2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

On computation of H_∞ norm for commensurate fractional order systems

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***Abstract*— This paper tackles the problem of H-infinity (H_∞) norm computation for a commensurate Fractional Order System (FOS). First, H_∞ norm definition is given for FOS and Hamiltonian matrix of a FOS is computed. Two methods based on this Hamiltonian matrix are then proposed to compute the FOS H_∞ norm: one based on a dichotomy algorithm and another one on LMI conditions. The LMI conditions are based on the Generalized LMI characterization of axes in the complex plane on which the Hamiltonian matrix eigenvalues must not appear to ensure a FOS norm less than predefined value. The accuracy of the proposed methods is proved on the computation of the modulus margin of a CRONE passive car suspension.**

I. INTRODUCTION

Many phenomena can be modeled with Fractional Order Systems (FOS). Thus, several studies have been made on FOS properties such as stability. Using pole location analysis [5] results have been obtained for commensurate FOS stability. The most well known stability result is Matignon's criteria [15] which enables to test FOS stability through the location of the state matrix eigenvalues in the complex plane. In the sequel, some LMI-based results have been proposed for commensurate FOS stability [12] [20] but several LMI-based Integer Order Systems (IOS) results still need to be extended to FOS.

LMIs have indeed many applications in control theory since they can effectively express various problems arising in that domain [7]. Moreover, compared with analytical methods (such as Riccati equations for instance), LMI approach has better flexibility allowing to tackle complicated robust control problems. One of the well known results for evaluating an integer system H_∞ norm using an LMI is the Bounded Real Lemma [6]. This LMI allows both to test system stability using Lyapunov theory and to determine an upper bound for its H_∞ norm.

Contrary to integer order systems very few results can be found on FOS H_∞ norm computation [17]. For IOS, most H_∞ norm computation methods are based on Riccati equations and their LMI counterparts. Generalization of these methods to FOS is a tedious task given that they simultaneously test the system stability and H_∞ norm, and that stability domain of FOS of order $0 < \nu \leq 1$ is not convex and thus not an LMI region. However Moze [17] proposed important basis to compute an upper bound of a FOS H_∞ norm even if these results do not prove the system stability.

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In this paper, another way is explored to compute FOS H_∞ norm. First, H_∞ norm definition is given for FOS and Hamiltonian matrix of a FOS is computed. Two methods based on this Hamiltonian matrix are then proposed to compute a FOS H_∞ norm: one based on a dichotomy algorithm and another one on LMI conditions. The LMI conditions are based on the Generalized LMI characterization of two half axes in the complex plane on which the Hamiltonian matrix eigenvalues must not appear to ensure a FOS norm less than predefined value. The accuracy of the proposed methods is proved on the computation of the modulus margin of a CRONE car suspension [18].

Notations: The transpose of a matrix A is denoted A^T, its conjugate $\bar{A}$ and its conjugate transpose A^*. For Hermitian matrices, $\succ$ ($\succeq$) denotes the Löwner partial order, i.e. $A \succ B$ iff $A - B$ is (semi) positive definite. Laplace transform of a transfer function $G(j\omega)$ is written $G(s)$ and its H_∞ norm is noted $\|G(s)\|_\infty$.

II. FRACTIONAL ORDER SYSTEMS

A. LTI Commensurate Fractional Order Systems

In this paper are considered LTI commensurate FOS admitting a pseudo state space representation of the form

$$\left\{ \begin{array}{rcl} D^\nu x(t) & = & A\,x(t) + B\,u(t) \\ y(t) & = & C\,x(t) + D\,u(t) \end{array} \right. \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the pseudo state vector, $u(t) \in \mathbb{R}^m$ is the input vector, $y(t) \in \mathbb{R}^p$ is the output vector, ν is the fractional order of the system and A, B, C and D are constant matrices. D^ν is the fractional differentiation operator of order ν (presented results are valid whatever definition used: Riemann-Liouville [16], Caputo [9] or others [21]). Transfer matrix is $G(s) = C\left(s^\nu I - A\right)^{-1} B + D$ and impulse response matrix is $g(t) = \mathcal{L}^{-1}\left(G(s)\right)$.

