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# On linear over-actuated regulation using input allocation

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*Abstract*— **We present in this paper preliminary results and research directions concerning the output regulation problem for over-actuated linear systems. The focus of the paper is on the characterization of the solution of the full-information regulator problem for systems which are right-invertible but not left-invertible, where the input operator is injective. The intrinsic redundancy in the plant model is exploited by parameterizing all solutions of the ensuing regulator equations and performing a static or dynamic optimization on the space of solutions. This approach effectively shapes the non-unique steady-state of the system so that the long-term behavior optimizes a given performance index. In particular, nonlinear cost functions that account for constraints on the inputs are considered. Examples are given to illustrate and validate the proposed methodology.**

## I. THE FULL INFORMATION REGULATOR PROBLEM FOR OVER-ACTUATED SYSTEMS

Consider the following linear plant:

$$\mathcal{P}: \quad \begin{array}{rcl} \dot{x} &=& Ax + Bu + Pw \\ e &=& Cx + Du + Qw \end{array} \tag{1}$$

with state $x \in \mathbb{R}^n$, control input $u \in \mathbb{R}^m$ and performance output $e \in \mathbb{R}^p$. The signal $w \in \mathbb{R}^q$ is assumed to be generated by the following exosystem

$$\mathcal{S}: \ \dot{w} \ = \ Sw. \tag{2}$$

Following standard regulation theory [1], [8], $(A, B, C, D)$ is refereed to as the realization of the plant model and $(S, P, Q)$ as the realization of the exosystem. The following assumptions define the class of plant and exosystem models considered in this paper:

*Assumption 1:*

1) *The plant model is* over-actuated*, that is,* $m > p$*;*
2) *The matrices* $B$ *and* $C$ *satisfy* $\operatorname{rank} B = m$ *and* $\operatorname{rank} C = p$*;*
3) *The pair* $(A, B)$ *is stabilizable;*
4) *The matrix* $S$ *is semi-simple (that is, it has only simple eigenvalues) and* $\operatorname{spec} S \subset \mathbb{C}^0$.

Item 2 of Assumption 1 is made to avoid trivialities and overlap with previous results. The case in which $B$ is rank-deficient, which corresponds to having the so-called *strong input redundancy* (see [9]) for $(A, B)$, can be handled separately from the *weak redundancy* exploited here.

To the best of our knowledge, the output regulation problem for linear over-actuated systems has been investigated first in [4] in the context of tracking control for a linearized model of a hypersonic aircraft, and later extended to encompass linear parameter-varying systems [5]. The steady-state optimization for an input-redundant linear system with nonlinear output function has been considered in [2], with exosystem model restricted to pure integrators. For the same type of exosystem, the results in [9], [6] provide a framework allowing for nonlinear dynamic allocation solutions.

### A. Problem Statement and Preliminaries.

The problem addressed in this paper is the design of a *full information* (possibly nonlinear) *regulator* that is capable of exploiting the redundancy stated at item 1 of Assumption 1 to induce a *desirable selection* of the control input $u$ (in a sense to be specified). As a possible selection of a function to be optimized, we consider in Section III function whose minimization corresponds to keeping the steady state input far from the saturation limits. As pointed out in [9], the use of input allocation should be seen as synergistic with anti-windup techniques, since the latter account for saturation during transients, whereas the former addresses steady-state saturations. This must be done while guaranteeing internal stability of the closed-loop system when the exosystem is disconnected and the asymptotic tracking requirement $\lim_{t\to\infty} e(t) = 0$ when the exosystem is active. As customary, by full information it means that both $x$ and $w$ are available for measurement. Here, it is also assumed that $\mathcal{P}$ and $\mathcal{S}$ are known exactly.

A standard sufficient condition for the solvability of the regulator problem (which becomes necessary if mild assumptions on parametric uncertainties affecting the plant matrices are considered) is given by the following:

*Assumption 2: [geometric version]*

1) *The plant model is right-invertible, i.e., the output* $e$ *is functionally controllable from the input* $u$*;*
2) *The set of transmission zeros of* $(A, B, C, D)$ *is disjoint from the spectrum of* $S$.

An algebraic version of Assumption 2, known as Davison condition, can also be formulated. Let $P_\Sigma(s)$ denote the *system matrix* of system (1), that is,

$$P_\Sigma(s) := \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix}. \tag{3}$$

Recall that system (1) is left invertible if and only if $\operatorname{rank} P_\Sigma(s) = n+m$ (as a polynomial matrix), and it is right invertible if and only if $\operatorname{rank} P_\Sigma(s) = n + p$. Obviously, system (1) is not left-invertible. The values of $\bar{s} \in \mathbb{C}$ for which the complex valued matrix $P_\Sigma(\bar{s})$ has rank less than the rank of $P_\Sigma(s)$ as a polynomial matrix constitute the *system zeros*, which include all the transmission zeros plus a subset of the *input decoupling zeros* (eigenvalues of the unreachable subsystem) and the *output decoupling zeros*

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(eigenvalues of the unobservable subsystem). The algebraic version of Assumption 2 is then stated as follows:

*Assumption 2: [algebraic version]* $\operatorname{rank} P_\Sigma(\lambda) = n + p$, *for all* $\lambda \in \operatorname{spec} S$.

