2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Updating the gain of global finite-time high gain observers

Laurent Burlion, Tarek Ahmed-Ali and Françoise Lamnabhi-Lagarrigue

Abstract— **Recently, new results about semi-global and global finite-time observation have been obtained by the use of continuous high gain observers. In this paper, we propose to extend these results by studying "time-varying high gain" observers and by providing new update laws : first, we adapt the law introduced by L. Praly in the case of asymptotic observation to the finite-time case and we prove that the updated high gain remains bounded. Secondly, we propose a new update law which guarantees the high gain's value tends asymptotically to 1.**

I. Introduction

Over the past few decades, the problem of finite-time observation of non linear systems has received a lot of attention. Except a few other results, there are mainly two classes of finite-time nonlinear observers that have been widely studied.
First, discontinuous finite-time sliding mode observers have deserved a lot of attention : a lot of papers using classical sliding observers can be found in the literature [14] – [17]. Sliding mode observers is still an active field of research and more recently higher order sliding observers have also been introduced (see for instance [18], [19]).
Secondly, spurred by the work of Bhat and Bernstein [1] on the finite time stabilization of a double integrator, a continuous homogeneous observer has been proposed for a large class of nonlinear second order systems [6]. This led to the development of the wide branch of continuous finite-time nonlinear observers.
The homogeneous domination approach was soon introduced in order to deal with higher dimensional uncertain nonlinear systems : its principle is to dominate the nonlinearities by introducing a scaling gain into the homogeneous observer. This enabled to solve the problem of finite-time output feedback stabilization of more and more complex classes of nonlinear systems [8], [13].
Meanwhile, another type of finite-time observer was introduced for a class of linearizable nonlinear system [10] and soon extended to observe uniformly observable systems in a semi-global [9] and global [12] ways . In all these design methods, the gain of the observer is fixed and must be chosen sufficiently large (so in a conservative way). However, in the asymptotic stabilization case, there exist some results to adapt the gain of high gain observers when the unknown nonlinear functions have an unknown growth rate [5] or when this rate depends on the measured output [4].

L. Burlion is with Onera - The French Aerospace Lab, F-31055 Toulouse, France. (Laurent.Burlion(at)onera.fr)
T. Ahmed-Ali is with University of Caen, UMR CNRS 6072 GREYC, bd du Marchal Juin , BP 5186, 14032 Caen Cedex, France
F. Lamnabhi-Lagarrigue is with L2S-CNRS, Supe elec, 3 rue Joliot Curie, 91192 Gif-sur-Yvette, France

Recently, in [11], an update law whose gain is an exponential function with arbitrary growth rate has been combined with the semi-global finite time observer of [9] to obtain a global result. Since, the resulted law is exponential and the uncertain nonlinear terms are bounded by a perfectly known Lipschitz condition, we think this observer deserves many extensions. Therefore, in this paper, we consider the problem of updating the gain of global finite time high-gain observers for a class of uncertain nonlinear system whose uncertain nonlinear terms have an unknown growth rate which depends on the output. We obtain two possible update laws : the first one is a natural extension of the law introduced in [4] at the difference that some terms of this law serve to dominate some homogeneous factors outside a compact set around the origin. It is proved that the high gain remains bounded when the state and input of the observed nonlinear system are bounded. Then, a second update law is proposed and provides a gain which not only remains bounded under the same conditions than above but also tends asymptotically to 1.
This paper is organized as follows. Some basic notations and definitions followed by the problem formulation are given in Section 2. In Section 3, we present our observer and some hypotheses. Then, we provide the adaptation laws and prove our main results in Section 4. We then illustrate our results on a numerical example in Section 5 and compare our adaptation laws.

II. Preliminaries

Let $\mathcal{R}$ (resp. $\mathcal{N}$) denote the set of real numbers (resp. natural integers). In this paper, we interest us to nonlinear SISO systems of dimension $n \in \mathcal{N}$.

A. Notations

Given $\alpha \in \mathcal{R}^+ \backslash \{0\}, y \in \mathcal{R}$, we note :

$$\lceil y \rfloor^{\alpha} := |y|^{\alpha} sign(y)$$

For a given vector $x := [x_1, \ldots, x_n]^T \in \mathcal{R}^n$, $\forall j \geq i$, we note

$$x_{i,j} := [x_i, x_{i+1}, \ldots, x_j]^T$$

We use the following notation for a given $n \times n$ diagonal matrix

$$diag(a_1, \ldots, a_n) := \begin{pmatrix} a_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & a_n \end{pmatrix}$$

B. Finite-Time Stability

In this paper, we interest us to finite time stability and stabilization results which have the merit to provide continuous time observer and controller by using a mix between Lyapunov theory and geometric homogeneity. We invite the reader which is not familiar with the basic definitions of homogeneity to look at the work of [2] or [12].
Nevertheless, let us briefly recall a basic result about Finite Time Stability (See [2] for more details) that will be useful in this paper.
Theorem : Suppose there exist a possibly non Lipschitz vector field f on $\mathcal{R}^n$, a C^1 positive definite function V : $\mathcal{R}^n \to \mathcal{R}$ and positive real numbers $a > 0, 1 > \beta > 0$ such that :

$$\forall x \in \mathcal{R}^n, \quad \dot{V}(x) = L_f V(x) \leq -aV^{\beta}(x)$$

then the origin is GFTS (Globally Finite Time Stable) under the vector field f.