Remark 1: For a FOS, the knowledge of $x(t_0)$ (t_0 being the initial time) is not sufficient to determine the future behavior of the system [13]. Consequently, vector x does not strictly represent the state of the system and is denoted "pseudo state" in this paper [19].

B. Stability of commensurate fractional order systems

Definition 1 (Matignon, 1996 [15]): A linear FOS defined by its impulse response g is Bounded-Input Bounded-Output (BIBO) stable iff $\forall\, u \in \mathcal{L}^\infty(\mathbb{R}^+, \mathbb{R}^m)$, $y = g \star u \in \mathcal{L}^\infty(\mathbb{R}^+, \mathbb{R}^p)$.

LTI IOS stability can be checked via the location of the state matrix A eigenvalues in the complex plane. This result was extended to LTI commensurate FOS of order $0 < \nu < 1$ by D. Matignon.

Theorem 1 (Matignon, 1996 [15]): System (1), with minimal triplet (A, B, C) and $0 < \nu < 1$, is BIBO stable iff

$$|\text{Arg}\,(\text{eig}(A))| > \nu\frac{\pi}{2}. \quad (2)$$

This result remains valid when $1 < \nu < 2$ as proved in [20]. Stability domain is thus defined as follows:

$$\mathcal{D}_s = \left\{ z \in \mathbb{C} \;:\; |\text{Arg}\,(z)| > \nu\frac{\pi}{2} \right\}. \quad (3)$$

Remark 2: Throughout the paper, triplet (A, B, C) is always supposed to be minimal.

III. H_∞ NORM OF A COMMENSURATE FRACTIONAL ORDER SYSTEM

As for LTI IOS [22], let us define the H_∞ norm of stable FOS (1) from its transfer function $G(s)$ as follows:

Definition 2: H_∞ norm of stable FOS system (1) is:

$$\|G\,(s)\|_\infty \triangleq \max_{\omega\in\mathbb{R}} \bar{\sigma}\,(G\,(j\omega))\,, \quad (4)$$

where $\bar{\sigma}\,(G\,(j\omega))$ is the largest singular value of $G\,(j\omega)$ at frequency ω:

$$\begin{aligned} \bar{\sigma}\,(G\,(j\omega)) &= \max_{i=\{1\cdots\min(m,p)\}} \sigma_i\,(G\,(j\omega)) & (5)\\ &= \max_{i=\{1\cdots\min(m,p)\}} \sqrt{\lambda_i\,(G(j\omega)^* G(j\omega))}. & (6)\end{aligned}$$

Steady state response of FOS (1) to sinusoidal input $u(j\omega)$ is $y(j\omega) = G(j\omega)u(j\omega)$. At frequency ω, the gain $\|y(j\omega)\|_2 / \|u(j\omega)\|_2$, depending on vector $u(j\omega)$ is:

$$\bar{\sigma}\,(G\,(j\omega)) = \max_{u(j\omega)\neq 0} \frac{\|y(j\omega)\|_2}{\|u(j\omega)\|_2}. \quad (7)$$

Worst case frequency gain is thus given by H_∞ norm of FOS:

$$\|G(s)\|_\infty = \max_{\omega\in\mathbb{R}} \max_{u(j\omega)\neq 0} \frac{\|y(j\omega)\|_2}{\|u(j\omega)\|_2}. \quad (8)$$

In time domain, equation (8) writes:

$$\|(G(s)\|_\infty = \max_{u(t)\neq 0} \frac{\|y(t)\|_2}{\|u(t)\|_2} = \max_{\|u(t)\|_2=1} \|y(t)\|_2\,. \quad (9)$$

Therefore, H_∞ norm can be interpreted in time domain as the largest energy among output signals resulting from all inputs of unit energy. Consequently, H_∞ norm physical interpretation, in frequency and time domains, is the same for FOS as for IOS.