Finally, we recall a few geometric concepts that will be used in the sequel. By $\mathcal{V}^* \subset \mathbb{R}^n$, we denote the *weakly unobservable subspace* for $\mathcal{P}$, that is, the set of initial conditions for which there exists an input function such that the ensuing output function is identically zero. It is well known [7] that $\mathcal{V}^*$ is the largest subspace $\mathcal{V} \subset \mathbb{R}^n$ such that

$$\begin{bmatrix} A \\ C \end{bmatrix} \mathcal{V} \subset (\mathcal{V} \times 0) + \operatorname{im} \begin{bmatrix} B \\ D \end{bmatrix}, \tag{4}$$

or equivalently the largest subspace $\mathcal{V} \subset \mathbb{R}^n$ such that there exists $F \in \mathbb{R}^{m\times n}$ ensuring

$$(A + BF)\mathcal{V} \subset \mathcal{V}, \quad (C + DF)\mathcal{V} = 0. \tag{5}$$

A matrix $F$ satisying (5) is called a *friend of* $\mathcal{V}$. Similarly, we denote by $\mathcal{R}^* \subset \mathbb{R}^n$ the *controllable weakly unobservable subspace*[1] of $\mathcal{P}$, that is, the set of initial conditions for which there exists an input function able to steer the state to zero in finite time while keeping the output function identically zero. Obviously, $\mathcal{R}^* \subset \mathcal{V}^*$; moreover, any friend of $\mathcal{V}^*$ is also a friend of $\mathcal{R}^*$ [7, Th. 7.14].

## II. Regulator architecture and properties

It is well known (see, for instance, [3, Chap. 1]) that the structure of a full-information regulator comprises:

1) a steady-state control action $u_{ss}(w)$ capable of inducing an identically zero output $e$ along a suitable steady-state trajectory $x_{ss}(w)$ of the plant, and
2) a stabilizing control action $\tilde{u}$ in feedback from the mismatch $\tilde{x} = x - x_{ss}(w)$, capable of stabilizing the steady-state trajectory at the previous item.

In the over-actuated case, since the plant model fails to have a unique inverse (recall that $\mathcal{P}$ is necessarily not left-invertible), redundancy can be exploited in the generation of the steady-state pair $(x_{ss}(w), u_{ss}(w))$. For linear models, this corresponds to selecting appropriately

$$x_{ss}(w) = \Pi w\,, \quad u_{ss}(w) = \Gamma w \tag{6}$$

among the infinitely many solutions $(\Pi, \Gamma)$ of the regulator (or Francis) equations:

$$\begin{aligned} \Pi S &= A\Pi + B\Gamma + P \\ 0 &= C\Pi + D\Gamma + Q\,. \end{aligned} \tag{7}$$

According to the Proposition 1, all steady-state pairs in (6) can be generated by exploiting a basis of the space of all solutions of the *homogeneous Francis equation*

$$\begin{aligned} \Pi S &= A\Pi + B\Gamma \\ 0 &= C\Pi + D\Gamma \end{aligned} \tag{8}$$

[1]When $D = 0$, $\mathcal{V}^*$ and $\mathcal{R}^*$ are usually termed respectively the *largest controlled-invariant subspace* and the *largest controllability subspace* contained in $\ker C$ (see [8]).

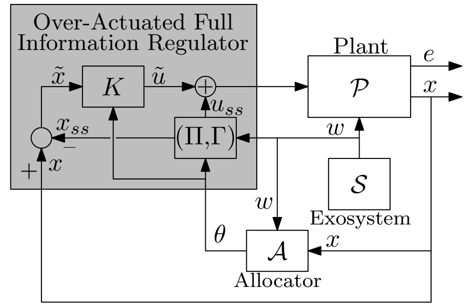


Fig. 1. The over-actuated regulator control scheme with dynamic input allocator.

*Proposition 1:* Under Assumption 1, all solutions to the Francis equations (7) are parametrized as

$$\Pi(\theta) = \Pi_p + \sum_{i=1}^{s} \theta_i \Pi_i\,, \quad \Gamma(\theta) = \Gamma_p + \sum_{i=1}^{s} \theta_i \Gamma_i \tag{9}$$

by the parameter vector $\theta = [\theta_1 \cdots \theta_s]^T \in \mathbb{R}^s$, where $s = (m - p)q$, $X_p = [\Pi_p; \Gamma_p]$ is any solution[2] of (7) whereas $X_i = [\Pi_i; \Gamma_i]$, $i = 1, \ldots, s$ are linearly independent matrices spanning the space of solutions of (8).

The next result is key to the selection of the stabilizing component of the regulator:

*Proposition 2:* Each solution $X_i = [\Pi_i; \Gamma_i]$ of (8) satisfies $\operatorname{im} \Pi_i \subset \mathcal{R}^*$, $i = 1, \ldots, s$.

