four cases, we also imposed LMI constraints to avoid closed-loop poles whose real parts are less than -100. The achieved γ values for the second and third controllers indicate that the conditions of Theorem 1 and Theorem 2 are potentially conservative. In this particular example, we observe that the conditions of Theorem 2 lead to a less conservative design.

The simulation results obtained for the particular initial condition $\tilde{x}(0) = [\,0\;\;0\;\;3\;\;0\;\;5\;\;1\;\;0\,]^T$ are presented in Figure 1. The top plot shows the uncontrolled system error, which clearly becomes sinusoidal after the transient period. The middle plot shows the errors for the four different controllers, whereas the bottom plot shows the associated control inputs during the transient period. The unrealistic control input demand of Σ_1 is evident from the bottom plot, whose scale is restricted to standard rudder deflection limits. Σ_2 leads to a better transient behavior than Σ_1 with much less control effort. Nevertheless, its performance is clearly worse than the Σ_3 and Σ_4, which is not surprising in view of the associated γ value. Σ_3 leads to a very similar decay

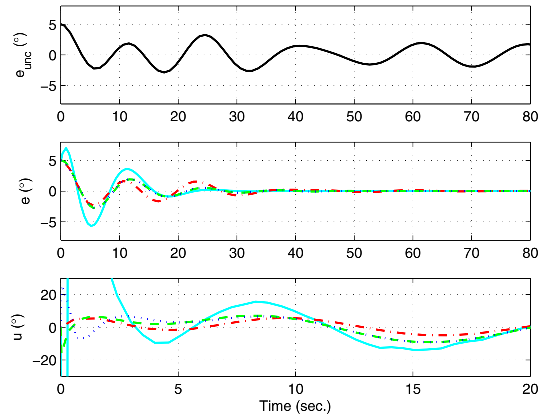

Fig. 1. Designs for the control of the roll angle: Σ_1:solid (cyan), Σ_2:dash-dotted (red), Σ_3:dashed (green), Σ_4:dotted (blue).

of the error with Σ_4. This is not surprising either, since the associated γ values are quite close to each other. Their different behaviors are visible from the bottom plots of the control inputs.

VI. Concluding Remarks

We have provided two different sets of LMI conditions for the synthesis of observer-based controllers for generalized asymptotic regulation with suboptimal transient response. Although the conditions are potentially conservative, we consider the results of the paper as an interesting contribution to LMI-based synthesis of restricted-structure controllers. It is to be explored whether the approach of this paper can be applied to derive similar results for other types of performance measures (e.g. $\mathscr{H}_\infty$).

Acknowledgement

We thank Carsten W. Scherer for the relevant discussions.

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