C. Problem position

Throughout this paper, we consider the following class of nonlinear systems :

$$\left\{\begin{array}{lcl} \dot{x}_1 & = & x_2 + \mathrm{f}_1(x_1) \\ \dot{x}_2 & = & x_3 + \mathrm{f}_2(x_1, x_2) \\ \vdots & & \\ \dot{x}_i & = & x_{i+1} + \mathrm{f}_i(x_1, x_{2,i}) \\ \vdots & & \\ \dot{x}_n & = & u + \mathrm{f}_n(x_1, x_{2,n}) \\ y & = & x_1 \end{array}\right. \tag{1}$$

where $x = [x_1, \ldots, x_n]^T \in \mathcal{R}^n$, $u \in \mathcal{R}$ and $y \in \mathcal{R}$ are respectively the state, input and output of this nonlinear system ; where the unknown functions f_i's are characterized by the following assumption [H1] :

$$|f_i(x_1, x_{2,i}) - f_i(x_1, x_{2,i} - z_{2,i})| \underset{\forall i \in [2,n]}{\leq} \gamma(y) \sum_{k=2:i} |z_k|$$

where $\gamma \geq 0$ with $\gamma(0) = 0$.

Our aim is to design a global finite-time observer for this class of nonlinear systems.

III. Global finite-time observer

Let us note

$$\xi_i := x_i - \hat{x}_i$$

where $\hat{x}$ denotes the state of a nonlinear high gain observer defined by:

$$\left\{\begin{array}{lcl} \dot{\hat{x}}_1 & = & \hat{x}_2 + \mathrm{f}_1(x_1) + \frac{k_{o,1}}{2} L(\lceil \xi_1 \rfloor^{\alpha_1} + \xi_1) \\ \dot{\hat{x}}_2 & = & \hat{x}_3 + \mathrm{f}_2(x_1, \hat{x}_2) + \frac{k_{o,2}}{2} L^2(\lceil \xi_1 \rfloor^{\alpha_2} + \xi_1) \\ \vdots & & \\ \dot{\hat{x}}_i & = & \hat{x}_{i+1} + \mathrm{f}_i(x_1, \hat{x}_{2,i}) + \frac{k_{o,i}}{2} L^i(\lceil \xi_1 \rfloor^{\alpha_i} + \xi_1) \\ \vdots & & \\ \dot{\hat{x}}_n & = & u + \mathrm{f}_n(x_1, \hat{x}_{2,n}) + \frac{k_{o,n}}{2} L^n(\lceil \xi_1 \rfloor^{\alpha_n} + \xi_1) \end{array}\right.$$

- where $[k_{o,1}, \ldots, k_{o,n}] \in \mathcal{R}^n$ will be defined later
- where $1 > (\alpha_1, \ldots, \alpha_n) > 0$ is decreasing and will be defined later.
- where $L \geq 1$ is time varying ($\dot{L}$ will be designed later)

Let us also introduce the following change of coordinates :

$$\forall i \in [1, n], \quad \varepsilon_i = \frac{\xi_i}{L^{i-1+b}}$$

We note : $\delta\mathrm{f}_i := f_i(x_1, x_{2,i}) - f_i(x_1, \hat{x}_{2,i})$.
We rewrite the hypothesis [H1] in the new coordinates (this will be useful in the proof of our main result)

$$\forall i \in [2, n], \quad |\delta\mathrm{f}_i| \leq \gamma(y) \sum_{k=2:i} L^{k-1+b} |\varepsilon_k|$$

where $b > 0$ is defined later.
In these coordinates, the dynamics of the observer error is written :

$$\left\{\begin{array}{lcl} \dot{\varepsilon}_1 & = & L\varepsilon_2 - \frac{k_{o,1}}{2} L(\, L^{(\alpha_1 - 1)b} \lceil \varepsilon_1 \rfloor^{\alpha_1} + \varepsilon_1) - b\frac{\dot{L}}{L}\varepsilon_1 \\ \dot{\varepsilon}_2 & = & L\varepsilon_3 - \frac{k_{o,2}}{2} L(\, L^{(\alpha_2 - 1)b} \lceil \varepsilon_1 \rfloor^{\alpha_2} + \varepsilon_1) + \frac{\delta\mathrm{f}_2}{L^{1+b}} \\ & & -(1+b)\frac{\dot{L}}{L}\varepsilon_2 \\ \vdots & & \\ \dot{\varepsilon}_i & = & L\varepsilon_{i+1} - \frac{k_{o,i}}{2} L(\, L^{(\alpha_i - 1)b} \lceil \varepsilon_1 \rfloor^{\alpha_i} + \varepsilon_1) + \frac{\delta\mathrm{f}_i}{L^{i-1+b}} \\ & & -(i-1+b)\frac{\dot{L}}{L}\varepsilon_i \\ \vdots & & \\ \dot{\varepsilon}_n & = & -\frac{k_{o,n}}{2} L(\, L^{(\alpha_n - 1)b} \lceil \varepsilon_1 \rfloor^{\alpha_n} + \varepsilon_1) + \frac{\delta\mathrm{f}_n}{L^{n-1+b}} \\ & & -(n-1+b)\frac{\dot{L}}{L}\varepsilon_n \end{array}\right. \tag{2}$$