The results of Propositions 1 and 2, as well as the general structure of a (static) full-information regulator mentioned above, suggest the architecture of the regulator with dynamic allocation shown in Figure 1. In particular, dynamic allocation of the control input $u$ is performed by acting on the *allocation parameter* $\theta$. Ideally, this must be accomplished without affecting the tracking performance – that is, preserving the asymptotic properties of the error signal $e(\cdot)$ as well as the controlled-invariance of the subspace $x = \Pi(\theta)w$. Once the steady-state trajectory ensuring $e(t) \equiv 0$ is computed as in (6), (9), it is then possible to design the feedback stabilizer, which according to the scheme in Figure 1, provides the feedback signal $\tilde{u}$ within the selection

$$u = u_{ss}(w, \theta) + \tilde{u}\ . \tag{10}$$

In particular, substituting (10), (6), (9) in the plant dynamics (1), exploiting (7), and defining $\tilde{x} = x - x_{ss}(w, \theta)$, the following dynamic equations are derived for the error system:

$$\begin{aligned} \dot{\tilde{x}} &= A\tilde{x} + B\tilde{u} - \sum_{i=1}^{s} \dot{\theta}_i \Pi_i w \\ e &= C\tilde{x} + D\tilde{u}. \end{aligned} \tag{11}$$

Given (11), it is quite natural to select the input $\tilde{u}$ as the linear feedback

$$\tilde{u} = K\tilde{x} = K(x - x_{ss}(w, \theta)), \tag{12}$$

where $K$ is designed in one of the following two ways:

[2]We use the notation $z = [x; y]$ to denote the vector (or matrix) $z = [x^T\ y^T]^T$.

1) as any stabilizing gain for the pair $(A, B)$, under the condition that $\dot{\theta}(t) \equiv 0$, that is the allocation parameter $\theta$ is statically optimized and is kept constant;
2) as a stabilizing gain for $(A, B)$ with the property that the transfer matrix from $\xi = [\xi_1 \cdots \xi_s]$ to $e$ for the system

$$\begin{aligned} \dot{\tilde{x}} &= A\tilde{x} + B\tilde{u} + \sum_{i=1}^{s} \Pi_i \xi_i \\ e &= C\tilde{x} + D\tilde{u}. \end{aligned} \quad (13)$$

is identically zero; this second choice will allow for nonstationary selections of the allocation parameter $\theta(t)$ without affecting the tracking performance.

The two possible selections above and the desirable properties of the ensuing "Over-Actuated Full Information Regulator" in Figure 1 are formally stated in Theorem 1.

*Theorem 1:* Consider the plant (1) and the exosystem (2) satisfying Assumptions 1 and 2. Moreover, consider the controller selection in (10), with (12) and (6), (9), where $(\Pi_p, \Gamma_p)$ and $(\Pi_i, \Gamma_i)$, $i = 1, \dots, s$ are computed as in Proposition 1. Consider the following two cases:

1) The matrix $K$ in (12) exponentially stabilizes the pair $(A, B)$ and the allocation parameter is constant ($\theta(t) = \overline{\theta}$ for all $t \geq 0$); alternatively,
2) the matrix $K$ in (12) exponentially stabilizes the pair $(A, B)$ and is a friend of the *controllable weakly unobservable subspace* $\mathcal{R}^*$ and the allocation parameter $\theta(t)$ is a differentiable and bounded signal for all $t \geq 0$.

For each of the above cases, given any initial condition $(x(0), w(0))$ of the plant-exosystem pair (1), (2), trajectories of the closed-loop system are bounded and the error output satisfies $\lim_{t\to\infty} e(t) = 0$. Furthermore, for all initial conditions satisfying $x(0) = \Pi(\theta(0))w(0)$ it holds that $e(t) \equiv 0$.

## III. Selection of the allocation policy

In the previous section we have introduced a design technique for the Over-Actuated Full Information Regulator of Figure 1 without commenting on the selection of the allocator block $\mathcal{A}$. In particular, via Theorem 1, we provided two design techniques for synthesizing the matrices of the regulator parametrically in $\theta$ in such a way that the tracking performance is preserved with constant selections of $\theta(t) = \overline{\theta}$ (item 1 of Theorem 1) and with differentiable selections of $\theta(t)$ (item 2 of Theorem 1). In this section, we will propose a few techniques to select the allocation parameter $\theta$ (namely the allocator block $\mathcal{A}$ in Figure 1), in such a way to induce desirable properties of the closed-loop signals. It should be recognized that the techniques proposed here are quite straigthforward and intuitive constructions, while more involved designs for $\theta$ might be achieved by possibly developing more in the direction highlighted in the next remark.