IV. H_∞ NORM COMPUTATION

A. FOS Hamiltonian matrix

The H_∞ norm of IOS is usually computed numerically from a state-space realization as the smallest value of γ such that the Hamiltonian matrix H_γ has no eigenvalue on the imaginary axis [23]. This section develops a similar result for FOS.

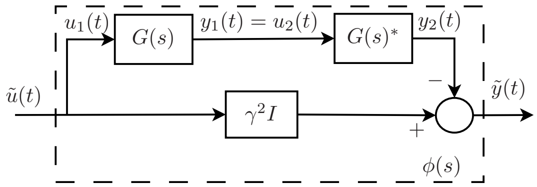

Fig. 1. Block diagram of $\phi(j\omega)$

1) H_∞ norm upper bound inequality: Definition 2 and relation (6) imply that H_∞ norm of FOS (1) is less than γ iff:

$$\forall\omega\in\mathbb{R}, \max_{i=\{1\cdots\min(m,p)\}} \sqrt{\lambda_i\,(G(j\omega)^* G(j\omega))} < \gamma. \quad (10)$$

The squared power of last inequality is:

$$\forall\omega\in\mathbb{R}, \max_{i=\{1\cdots\min(m,p)\}} \lambda_i\,(G(j\omega)^* G(j\omega)) < \gamma^2. \quad (11)$$

Using eigenvalues properties, relation (11) becomes:

$$\forall\omega\in\mathbb{R}, \max_{i=\{1\cdots\min(m,p)\}} \lambda_i\big(\gamma^2 I - G(j\omega)^* G(j\omega)\big) > 0, \quad (12)$$

which is equivalent to the H_∞ upper bound inequality:

$$\forall\omega\in\mathbb{R}, \big(\gamma^2 I - G(j\omega)^* G(j\omega)\big) \succ 0. \quad (13)$$

Relation (13) is an infinite dimension inequality since it depends on $\omega \in \mathbb{R}$. The next two subsections show how to make this relation independent of ω. The first step is to build a pseudo state-space representation of $\phi(s) = \gamma^2 I - G(s)^* G(s)$.

2) Pseudo state-space representation of $\phi(s)$: Transfer matrix $\phi(s)$ can be seen as the interconnection represented by block diagram of Fig. 1.

According to relation (1), system $G(s)$ of Fig. 1 admits the following pseudo state-space representation:

$$\left\{ \begin{array}{rcl} D^\nu x_1(t) &=& A\; x_1(t) + B\; u_1(t) \\ y_1(t) &=& C\; x_1(t) + D\; u_1(t) \end{array} \right. . \quad (14)$$

In order to derive a pseudo state-space representation for $G(s)^*$, it must be proved that conjugate of any commensurate fractional order transfer function evaluated at frequency ω is equal to the transfer function evaluated at $-\omega$:

$$\overline{G_{ij}(j\omega)} = G_{ij}(-j\omega), \quad (15)$$

where $G_{ij}(s)$ is the transfer function between input i and output j. Commensurate fractional order transfer function $G_{ij}(s)$ can be rewritten as:

$$G_{ij}(s) = K_e \frac{\prod_{q=1}^{N_z}(a_q + s^\nu)}{\prod_{q=1}^{N_p}(b_q + s^\nu)} = K_e \frac{\prod_{q=1}^{N_z} z_q(s)}{\prod_{q=1}^{N_p} p_q(s)}, \quad (16)$$

where N_p and N_z, $N_p \geq N_z$, are respectively the number of poles and zeros of $G_{ij}(s)$ and ν is the commensurate order.