A. Additional Hypotheses :

Let us note $C = [1 \ 0 \ \ldots \ 0]$ and A the matrix defined by $(A)_{i,j} = \delta_{i,j-1}$ (where $\delta_{i,j}$ is the Kronecker delta).
Let $a > 0$, we choose $Q = Q^T > 0$ such that there exists $1 > q > 0$ s.t

$$A^T Q + QA - C^T C \leq -2aQ \quad ; \quad qI_n \leq Q \leq I_n \tag{3}$$

We then define $(k_{o,1}, \ldots, k_{o,n}) > 0$ by:

$$[k_{o,1}, \ldots, k_{o,n}]^T = Q^{-1} C^T$$

Let us also note $D = diag(0, 1, \ldots, n-1)$. We suppose there exists $b > 0$ such that:

$$-bQ \leq D^T Q + QD \leq bQ \tag{4}$$

Remark : in practice, we solve the LMIs associated to hypothesis (3) and then we search $b > 0$ in order to satisfy hypothesis (4).

IV. Main Results

Let us note :

$$|\varepsilon_1|^{\#} := \max\{|\varepsilon_1|^{\alpha_1}, |\varepsilon_1|^{\alpha_n}\}$$

$$c_{1,n} := \min\{3c_1, c_n\} \quad \text{and} \quad c^{1,n} := \max\{3c_1, c_n\}$$

$$k_B := \frac{3n}{a} \max_{i \in [1,n]} k_{o,i}$$

$$c_B := \frac{(n-1)^2}{\sqrt{q}}\left(1+\max_{i\in[2,n]}\frac{1}{c_i}\right)$$

Let us now present our first result :

Theorem 1: *Let us suppose the state of system (1) remains bounded by a given bounded controller u ; under the hypotheses [H1], (3), (4), there exists $\alpha < 1$ such that system (2) is Globally Finite Time Stable when we use the following adaptation law :*

$$\dot{L} = -\frac{1}{b}L\left(\frac{a}{3}(L-1-L|\varepsilon_1|^{\#}) - \left(\frac{c_B}{c_{1,n}} + 2\frac{n-1}{\sqrt{q}}\right)\gamma(y)\right) \tag{5}$$

with $L(0)=1$.
Moreover, this law guarantees that L remains bounded.

proof : it is somewhat technical but relies on the fact that we prove these results in several steps, taking in mind the fact that if we prove global asymptotic stability and finite-time stability in a neighborhood of the origin, we obtain global finite time stability. (See for instance, Lemma 1 of [12])

A. Global Asymptotic Stability

Under the hypotheses [H1], (3), (4), let us first study the asymptotic stability.

Let us consider the following candidate Lyapunov function

$$V_o(\varepsilon) = \varepsilon^T Q \varepsilon$$

where Q satisfies (3).
$\forall i, \alpha_i \le 1$, we interest us to the following quantity

$$B(\varepsilon) := \varepsilon^T Q \begin{pmatrix} \varepsilon_2 - k_{o,1}L^{(\alpha_1-1)b}\lceil\varepsilon_1\rfloor^{\alpha_1} \\ \dots \\ \varepsilon_{i+1} - k_{o,i}L^{(\alpha_i-1)b}\lceil\varepsilon_1\rfloor^{\alpha_i} \\ \dots \\ -k_{o,n}L^{(\alpha_n-1)b}\lceil\varepsilon_1\rfloor^{\alpha_n}\end{pmatrix}$$

- When $\varepsilon \in \left\{\varepsilon \in \mathcal{R}^n \ s.t\ \sqrt{V_o(\varepsilon)} > k_B\right\}$ we have:

$$\begin{aligned} B(\varepsilon) &\le \varepsilon^T QA\varepsilon + n\sqrt{V_o}\max_{i\in[1,n]} k_{o,i}|\varepsilon_1|^{\#} \\ &\le \varepsilon^T QA\varepsilon + \frac{a}{3}|\varepsilon_1|^{\#}V_o \end{aligned}$$

- When ε is in the following complementary set

$$\mathcal{C}_1 := \left\{\varepsilon \in \mathcal{R}^n \ s.t\ \sqrt{V_o(\varepsilon)} \le k_B\right\}$$

  Since when $\alpha = 1$, $\forall \varepsilon \in \mathcal{R}^n, B(\varepsilon) \le 0$ and $\mathcal{C}_1$ is a compact set, we know from the tube Lemma [3] that there exists $1 > \bar{\alpha}_1 > 0$ such that $\forall \alpha \in]1-\bar{\alpha}_1, 1[$,

$$\forall \varepsilon \in \mathcal{C}_1, B(\varepsilon) \le 0$$

So we conclude, that there exists $1 > \bar{\alpha}_1 > 0$ such that $\forall \alpha \in]1-\bar{\alpha}_1, 1[$,

$$\forall \varepsilon \in \mathcal{R}^n, \quad B(\varepsilon) \le \max\{0, \varepsilon^T QA\varepsilon\} + \frac{a}{3}|\varepsilon_1|^{\#}V_o$$