*Remark 1:* Consider the (parametric in $\theta$) control law (10), (12), (6) proposed in the previous section. Taking into account the matrix selection in (9), this control law can be rewritten as the following affine function of $\theta$:

$$\begin{aligned} u &= \Gamma(\theta)w + K(x - \Pi(\theta)w) \\ &= Kx + (\Gamma_p - K\Pi_p)w + \Psi(w)\theta, \end{aligned} \quad (14a)$$

where

$$\Psi(w) = \left[\ (\Gamma_1 - K\Pi_1)w \quad \cdots \quad (\Gamma_s - K\Pi_s)w\ \right]. \quad (14b)$$

Based on the representation (14) of the control input $u$, it is possible to design static or dynamic selections of $\theta$ aiming at keeping the control $u$ as small as possible, possibly based on its saturation limits or based on other plant input performance specifications. In this paper, two static selections of $\theta$ will be considered. In the first one a constant value of $\theta$ is pre-computed off-line by optimizing a suitable performance index. In the second one, an (almost) piecewise constant $\theta(t)$ is obtained by an on-line optimization algorithm which adapts $\theta(t)$ with the goal of reaching the minimum of the same performance index. This give raise to a time-varying selection of $\theta(t)$ which converges in finite time to a constant value. More general time-varying selections of $\theta$ and dynamic selections along similar lines to those developed in [9], [6] will be investigated in future work. ⌟

For simplicity, the following assumption is introduced. Its satisfaction can be easily verified by checking if the ratios among (the imaginary parts of) the eigenvalues of $S$ are all rational numbers.

*Assumption 3:* The exosystem (2) generates periodic responses, that is, there exists $T > 0$ such that for any $w(0)$ and for all $t \geq 0$, $w(t + T) = w(t)$.

### A. An off-line selection of a constant $\theta$

The first strategy for optimizing the input allocation whenever Assumption 3 holds arises from recognizing that for each initial condition $w_0$ of the exosystem (2) (and for each value of $\theta$), a unique periodic steady state control input $u_{ss}(t) = \Gamma(\theta)w$ is defined whenever $\theta$ is kept constant. Indeed, due to Assumption 3, the response $w(t)$ is periodic and only depends on the initial condition.

A sensible problem then corresponds to the one of selecting $\theta$ as the minimizer of the following cost function:

$$J(\theta, w(t)) = \max_{\substack{t \in [0,T] \\ i \in \{1,\dots,m\}}} \left(1 - \frac{u_{ss,i}(w(t))}{\underline{u}_i}\right)\left(\frac{u_{ss,i}(w(t))}{\overline{u}_i} - 1\right), \quad (15)$$

which satisfies $J(\theta, w(t)) = J(\theta, w(0))$ for all $\theta$ and for all $t \geq 0$, because $w(t) = w(t + T)$ and because the function corresponds to the maximum over the whole period.

The rationale behind the cost (15) is to maximize the worst case distance of any input from its saturation level. This is actually carried out by normalizing the saturation levels so that the percentage of the available input range at all inputs is maximized. Note that the cost function takes into account the possible periodic nature of the steady-state input by computing the worst case distance over the whole period of the steady-state plant input.

Based on the cost function (15), the approach proposed here is to optimize the selection of $\theta$ based on the measurement of the initial value of $w$ (or, equivalently, its value at any time $t \geq 0$). In particular, assuming that it is possible to solve offline the following optimization problem:

$$\theta^*(w_0) = \arg\min_{\theta \in \mathbb{R}^s} J(\theta, w_0), \quad (16)$$

a constant value of $\theta(t) = \theta^*(w_0)$ can be adopted and, according to item 1 of Theorem 1, for any $K$ stabilizing the pair $(A, B)$, the asymptotic tracking performance is retained and the worst distance of the worst input from its saturation value is maximized. This fact is illustrated in Example 1 in the next section.

### B. An on-line iterative selection of $\theta$

The constant selection of $\theta$ proposed above suffers from two main issues: 1) it might be nontrivial, in general, to explicitly compute the maximizer of the cost function (15); 2) even in cases where this maximizer could be computed, this should be parametrized with respect to $w$, and this might be expensive in terms of storage of a suitable grid of optimal values for $\theta$. To address the above problems, we propose here a gradient based strategy which allows to optimize online the value of $\theta(t)$.

The functional dependence of $\theta$ on time is restricted to be a smooth but almost piecewise constant signal. In particular, we fix a small scalar $\delta \ll T$ denoting the time used to smoothly transition between two subsequent periods (namely, time intervals of length $T$) where $\theta(t) = \theta_{[k]}$ is constant. Then we define:

$$\begin{aligned} \theta(t) &:= \theta_{[k]}, \quad \forall t \in \mathcal{T}_k,\ k \in \mathbb{N}, \\ \mathcal{T}_k &:= [(T+\delta)k, (T+\delta)k + T]. \end{aligned} \tag{17}$$