At frequency ω, $z_q(s)$ becomes:

$$z_q(j\omega) = a_q + \omega^\nu j^\nu, \quad (17)$$

$$z_q(j\omega) = a_q + \omega^\nu cos\left(\nu\frac{\pi}{2}\right) + j\omega^\nu sin\left(\nu\frac{\pi}{2}\right). \quad (18)$$

$z_q(-j\omega)$ can be written similarly by noticing that:

$$(-j)^\nu = cos\left(\nu\frac{\pi}{2}\right) - jsin\left(\nu\frac{\pi}{2}\right), \quad (19)$$

$$z_q(-j\omega) = a_q + \omega^\nu cos\left(\nu\frac{\pi}{2}\right) - j\omega^\nu sin\left(\nu\frac{\pi}{2}\right). \quad (20)$$

That is to say:

$$z_q(-j\omega) = \overline{z_q(j\omega)}. \quad (21)$$

Applying a same analysis for poles and given that the product of conjugate complex numbers is equal to the conjugate of the complex numbers product leads to:

$$\overline{G_{ij}(j\omega)} = K_e \frac{\overline{\prod_q z_q(j\omega)}}{\overline{\prod_q p_q(j\omega)}} = K_e \frac{\prod_q z_q(-j\omega)}{\prod_q p_q(-j\omega)} = G_{ij}(-j\omega), \quad (22)$$

and that for a complex number $\bar{z}^{-1} = \overline{z^{-1}}$, relation (15) thus holds for FOS.

Thanks to relation (15), the transfer matrix associated with pseudo state-space representation (1) evaluated at $-j\omega$ is:

$$G(-j\omega) = C\left((-j\omega)^\nu I - A\right)^{-1} B + D, \quad (23)$$

$$G(-j\omega)^T = B^T\left(e^{-\nu j\pi}(j\omega)^\nu I - A^T\right)^{-1} C^T + D^T, \quad (24)$$

$$G(-j\omega)^T = e^{\nu j\pi} B^T\left((j\omega)^\nu I - e^{\nu j\pi} A^T\right)^{-1} C^T + D^T. \quad (25)$$

A pseudo state-space representation of $G(s)^*$ is thus:

$$\begin{cases} D^\nu x_2(t) = e^{\nu j\pi} A^T x_2(t) + e^{\nu j\pi} C^T u_2(t) \\ y_2(t) = B^T x_2(t) + D^T u_2(t) \end{cases}. \quad (26)$$

Therefore a pseudo state-space representation of the series interconnection $G(s)^*G(s)$ is:

$$\begin{cases} D^\nu \hat{x}(t) = \begin{pmatrix} A & 0 \\ e^{\nu j\pi} C^T C & e^{\nu j\pi} A^T \end{pmatrix} \hat{x}(t) + \begin{pmatrix} B \\ e^{\nu j\pi} C^T D \end{pmatrix} u_1(t) \\ y_2(t) = \begin{pmatrix} D^T C & B^T \end{pmatrix} \hat{x}(t) + D^T D u_1(t) \end{cases} \quad (27)$$

where $\hat{x} = \begin{pmatrix} x_1^T & x_2^T \end{pmatrix}^T$.

Thanks to relation (27), a pseudo state-space representation of $\phi(s)$ is:

$$\begin{cases} D^\nu \tilde{x}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}\tilde{u}(t) \\ \tilde{y}(t) = \tilde{C}\tilde{x}(t) + \tilde{D}\tilde{u}(t) \end{cases}, \quad (28)$$

where:

$$\begin{array}{ll} \tilde{A} = \begin{pmatrix} A & 0 \\ e^{\nu j\pi} C^T C & e^{\nu j\pi} A^T \end{pmatrix} & \tilde{B} = \begin{pmatrix} B \\ e^{\nu j\pi} C^T D \end{pmatrix}, \\ \tilde{C} = -\begin{pmatrix} D^T C & B^T \end{pmatrix} & \tilde{D} = \left(\gamma^2 I - D^T D\right). \end{array} \quad (29)$$

Combining the properties of H_∞ norm and the study of $\phi(s)$ pseudo state-space representation lead to one of the main results of this paper.