Therefore, when $\alpha \in]1-\bar{\alpha}_1, 1[$, we can write :

$$\begin{aligned} \dot{V}_o &\le L\varepsilon^T(QA - C^TC)\varepsilon + LB(\varepsilon) \\ &\quad + 2\varepsilon^T Q \begin{pmatrix} 0 \\ \dots \\ \frac{\delta f_i}{L^{i-1+b}} - (i-1)\frac{\dot{L}}{L}\varepsilon_i \\ \dots \\ \frac{\delta f_n}{L^{n-1+b}} - (n-1)\frac{\dot{L}}{L}\varepsilon_n \end{pmatrix} - 2b\frac{\dot{L}}{L}\varepsilon^T Q\varepsilon \\ &\le L\varepsilon^T(QA - C^TC)\varepsilon + L\max\{0, \varepsilon^T QA\varepsilon\} \\ &\quad + \frac{a}{3}L|\varepsilon_1|^{\#}V_o + 2\frac{n-1}{\sqrt{q}}\gamma(y)V_o \\ &\quad - 2\frac{\dot{L}}{L}\varepsilon^T(QD + bQ)\varepsilon \\ &\le L\max\left\{-aV_o - \frac{1}{2}\varepsilon^TC^TC\varepsilon, -2aV_o\right\} \\ &\quad + \frac{a}{3}L|\varepsilon_1|^{\#}V_o + 2\frac{n-1}{\sqrt{q}}\gamma(y)V_o \\ &\quad - 2\frac{\dot{L}}{L}\varepsilon^T(QD + bQ)\varepsilon \\ &\le -\left[a\left(1 - \frac{1}{3}|\varepsilon_1|^{\#}\right)L - 2\frac{n-1}{\sqrt{q}}\gamma(y)\right]V_o \\ &\quad - \frac{\dot{L}}{L}b\left[2 - sign\left(\frac{\dot{L}}{L}\right)\right]V_o \end{aligned}$$

If we apply the adaptation Law (5), we get after a few computations (remark that it is easy to prove that this adaptation law guarantees that $\forall t \ge 0, L(t) \ge 1$ and so we can use $-(2L+1) \le -3$):

$$\begin{aligned} \dot{V}_o &\le -aV_o - \frac{c_B}{c_{1,n}}\gamma(y)V_o \\ &\le -aV_o \end{aligned}$$

so the system is GAS (and so, it will stay inside the ball $\mathcal{B}_1$ defined below within a finite time)

B. Proof inside $\mathcal{B}(1) := \{\varepsilon \ s.t\ |\varepsilon_i| \le 1\}$

Secondly, we interest us to the ball $\mathcal{B}(1)$

1) First subsystem: First we consider the following vector field f_α:

$$\left\{\begin{aligned} \dot{\varepsilon}_1 &= \varepsilon_2 - k_{o,1}L^{(\alpha_1-1)b}\lceil\varepsilon_1\rfloor^{\alpha_1} \\ \dot{\varepsilon}_2 &= \varepsilon_3 - k_{o,2}L^{(\alpha_2-1)b}\lceil\varepsilon_1\rfloor^{\alpha_2} \\ &\vdots \\ \dot{\varepsilon}_i &= \varepsilon_{i+1} - k_{o,i}L^{(\alpha_i-1)b}\lceil\varepsilon_1\rfloor^{\alpha_i} \\ &\vdots \\ \dot{\varepsilon}_n &= -k_{o,n}L^{(\alpha_n-1)b}\lceil\varepsilon_1\rfloor^{\alpha_n} \end{aligned}\right.$$

As in [10], we define, $(r_1, \dots, r_n) > 0$ and $(\alpha_1, \dots, \alpha_n) > 0$ such that :

$$\left\{\begin{aligned} r_{i+1} &= r_i + d, \quad 1 \le i \le n-1 \\ \alpha_i &= \tfrac{r_{i+1}}{r_1}, \quad 1 \le i \le n-1 \\ \alpha_n &= \tfrac{r_n + d}{r_1} \end{aligned}\right. \tag{6}$$

These equations simply say that the vector field f_α is homogeneous of degree d with respect to the weights

$(r_1, \dots, r_n)$.

Let us take a real number $0 < \alpha < 1$.
We set: $r_1 = 1$, $r_2 = \alpha$, $d = r_2 - r_1 = \alpha - 1$ and we recursively prove that: $r_i = (i-1)\alpha - (i-2) \quad 1 < i \le n$ and, $\alpha_i = i\alpha - (i-1) \quad 1 < i \le n$.
Since $r_1 > \dots > r_n$, in order to guarantee $\forall i, r_i > 0$, we will also need to assume that :

$$\alpha > \frac{n-2}{n-1} = 1 - \frac{1}{n-1}$$

Let us now define :$V_z(\alpha, \varepsilon) := z^T Q z$ where $z_i = \lceil \varepsilon_i \rfloor^{\frac{1}{r_i \pi_r}}$ and where $\pi_r = \prod_{i=1:n-1} r_i$.
Let us also note : $c_i = \frac{1}{r_i \pi_r}$
It is obvious that V_z is homogeneous of degree $\frac{2}{\pi_r}$ with respect to the weights $(r_1, \dots, r_n)$. and that $L_{f_\alpha} V_z$ is homogeneous of degree $\frac{2}{\pi_r} + degree(f_\alpha) = \frac{2}{\pi_r} + d = \frac{2}{\pi_r} + \alpha - 1$ with respect to the weights $(r_1, \dots, r_n)$.
By using Lemma 4.2 of [2],