Consider the index

$$J_{[k]} := J(\theta, w(t), k) = \max_{\substack{t \in \mathcal{T}_k \\ i \in \{1,\ldots,m\}}} J_i(\theta, w(t)), \tag{18a}$$

$$J_i(\theta, w(t)) = \left(1 - \frac{u_{ss,i}(w(t))}{\underline{u}_i}\right)\left(\frac{u_{ss,i}(w(t))}{\overline{u}_i} - 1\right) \tag{18b}$$

which corresponds to (15) but evaluated during the $k$−th period. In the $\delta$-small interval $[(T+\delta)k+T, (T+\delta)(k+1)]$, between the end of the $k$-the period and the beginning of the $(k+1)$-the period, $\theta(t)$ is smoothly transferred from $\theta_{[k]}$ to:

$$\theta_{[k+1]} = \theta_{[k]} - \alpha_{[k]} d_{[k]}, \tag{19}$$

where, letting $i^*_{[k]} \in \{1, \ldots, m\}$ denote the index of the input component such that the maximum in (18) is achieved and letting $t_{[k]}$ denote the corresponding time, the update direction $d_{[k]}$ and the update step length $\alpha_{[k]}$ are chosen as

$$\begin{aligned} d_{[k]} &= \frac{\nabla_\theta J_{i^*_{[k]}}(\theta_{[k]}, w(t_{[k]}))}{\left\| \nabla_\theta J_{i^*_{[k]}}(\theta_{[k]}, w(t_{[k]})) \right\|}, \\ \beta_{[k]} &= \tau (J_{[k]} + 1)^2 \left\| \nabla_\theta J_{i^*_{[k]}}(\theta_{[k]}, w(t_{[k]})) \right\|, \\ \alpha_{[k]} &= \begin{cases} \beta_{[k]}, & \beta_{[k]} \geq \underline{\alpha}, \\ 0, & \beta_{[k]} < \underline{\alpha}, \end{cases} \end{aligned}$$

with $\tau$, $\underline{\alpha}$ being positive constants. The factor $(J_{[k]} + 1)^2$ is useful to get smoother convergence, since it gets closer to zero (thus reducing the step length and inducing more cautious updates) when the components $u_{ss,i}$ get closer to zero, that is far from the bounds on the input. The parameter $\underline{\alpha}$ is used to stop the updates when the update step becomes sufficiently small; in this way, after a finite number of iterations the value of $\theta$ remains constant and the result in Theorem 1 applies, yielding asymptotic tracking with a reduced excursion of the steady-state input. Example 2 in Section IV shows the effectiveness of the proposed recipe.

## IV. Simulation examples

A first numerical example is presented to illustrate the item 1 of Theorem 1.

*Example 1:* Consider the plant and exosystem described by the matrices:

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{ccc|cc} -0.02 & 0.41 & 0.35 & -0.76 & -0.31 \\ -0.10 & 0.50 & 0.31 & -0.00 & 0.17 \\ 0.29 & -0.44 & -0.67 & 0.91 & -0.55 \\ \hline 0.50 & -0.48 & 0.01 & 0 & 0 \end{array}\right]$$

$$S = \begin{bmatrix} 0 & 15.78 \\ -15.78 & 0 \end{bmatrix}$$

$$\left[\begin{array}{c} P \\ \hline Q \end{array}\right] = \left[\begin{array}{cc} 0.78 & -0.72 \\ 0.91 & -0.70 \\ 0.09 & -0.48 \\ \hline 0.68 & -0.49 \end{array}\right].$$

All the matrices are generated randomly, except for the matrix $S$, which is characterized by two imaginary eigenvalues $\lambda_1 = \lambda_2^* = j\frac{2\pi}{T}$, with $T$ generated randomly and given by $T = 0.39$. Moreover, select $\underline{u} = [-30; -20]$ and $\overline{u} = [30; 30]$ in (18). Assumptions 1 and 2 are both satisfied, so the results of Theorem 1 can be applied. The dimension of the free parameter $\theta$ is given by $s = (m - p)q = (2 - 1)2 = 2$. Solving the Francis equation, the particular and homogeneous solutions can be found to be

$$\left[\begin{array}{c} \Pi_p \\ \hline \Gamma_p \end{array}\right] = \left[\begin{array}{cc} -1.47 & 0.91 \\ -0.08 & -0.09 \\ 1.72 & -1.11 \\ \hline 20.70 & 29.03 \\ 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{c|c} \Pi_1 & \Pi_2 \\ \hline \Gamma_1 & \Gamma_2 \end{array}\right] = \left[\begin{array}{cc|cc} -0.43 & -0.93 & 0.93 & -0.43 \\ -0.37 & -0.82 & 0.82 & -0.37 \\ 2.95 & 5.39 & -5.39 & 2.95 \\ \hline -47.82 & 28.98 & -28.98 & -47.82 \\ 70.93 & -42.44 & 42.44 & 70.93 \end{array}\right] 10^{-2}.$$