3) Hamiltonian matrix theorem:

Theorem 2: Let $\gamma > \bar{\sigma}(D)$ be a positive real number. Then $\|G(s)\|_\infty < \gamma$ iff fractional order Hamiltonian matrix:

$$H_\gamma = \begin{pmatrix} A + BRD^T C & BRB^T \\ e^{\nu j\pi} C^T\left(I + DRD^T\right) C & e^{\nu j\pi}\left(A^T + C^T DRB^T\right) \end{pmatrix} \quad (30)$$

where $R = \left(\gamma^2 I - D^T D\right)^{-1}$ has no eigenvalue in set $\mathbb{C}_{\nu 0} = \left\{(j\omega)^\nu = \omega^\nu e^{\nu j\frac{\pi}{2}}, \omega \in \mathbb{R}\right\}$.

Proof: Let us consider $\phi(s) = \gamma^2 I - G(-s)^T G(s)$. It can be noticed that the H_∞ norm of $G(s)$ is bounded by scalar γ i.e. $\|G(s)\|_\infty < \gamma$ iff $\phi(j\omega) \succ 0$ for all $\omega \in \mathbb{R}$. Considering that the limit of $G(-j\omega)^T G(j\omega)$ as ω tends towards infinity is given by the direct term of the pseudo state-space representation (27)

$$\lim_{\omega\to\infty} G(-j\omega)^T G(j\omega) = D^T D, \quad (31)$$

and given that $\gamma > \bar{\sigma}(D)$ implies:

$$\gamma^2 > \sigma_{max}\left(D^T D\right), \quad (32)$$

it can be noticed that:

$$\lim_{\omega\to\infty} \phi(j\omega) = \gamma^2 - \sigma_{max}\left(D^T D\right) > 0. \quad (33)$$

Moreover $\phi(j\omega)$ is a continuous function of ω. Therefore $\phi(j\omega) \succ 0$ for all $\omega \in \mathbb{R}$ iff $\phi(j\omega)$ is non singular for all $\omega \in \mathbb{R} \cup \{\infty\}$. For an IOS, that implies that $\phi(s)$ has no pure imaginary zero or $\phi(s)^{-1}$ has no pure imaginary pole. But since $G(s)$ is a commensurate FOS of order ν, $\phi(j\omega)$ is non singular for all $\omega \in \mathbb{R} \cup \{\infty\}$ iff the zeros of $\phi(s)$ or the poles of $\phi(s)^{-1}$ do not belong to the imaginary axis. Given the relations between eigenvalues and poles of a commensurate FOS [19], $\phi(s)^{-1}$ has no pole on the imaginary axis iff $\phi(s)^{-1}$ pseudo state matrix has no eigenvalue in $\mathbb{C}_{\nu 0} = \left\{(j\omega)^\nu = \omega^\nu e^{\nu j\frac{\pi}{2}}, \omega \in \mathbb{R}\right\}$.

A pseudo state-space representation of $\phi(s)^{-1}$ is found by inverting pseudo state-space relation (28) of $\phi(s)$. Let the input, the output and the pseudo state vector of $\phi(s)^{-1}$ be respectively $u_\gamma(t) = \tilde{y}(t)$, $y_\gamma(t) = \tilde{u}(t)$ and $x_\gamma(t) = \tilde{x}(t)$. Then:

$$y_\gamma(t) = \tilde{u}(t) = \tilde{D}^{-1}\left(\tilde{y}(t) - \tilde{C}\tilde{x}(t)\right), \quad (34)$$

$$y_\gamma(t) = -\tilde{D}^{-1}\tilde{C}x_\gamma(t) + \tilde{D}^{-1}u_\gamma(t). \quad (35)$$

The state equation is:

$$D^\nu \tilde{x}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}y_\gamma(t). \quad (36)$$