$$-\beta_1(\alpha, L) V_z^\beta(\varepsilon) \le L_{f_\alpha} V_z(\varepsilon) \le -\beta_2(\alpha, L) V_z^\beta(\varepsilon)$$

where $\beta := \frac{deg(L_{f_\alpha} V_z)}{deg(V_z)} = 1 + \frac{\pi_r}{2}(\alpha - 1) < 1$ because $\alpha < 1$ and where $\beta_1(\alpha, L) := -\min_{\{z \ s.t \ V_z(\alpha,\varepsilon)=1\}} L_{f_\alpha} V_z(\alpha, \varepsilon) > 0$ and $\beta_2(\alpha, L) := -\max_{\{z \ s.t \ V_z(\alpha,\varepsilon)=1\}} L_{f_\alpha} V_z(\alpha, \varepsilon) > 0$. (we easily prove it when $\alpha = 1$ (asymptotic case) and we then use the same proof than [11])

2) Full system: Since $\forall \varepsilon$, the following quantity is negative when $\alpha = 1$

$$B_z(\varepsilon) := z^T Q \begin{pmatrix} c_1 |\varepsilon_1|^{c_1 - 1} (\varepsilon_2 - k_{o,1} \varepsilon_1) \\ \dots \\ c_i |\varepsilon_i|^{c_i - 1} (\varepsilon_{i+1} - k_{o,i} \varepsilon_1) \\ \dots \\ c_n |\varepsilon_n|^{c_n - 1} (-k_{o,n} \varepsilon_1) \end{pmatrix}$$

moreover, since $\mathcal{B}(1)$ is compact, we know from the tube Lemma [3] that there exists $1 > \bar{\alpha}_2 > 0$ such that:

$$\forall \varepsilon \in \mathcal{B}(1),\ \forall \alpha \in]1 - \bar{\alpha}_2, 1[, \quad B_z(\varepsilon) \le 0 \tag{7}$$

Therefore, if $1 > \alpha > \max\Big(1 - \frac{1}{n-1}, 1 - \bar{\alpha}_2\Big)$, we can write :

$$\begin{aligned}
\dot{V}_z \quad \le \quad & -\beta_2(\alpha, L) V_z^\beta(\varepsilon) + L B_z(\varepsilon) \\
& + 2 z^T Q \begin{pmatrix} c_1 |\varepsilon_1|^{c_1 - 1} \left(-b \frac{\dot{L}}{L} \varepsilon_1\right) \\ \dots \\ c_i |\varepsilon_i|^{c_i - 1} \left(\frac{\delta \mathrm{f}_i}{L^{i-1+b}} - (i-1+b) \frac{\dot{L}}{L} \varepsilon_i\right) \\ \dots \\ c_n |\varepsilon_n|^{c_n - 1} \left(\frac{\delta \mathrm{f}_n}{L^{n-1+b}} - (n-1+b) \frac{\dot{L}}{L} \varepsilon_n\right) \end{pmatrix} \\
\le \quad & -\beta_2(\alpha, L) V_z^\beta(\varepsilon) - 2 \frac{\dot{L}}{L} z^T (QD + bQ) \times \\
& \begin{pmatrix} c_1 & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & c_n \end{pmatrix} z + 2 z^T Q \begin{pmatrix} 0 \\ \dots \\ c_i |\varepsilon_i|^{c_i - 1} \left(\frac{\delta \mathrm{f}_i}{L^{i-1+b}}\right) \\ \dots \\ c_n |\varepsilon_n|^{c_n - 1} \left(\frac{\delta \mathrm{f}_n}{L^{n-1+b}}\right) \end{pmatrix}
\end{aligned}$$

First, since the c_is are positive and increasing and by using (4), we have:

- if $\frac{\dot{L}}{L} \ge 0$

$$-2 \frac{\dot{L}}{L} z^T (QD + bQ) \begin{pmatrix} c_1 & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & c_n \end{pmatrix} z \le -c_n b \frac{\dot{L}}{L} V_z$$

- if $\frac{\dot{L}}{L} \le 0$

$$-2 \frac{\dot{L}}{L} z^T (QD + bQ) \begin{pmatrix} c_1 & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & c_n \end{pmatrix} z \le -3 c_1 b \frac{\dot{L}}{L} V_z$$

Due to space restriction, the right term of the last two inequalities is given by $-b c^\# \frac{\dot{L}}{L} V_z$ where $c^\# := c_n + \frac{1}{2}\Big(1 - sign(\frac{\dot{L}}{L})\Big)(3c_1 - c_n)$.
By Young's inequality:

$$|\varepsilon_i|^{c_i - 1} |\varepsilon_k| \le \frac{c_i - 1}{c_i} |\varepsilon_i|^{c_i} + \frac{1}{c_i} |\varepsilon_k|^{c_i}$$

Hence, $\forall i \in [2, n]$:

$$\begin{aligned}
|\varepsilon_i|^{c_i - 1} \frac{\delta \mathrm{f}_i}{L^{i-1+b}} \quad & \le \quad \gamma(y) \sum_{k=2:i} L^{k-i-2} |\varepsilon_i|^{c_i - 1} |\varepsilon_k| \\
& \le \quad \gamma(y) \left(\frac{c_i - 1}{c_i}(i-1)|\varepsilon_i|^{c_i} + \frac{1}{c_i} \sum_{k=2}^{i} |\varepsilon_k|^{c_i}\right) \\
& \le \quad \gamma(y) \left((n-1)\sqrt{z'z} + \frac{1}{c_i} \sum_{k=2}^{i} |\varepsilon_k|^{c_i}\right)
\end{aligned}$$