A first stabilizing matrix

$$K_{NF} = \begin{bmatrix} -68.49 & -380.02 & -80.01 \\ 0.06 & -65.40 & -2.68 \end{bmatrix}$$

is computed using the Matlab command `place` with the goal of assigning the eigenvalues of the closed-loop matrix $(A + BK_{NF})$ at $\{-8, -10, -12\}$. Then, Theorem 1 guarantees that using $K = K_{NF}$, for any constant value of $\theta$, asymptotic tracking is ensured. In particular, according to the construction given in Section III-A, we select a constant value of $\theta$ optimizing the performance index $J(\theta)$ in (15) when initializing the exosystem (2) from the initial condition $w_0 = [1; 0]$. Figure 2 shows a numerical evaluation of the optimal value of $\theta$, based on the level sets of $J(\theta)$. Note that the minimizer is $\theta^*(w_0) = [6; 18]$. This proves that the redundancy in the regulator problem can be effectively used to obtain an improved usage of the steady-state plant inputs. This is even more evident in Figure 3, where the steady-state input trajectory corresponding to $w_0$ is plotted on the inputs space for the two cases of $\theta = 0$ (blue circles) and $\theta = \theta^*$ (red dots). In the former case, with $\theta = 0$, the input cannot satisfy the saturation constraint (represented by the green dashed lines), while in the latter case, with $\theta = \theta^*$, the input

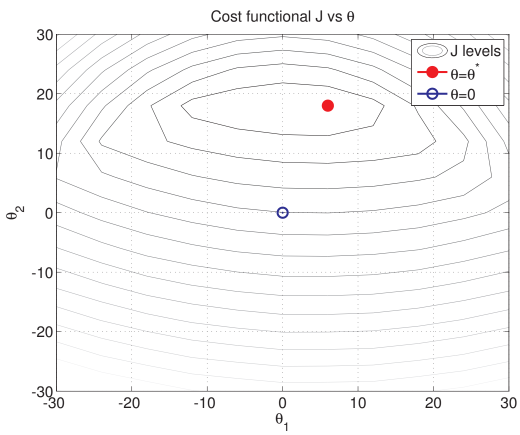


Fig. 2. Example 1. The cost $J$ in (15) as a function of the parameter $\theta \in \mathbb{R}^2$. The minimizer is $\theta^* = [0; 20]$ (red dot), different from zero (blue circle).

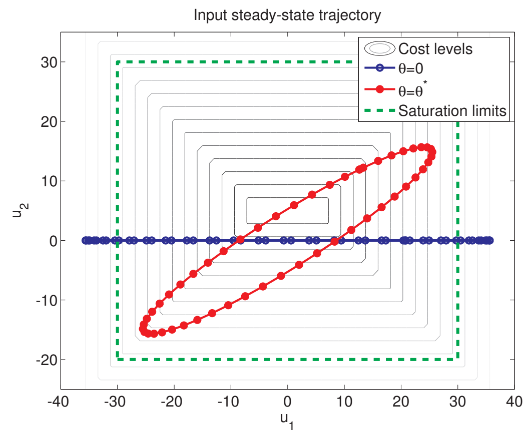


Fig. 3. Example 1. The values of $u_{ss}$ in the case $\theta = 0$ (blue circles), and for the optimal case $\theta = \theta^*$ (red dots). In the former case the steady-state trajectory is infeasible for the input constraints while in the latter case it is feasible.

trjectory lies entirely in the available range and maximizes the worst case distance from the saturation limits.

A second stabilizing matrix

$$K_F = \begin{bmatrix} -217.50 & -240.26 & -84.16 \\ 337.82 & 339.42 & 124.90 \end{bmatrix}$$

is computed to assign the same set of eigenvalues assigned by $K_{NF}$ while being a friend of $\mathcal{R}^*$. With $K = K_F$, the subspace spanned by the matrix

$$R = \begin{bmatrix} -0.69 & -0.00 \\ -0.71 & 0.02 \\ 0.01 & 0.99 \end{bmatrix}$$

is invariant.

In Figure 4 two different simulations, respectively with $K = K_{NF}$ (black dash-dotted) and $K = K_F$ (blue solid), are shown. In both of them the exosystem starts from the same initial condition $w_0 = [1; 0]$ used in the previous steps, but the plant starts from an initial condition $x_0 = x_{ss}(w_0, \theta) + \tilde{x}_0$, with $\tilde{x}_0 = C^T = [0.50; -0.48; 0.01]$, which does not belong to the invariant subspace of the steady-state trajectories. In both simulations $\theta = \theta^*$ is used, so that the steady-state trajectories of both the input components are within their saturation limits (green dashed horizontal lines). Since $\theta$ is constant, item 1 of Theorem 1 applies and so both with the friend matrix $K_F$ and with the non friend one $K_{NF}$, the inputs converge, after a transient, to the steady-state trajectory $u_{ss}(w)$ (red dashed curves) and the error $e$ converges to 0 (lower trace in the figure).

*Example 2:* Consider the plant and exosystem described by the matrices

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{ccc|cc} -0.01 & 0.80 & 0.56 & -0.19 & 0.88 \\ -0.02 & -0.26 & -0.22 & -0.80 & 0.91 \\ -0.32 & -0.77 & -0.51 & -0.73 & 0.15 \\ \hline -0.88 & -0.53 & -0.29 & 0 & 0 \end{array}\right]$$

$$S = \left[\begin{array}{c|cc|cc} 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 7.65 & 0 & 0 \\ 0 & -7.65 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 3.82 \\ 0 & 0 & 0 & -3.82 & 0 \end{array}\right]$$

$$\left[\begin{array}{c} P \\ \hline Q \end{array}\right] = \left[\begin{array}{ccccc} -0.96 & 0.29 & -0.09 & 0.48 & -0.63 \\ -0.91 & 0.46 & 0.09 & -0.62 & -0.26 \\ -0.66 & 0.29 & -0.40 & 0.37 & 0.25 \\ \hline 0.56 & -0.83 & 0.85 & 0.55 & -0.02 \end{array}\right].$$