Using relation (35) in pseudo state equation (36) gives:

$$D^\nu \tilde{x}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}\left(-\tilde{D}^{-1}\tilde{C}x_\gamma(t) + \tilde{D}^{-1}u_\gamma(t)\right), \quad (37)$$

$$D^\nu \tilde{x}(t) = \left(\tilde{A} - \tilde{B}\tilde{D}^{-1}\tilde{C}\right)\tilde{x}(t) + \tilde{B}\tilde{D}^{-1}u_\gamma(t). \quad (38)$$

Therefore a state-space representation of $\phi(s)^{-1}$ is:

$$\begin{cases} D^\nu x_\gamma(t) = H_\gamma x_\gamma(t) + B_\gamma u_\gamma(t) \\ y_\gamma(t) = C_\gamma x_\gamma(t) + D_\gamma u_\gamma(t) \end{cases} \quad (39)$$

where

$$H_\gamma = \tilde{A} - \tilde{B}\tilde{D}^{-1}\tilde{C}$$
$$= \begin{pmatrix} A + BRD^T C & BRB^T \\ e^{\nu j\pi} C^T \left(I + DRD^T\right) C & e^{\nu j\pi}\left(A^T + C^T DRB^T\right) \end{pmatrix}, \quad (40)$$

$$B_\gamma = \tilde{B}\tilde{D}^{-1} = \begin{pmatrix} BR \\ e^{\nu j\pi} C^T DR \end{pmatrix}, \quad (41)$$

$$C_\gamma = -\tilde{D}^{-1}\tilde{C} = \begin{pmatrix} RD^T C & RB^T \end{pmatrix}, \quad (42)$$

$$D_\gamma = \tilde{D}^{-1} = R, \quad (43)$$

with $R = \left(\gamma^2 I - D^T D\right)^{-1}$.
$\phi(s)$ is thus not singular iff matrix H_γ eigenvalues do not belong to the set $\mathbb{C}_{\nu 0}$. That proves theorem 2. ■

Remark 3: Please note that fractional order Hamiltonian matrix H_γ is a complex matrix and does not have the properties of the integer order Hamiltonian matrix (for instance, its spectrum is not symmetric with respect to imaginary axis). Moreover, when system H_∞ norm is greater than γ, H_γ has eigenvalues in $\mathbb{C}_{\nu 0}$ and not on the imaginary axis as for IOS.

There are several ways to compute H_∞ norm using theorem 2. The next sections provide two pertinent methods.

B. γ-iteration

This method for H_∞ norm computation is a dichotomous optimization process directly derived from theorem 2. The following algorithm shows γ-iteration for a FOS:

1) Choose $[\gamma_{min}, \gamma_{max}]$ such that $\gamma_{min} > \bar{\sigma}(D)$.
2) For $\gamma = (\gamma_{min} + \gamma_{max})/2$, determine H_γ eigenvalues.
   – If the eigenvalues are not in set $\mathbb{C}_{\nu 0}$, γ is reduced by taking a new interval $[\gamma_{min}, \gamma]$.
   – If the eigenvalues are in set $\mathbb{C}_{\nu 0}$, γ is increased by taking a new interval $[\gamma, \gamma_{max}]$.
3) Step 2 is repeated until γ gives a satisfactory approximation of H_∞ norm.

Such method is implemented for IOS in SLICOT library [4] and in many numerical computing softwares such as MATLAB [3] and SCILAB [8]. This method is being implemented for FOS in CRONE toolbox [14] and will be released in a future version.

C. LMI-based H_∞ norm computation

Based on theorem 2, an LMI-based method for H_∞ norm computation is presented in this section. Such method is proposed as a first step to tackle the controller synthesis problem.