Since, $\forall i \ge k, c_i \ge c_k \ge 1$ and $\varepsilon \in \mathcal{B}(1)$, we have :

$$|\varepsilon_k|^{c_i} \le |\varepsilon_k|^{c_k}$$

So, $\forall i \in [2, n]$:

$$|\varepsilon_i|^{c_i - 1} \frac{\delta \mathrm{f}_i}{L^{i-1+b}} \quad \le \quad \gamma(y) \left(1 + \max_{i \in [2,n]} \frac{1}{c_i}\right)(n-1)\sqrt{z^T z}$$

Thus, we obtain:

$$\begin{aligned}
\dot{V}_z \quad \le \quad & -\beta_2(\alpha, L) V_z^\beta(\varepsilon) - b c^\# \frac{\dot{L}}{L} V_z \\
& + \gamma(y) \frac{(n-1)^2}{\sqrt{q}} \left(1 + \max_{i \in [2,n]} \frac{1}{c_i}\right) V_z
\end{aligned}$$

We identify the coefficient c_B and we thus obtain :

$$\dot{V}_z \le -\beta_2(\alpha, L) V_z^\beta(\varepsilon) - \left(b c^\# \frac{\dot{L}}{L} - c_B \gamma(y)\right) V_z \tag{8}$$

C. Conclusion of the proof of Theorem 1

To sum up, we applied the adaptation law (5):

$$\dot{L} = -\frac{1}{b} L \left(\frac{a}{3}(L - 1 - L|\varepsilon_1|^\#) - \left(\frac{c_B}{c_{1,n}} + 2\frac{n-1}{\sqrt{q}}\right)\gamma(y)\right)$$

with $L(0) = 1$.
Moreover, there exists $1 > \alpha > \max\Big(1 - \frac{1}{n-1}, 1 - \bar{\alpha}_1, 1 - \bar{\alpha}_2\Big) > 0$ such that:

- system (2) adapted by this law is GES.
- $\forall \varepsilon \in \mathcal{B}_1$, applying the adaption law to equation the (8), we obtain:

$$\begin{array}{rcl} V_z & := & z(\varepsilon)^T Q z(\varepsilon) \\ \dot{V}_z & \leq & -\beta_2(\alpha, L) V_z^{\beta}(\varepsilon) + \frac{a}{3} c^{\#}(L - L|\varepsilon_1|^{\#} - 1) V_z \\ & & -2c_{1,n} \frac{n-1}{\sqrt{q}} \gamma(y) V_z \\ & \leq & -\beta_2(\alpha, L) V_z^{\beta}(\varepsilon) + \frac{ac^{1,n}}{3} L V_z \end{array}$$

First, using the first Lyapunov function V_o, we proved the system is GES so there exists a finite time such that the system enters the ball $\mathcal{B}_1$ without leaving it anymore.
Inside this ball :

- we need to consider the following set :

$$\mathcal{C}_2 := \left\{ \varepsilon \ s.t \ \frac{ac^{1,n}}{3} V_z(\varepsilon) \leq \min_{L \geq 1} \frac{\beta_2(\alpha, L)}{2L} V_z^{\beta}(\varepsilon) \right\}$$

  If L remains bounded (as we will prove after), this set is not a zero measured set (i.e it is bigger than $\{0_n\}$). Since the system is GES, there exists a finite time such that the system enters the set $\mathcal{B}_1 \bigcap \mathcal{C}_2$ without leaving it anymore.
  Since in this set we have:

$$\dot{V}_z \leq -\frac{\beta_2(\alpha, L)}{2} V_z^{\beta}$$

  we conclude that system (2) converges to the origin in a finite time.

Thus, we proved the system is GFTS.
Moreover, provided y remains bounded and since $L(0) = 1$, the adaptation law form guarantees that L remains bounded. Indeed, since ε_1 tends to 0 within a finite time, L can increase with a bounded rate within a finite time and after, $L\dot{L} < 0$ if

$$L > 1 + \left(\frac{c_B}{c_{1,n}} + 2\frac{n-1}{\sqrt{q}} \right) \gamma(y(t))$$

the right term is bounded because $\gamma(y)$ remains bounded, so after a finite time L decreases if it is superior to a bounded value.
Let us now stem our second result which is an extension of the previous one :

Theorem 2: *Let us suppose the state of system (1) remains bounded by a given bounded controller u ; if the hypotheses [H1], (3), (4) are satisfied, there exists $\alpha < 1$ such that system (2) is Globally Finite Time Stable when we use the adaptation law (5) when $\varepsilon_1 \neq 0$ and :*

$$\dot{L} = -\frac{L}{b} \left[\frac{a}{3}(L-1) \right] \quad \text{when } \varepsilon_1 = 0 \tag{9}$$

with $L(0) = 1$.
Moreover, this law guarantees that L tends asymptotically to 1.

Sketch of the proof: We use almost the same computations than the ones carried in the proof of theorem 1.