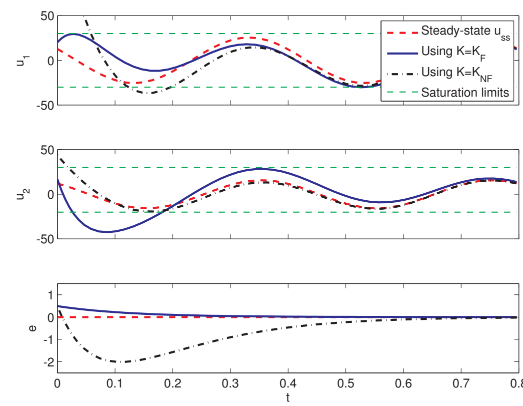


Fig. 4. Response using $\theta = \theta^*$ with $K = K_F$ (blue solid) and with $K = K_{NF}$ (black dash-dotted). In both cases the error $e$ converges to 0 (lower plot) and $u$ converges to $u_{ss}(w)$ (red dashed), because $\theta$ is constant.

All the matrices are generated randomly, except for the matrix $S$, which is characterized by an eigenvalue in the origin $\lambda_1 = 0$ and two pairs of imaginary eigenvalues $\lambda_2 = \lambda_3^* = j\frac{2\pi}{T_1}$ and $\lambda_4 = \lambda_5^* = j\frac{2\pi}{T_2}$ with $T_2 = 2T_1$ and $T_1 = 0.82$ generated randomly. Moreover, select $\underline{u} = [-30; -20]$ and $\overline{u} = [30; 30]$ in (18). Assumptions 1 and 2 are both satisfied, so the results of Theorem 1 can be applied. The dimension of the free parameter $\theta$ is given by $s = (m - p)q = (2 - 1)5 = 5$. Solving the Francis equation, the particular and homogeneous solutions can be found to be

$$\left[\begin{array}{c} \Pi_p \\ \hline \Gamma_p \end{array}\right] = \left[\begin{array}{ccccc} -0.26 & -0.52 & 0.65 & 0.35 & -0.18 \\ 1.49 & -0.58 & 0.54 & 0.39 & 0.23 \\ 0 & -0.23 & -0.03 & 0.10 & 0.03 \\ \hline -2.52 & 0 & 0 & 0 & 0 \\ -0.81 & -5.34 & -4.86 & -0.17 & 2.00 \end{array}\right]$$

$$\left[\begin{array}{c} \Pi_1 \\ \hline \Gamma_1 \end{array}\right] = \left[\begin{array}{ccccc} 3.65 & 0.09 & -3.65 & -0.13 & 0.79 \\ -20.83 & -0.24 & 2.43 & 0.16 & -0.51 \\ 26.66 & 0.14 & 6.56 & 0.10 & -1.44 \\ \hline 2.18 & 78.35 & -8.60 & -8.84 & 0.85 \\ 2.49 & 48.83 & -7.46 & -5.58 & 0.99 \end{array}\right] 10^{-2}$$

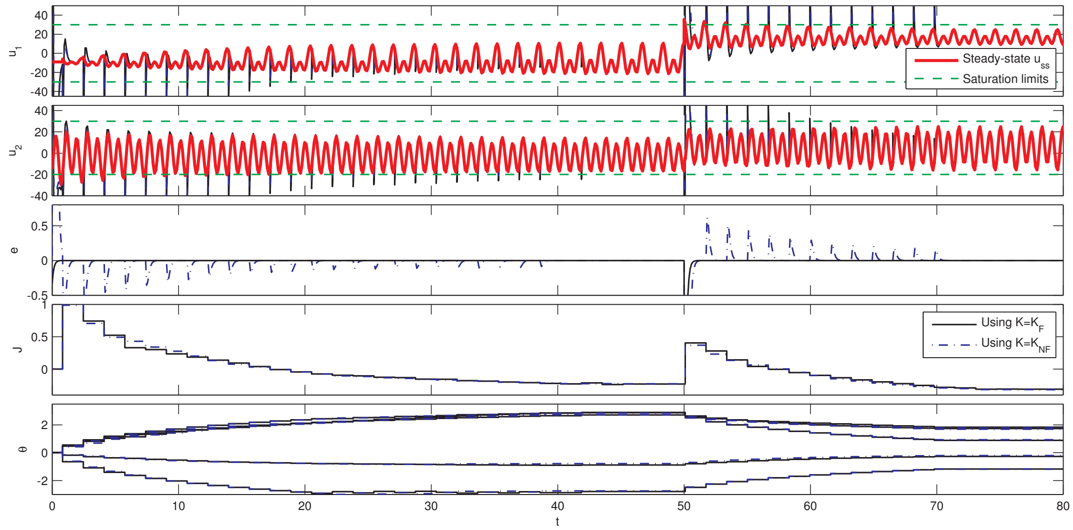


Fig. 5. Example 2. Inputs $u$ and $u_{ss}$ (top two plots), error $e$ (middle trace), cost function $J$ in (18) (middle-bottom trace) and parameter $\theta$ (bottom trace).

$$\begin{bmatrix}\Pi_2\\ \hline \Gamma_2\end{bmatrix} = \left[\begin{array}{rrrrr} -7.49 & 2.34 & -0.81 & -1.48 & 1.97\\ 42.70 & -1.60 & 0.42 & 1.18 & -1.17\\ -54.67 & -4.13 & 1.66 & 2.30 & -3.80\\ \hline -4.48 & 24.20 & 47.65 & -25.38 & -9.99\\ -5.12 & 16.42 & 29.20 & -16.65 & -5.07 \end{array}\right] 10^{-2}$$

$$\begin{bmatrix}\Pi_3\\ \hline \Gamma_3\end{bmatrix} = \left[\begin{array}{rrrrr} 5.60 & 3.10 & 0.72 & -0.00 & -1.93\\ -31.92 & -2.03 & -0.63 & -0.18 & 1.29\\ 40.86 & -5.62 & -1.02 & 0.34 & 3.46\\ \hline 3.35 & -6.34 & 68.26 & 20.60 & -5.69\\ 3.82 & -2.16 & 42.90 & 12.81 & -4.65 \end{array}\right] 10^{-2}$$

$$\begin{bmatrix}\Pi_4\\ \hline \Gamma_4\end{bmatrix} = \left[\begin{array}{rrrrr} -2.24 & 0.41 & -0.44 & 7.15 & -1.37\\ 12.77 & -0.30 & 0.27 & -4.91 & 0.22\\ -16.35 & -0.70 & 0.83 & -12.55 & 3.71\\ \hline -1.34 & 10.68 & 7.61 & 35.50 & 72.01\\ -1.53 & 6.89 & 4.48 & 26.20 & 44.01 \end{array}\right] 10^{-2}$$

$$\begin{bmatrix}\Pi_5\\ \hline \Gamma_5\end{bmatrix} = \left[\begin{array}{rrrrr} 2.88 & 0.27 & 0.77 & 1.88 & 6.83\\ -16.44 & -0.14 & -0.53 & -0.59 & -4.75\\ 21.04 & -0.55 & -1.36 & -4.56 & -11.89\\ \hline 1.72 & -15.77 & 8.08 & -67.15 & 39.98\\ 1.97 & -9.66 & 5.48 & -40.69 & 28.81 \end{array}\right] 10^{-2}$$

Two different stabilizing matrices,

$$K_{NF} = \begin{bmatrix} 224.99 & -259.22 & 251.22\\ 105.78 & -134.41 & 125.14 \end{bmatrix}$$

$$K_F = \begin{bmatrix} 160.13 & -412.32 & 275.49\\ 94.41 & -260.23 & 169.56 \end{bmatrix},$$

are computed: the former using the Matlab command `place` with the goal of assigning the eigenvalues of the closed-loop matrix $(A + BK_{NF})$ at $\{-8, -10, -12\}$ ; the latter assigning the same eigenvalues while being a friend of $\mathcal{R}^*$. With $K = K_F$, the subspace spanned by the matrix

$$R = \begin{bmatrix} -0.56 & -0.00\\ 0.72 & 0.48\\ 0.39 & -0.87 \end{bmatrix}$$

is invariant. Then, Theorem 1 guarantees that using $K = K_{NF}$, for any constant value of $\theta$, asymptotic tracking is ensured. In particular, we select $\theta$ according to the construction given in Section III-B. Figure 5 shows a numerical simulation of the closed-loop system, with initial conditions $x_0 = [0.63; 0.58; 0.28]$, $w_0 = [3.6; 3; 0; 2.12; 2.12]$ and $\theta_0 = 0$, repeated in both the cases with $K = K_{NF}$ (blue dashed-dotted trace) and $K = K_F$ (black solid trace). In order to show how the allocation law (19) responds to disturbances, at time $t = 50$ the first component of the exosystem state $w_1$ jumps to the value $w_1 = -6.4$. Both the initial condition $w0$ and the value of $w$ after the jump, define infeasible $(J > 0)$ steady-state trajectories $u_{ss}$ (red solid bold trace) for the input $u$. The allocator block changes the free parameter $\theta$ and makes the actual cost index $J(\theta)$ decrease to negative values. Indeed $u_{ss}$ moves back into the feasible input range. With both the choices for the stabilizer $K$, the actual input $u$ reaches the steady-state trajectory $u_{ss}$ after a transient due to both the initial condition $\tilde{x}_0 \neq 0$ and to the adaptation of $\theta$, as described by (13). The advantage in using the friend matrix and relying on item 2 of Theorem 1 appears from the middle plot of the figure, which shows that the friend matrix does not cause the impulsive transients experienced on the error $e$ when $\theta$ varies according to the adaptation law.

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