If we apply the adaptation Law (5) when $\varepsilon_1 \neq 0$ and (9) otherwise, after some computations, we get:

$$\left\{ \begin{array}{lcll} \dot{V}_o & \leq & -aV_o & \text{if } \varepsilon_1 \neq 0 \\ \dot{V}_o & \leq & -\left(a - \frac{2(n-1)}{\sqrt{q}} \gamma(y) \right) V_o & \text{if } \varepsilon_1 = 0 \end{array} \right.$$

When $\varepsilon_1 = 0$, the Lyapunov function can increase but the system can not remain on the set $\mathcal{S}_c = \{\varepsilon_1 = 0 \ ; \ \varepsilon_{2,n} \neq 0\}$ during a non zero length time interval. We prove it by contradiction : suppose the system remains on this set during a non zero measured time interval, we have $\varepsilon_1 = \dot{\varepsilon}_1 = \ldots = \varepsilon_1^{(n-1)} = 0$ but because of equation (2), this implies that $\varepsilon_2 = \ldots = \varepsilon_n = 0$ which contradicts the fact that $\varepsilon \in \mathcal{S}_c$. We say that the error system 'globally decreases at almost every time'.

As for Theorem 1, we will finally come to the following conclusions : there exists $1 > \alpha > \max\left(1 - \frac{1}{n-1}, 1 - \bar{\alpha}_1, 1 - \bar{\alpha}_2\right) > 0$ such that:

- system (2) adapted by the law of theorem 2 GA decreases at almost every time.
- $\forall \varepsilon \in \mathcal{B}_1$, applying the adaption law to equation (8), we obtain :

$$\dot{V}_z \quad \leq \quad -\beta_2(\alpha, L) V_z^{\beta}(\varepsilon) + \frac{ac^{1,n}}{3} L V_z \quad \text{if } \varepsilon_1 \neq 0$$

  If $\varepsilon_1 = 0$, we also have:

$$\begin{array}{rcl} \dot{V}_z & \leq & -\beta_2(\alpha, L) V_z^{\beta}(\varepsilon) + \left(\frac{ac^{1,n}}{3} L + c_B \gamma(y) \right) V_z \\ & \leq & -\beta_2(\alpha, L) V_z^{\beta}(\varepsilon) + L \left(\frac{ac^{1,n}}{3} + c_B \gamma(y) \right) V_z \end{array}$$

  Let us define the following set:

$$\begin{array}{rcl} \mathcal{C}_3 & := & \left\{ \varepsilon \ s.t \ \frac{ac^{1,n}}{3} V_z(\varepsilon) + c_B \max_{t \geq 0} \gamma(y(t)) \leq \right. \\ & & \left. \min_{L \geq 1} \frac{\beta_2(\alpha, L)}{2L} V_z^{\beta}(\varepsilon) \right\} \end{array}$$

  Since the system GA decreases at almost every time, it will enter the set $\mathcal{B}_1 \bigcap \mathcal{C}_3$ in a finite time without leaving it anymore. Since in this set, we have :

$$\forall \varepsilon_1, \quad \dot{V}_z \leq -\frac{\beta_2(\alpha, L)}{2} V_z^{\beta}(\varepsilon)$$

  we conclude that system (2) converges to the origin in a finite time.

Thus, we proved the system is GFTS. Moreover, provided y remains bounded and since $L(0) = 1$, we prove that L remains bounded as in the proof of theorem 1. Moreover, since ε_1 and its time derivatives are equal to 0 after a finite time, the adaptation law (9) guarantees that L tends to 1.

V. ILLUSTRATIVE EXAMPLE

We illustrate the effectiveness of our design on the following two dimensional nonlinear system:

$$\left\{ \begin{array}{lcl} \dot{x}_1 & = & x_2 + x_1^2 \\ \dot{x}_2 & = & u + x_1^2 \cos x_2 \\ y & = & x_1 \end{array} \right.$$

Choosing $\gamma(y) = y^2$, it is easy to see that assumption [H1] holds.

Then we choose $\alpha = \frac{4}{5}$, $a = \frac{1}{2}$, we solve the LMIs given by (3) and (4) and we obtain :

$$Q = \begin{bmatrix} 0.5865 & -0.3787 \\ -0.3787 & 0.6530 \end{bmatrix}$$

and $q = 0.1054$, $k_{o1} = 2.7261$, $k_{o2} = 1.5811$, $b = 2.2646$

We apply an input u such that the state and the input of the system we want to observe remain bounded.

- Figure 1 shows the observer error when we apply the adaptation 1 : the error goes to 0 in finite time and L remains bounded. More specifically, in this example, L oscillates like y (which oscillates between two bounds).

- Figure 2 shows the observer error when we apply the adaptation 2 : the error goes to 0 in finite time and the fact that $\gamma(y)$ does not go to 0 does not prevent L from asymptotically tending to 1.

(Remark : if the initial error of the observer is large, we will need to choose bigger value of α sufficiently close to 1 (but still inferior to 1) so that the observer still converges).

So, this numerical result illustrates what we have theoretically proved in this paper.

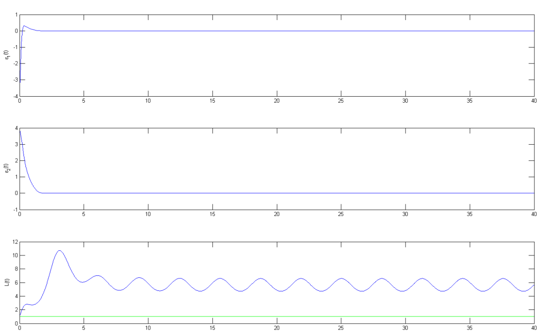

Fig. 1. With the adaptation Law 1

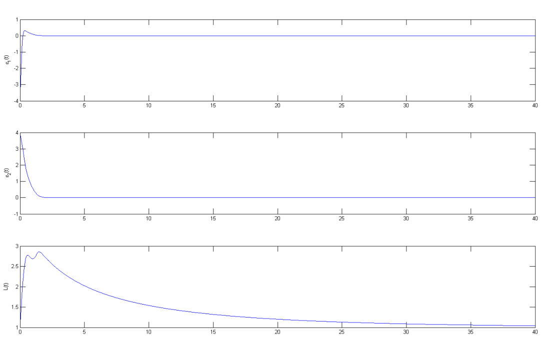

Fig. 2. With the adaptation Law 2

VI. CONCLUSIONS AND FUTURE WORK

This paper has addressed the problem of updating the gain of a global finite-time high gain observer. Using the separation principle for this specific class of triangular systems and adding a few hypotheses on the f_is, it may be quite straightforward to build a finite time controller and to combine it with our observer in order to get an output feedback which renders the system GFTS and such that the time-varying high gains of both controller and observer are bounded (and can even tend to 1).

REFERENCES

[1] S.P. Bhat and D.S. Bernstein, *Finite time stability of continuous autonomous systems*, SIAM J. Control. Optim., vol. 38(3), 1998, pp. 751-766.

[2] S.P. Bhat and D.S. Bernstein, *Geometric homogeneity with applications to finite-time stability*, Math. Control Signals Systems, vol. 17, 2005, pp. 101-127.

[3] J.P. Gauthier, H. Hammouri and S. Othman, *A simple observer for nonlinear systems applications to bioreactors*, IEEE TAC, vol. 37(6), 1992, pp. 875-880.

[4] L. Praly, *Asymptotic stabilization via output feedback for lower triangular systems with output dependent incremental rate*, IEEE TAC, vol. 48(6), 2003, pp. 1103-1108.

[5] H. Lei and W. Lin, *Universal adaptive control of nonlinear systems with unknown growth rate by output feedback*, Automatica, vol. 42, 2006, pp. 1783-1789.

[6] Y. Hong, J. Huang and Y. Xu, *On an output feedback finite-time stabilization problem*, IEEE TAC, vol. 46(2), 2001, pp. 305-309.

[7] X. Huang, W. Lin and B. Yang, *Global finite-time stabilization of a class of uncertain nonlinear systems*, Automatica, vol. 41, 2005, pp. 881-888.

[8] J. Li and C. Qian, *Global finite-time stabilization of a class of uncertain nonlinear systems using output feedback*, Proc. of the 44th Conference on Decision and Control, Seville, 2005, pp. 2652-2657.

[9] Y. Shen and X. Xia, *Semi-global finite-time observers for nonlinear systems*, Automatica, vol. 44, 2008, pp. 3152-3156.

[10] W. Perruquetti, T. Floquet and E. Moulay, *Finite-time observers : application to secure communication*, IEEE TAC, vol. 53(1), 2008, pp. 356-360.

[11] Y. Shen and Y. Huang, *Uniformly observable and globally lipschitzian nonlinear systems admit global finite-time observers*, IEEE TAC, vol. 54(11), 2009, pp. 2621-2625.

[12] T. Menard, E. Moulay and W. Perruquetti, *A global high-gain finite-time observer*, IEEE TAC, vol. 55(6), 2010, pp. 1500-1506.

[13] R. Jia and C. Qian, *Semi-global finite-time stabilization of a class of nonlinear systems by output feedback*, Proc. of the 49th Conference on Decision and Control, Atlanta, 2010, pp. 2169-2174.

[14] S. V. Drakunov, *Sliding mode observers based on equivalent control method*, Proc. of the 31st Conference on Decision and Control, Tucson, 1992, pp. 23682369.

[15] S. V. Drakunov and V. Utkin, *Sliding mode observers. Tutorial*, Proc. of the 34th Conference on Decision and Control, New Orleans, 1995, pp. 33763378.

[16] T. Ahmed-Ali and F. Lamnabhi-Lagarrigue, *Sliding observer-controller design for uncertain triangular nonlinear systems*, IEEE TAC, vol. 44(6), 1999, pp. 1244-1249.

[17] Y. Xiong and M. Saif, *Sliding mode observer for nonlinear uncertain systems*, IEEE TAC, vol. 46(12), 2001, pp. 2012-2017.

[18] J. A. Davila, L. M. Fridman and A. Levant, *Second-order sliding-mode observer for mechanical systems*, IEEE TAC, vol. 50(11), 2005, pp. 1785-1789.

[19] Yu. B. Shtessel, S. Baev, C. Edwards and S. Spurgeon, *HOSM Observer for a Class of Non-Minimum Phase Causal Nonlinear MIMO Systems*, IEEE TAC, vol. 55(2), 2010, pp. 543-548.