Theorem 3: The H_∞ norm of stable commensurate FOS (1) is bounded by scalar γ iff there exist three positive definite hermitian matrices X_1, X_2 and X_3 $\in \mathbb{C}^{2n\times 2n}$ such that

$$r_1 H_\gamma X_1 + \bar{r_1} X_1 H_\gamma^* + r_2 H_\gamma X_2 + \bar{r_2} X_2 H_\gamma^* - H_\gamma X_3 - X_3 H_\gamma^* \prec 0 \quad (44)$$

and three positive definite hermitian matrices X_4, X_5 and X_6 $\in \mathbb{C}^{2n\times 2n}$ such that

$$\bar{r}_1 H_\gamma X_4 + r_1 X_4 H_\gamma^* + \bar{r}_2 H_\gamma X_5 + r_2 X_5 H_\gamma^* - H_\gamma X_6 - X_6 H_\gamma^* \prec 0 \quad (45)$$

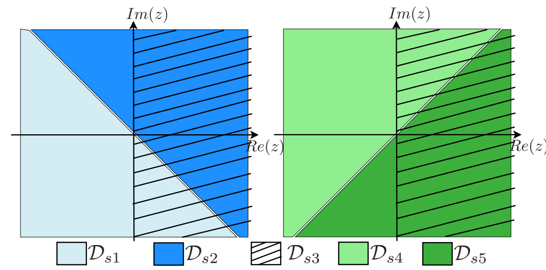

Fig. 2. $\mathbb{C} \setminus \mathbb{C}_{\nu 0}^+$ (left) and $\mathbb{C} \setminus \mathbb{C}_{\nu 0}^-$ (right) as unions of three half planes

where $r_1 = e^{j(1-\nu)\frac{\pi}{2}}$, $r_2 = e^{-j(1+\nu)\frac{\pi}{2}}$, and matrix H_γ is defined by (30).

Proof: The proof is based on the formalism introduced in [1], [2] and [10] on the concept of Generalized LMI (GLMI) regions, now introduced.

Definition 3: [10] A region $\mathcal{D}$ of the complex plane is a GLMI region of order l if $\exists\ \theta_k \in \mathbb{C}^{l\times l}$, $\psi_k \in \mathbb{C}^{l\times l}$, $H_k \in \mathbb{C}^{l\times l}$ and $J_k \in \mathbb{C}^{l\times l}$ ($\forall\ k \in \{1, \cdots, m\}$), s.t.

$$\mathcal{D} = \Big\{ z \in \mathbb{C}\ :\ \exists w = \begin{bmatrix} w_1 & \cdots & w_m \end{bmatrix}' \in \mathbb{C}^m \\ \text{s.t. } f_{\mathcal{D}}(z, w) < 0,\ g_{\mathcal{D}}(w) = 0 \Big\}, \quad (46)$$

where $f_{\mathcal{D}}(z,w) = \sum_{k=1}^m (\theta_k w_k + \theta_k^* \bar{w}_k + \psi_k z w_k + \psi_k^* \bar{w}_k \bar{z})$ and $g_{\mathcal{D}}(w) = \sum_{k=1}^m (H_k w_k + J_k \bar{w}_k)$.

It was previously mentioned that system (1) H_∞ norm is bounded by scalar γ iff matrix H_γ eigenvalues do not belong to the set $\mathbb{C}_{\nu 0} = \left\{(j\omega)^\nu = \omega^\nu e^{\nu j\frac{\pi}{2}}, \omega \in \mathbb{R}\right\}$. $\mathbb{C}_{\nu 0}$ can be decomposed in two domains $\mathbb{C}_{\nu 0}^+ = \{(j\omega)^\nu, \omega \in \mathbb{R}^+\}$ and $\mathbb{C}_{\nu 0}^- = \{(j\omega)^\nu, \omega \in \mathbb{R}^-\}$. Domain $\mathbb{C} \setminus \mathbb{C}_{\nu 0}^+$ can be viewed as the union of three half planes, denoted $\mathcal{D}_{s1}$, $\mathcal{D}_{s2}$ and $\mathcal{D}_{s3}$. The left half plane is obtained by rotating the first two of angles $\varphi_1 = (1-\nu)\frac{\pi}{2}$ and $\varphi_2 = -(1+\nu)\frac{\pi}{2}$ respectively and $\mathcal{D}_{s3}$ is the right half plane i.e. $\varphi_3 = \pi$ as shown in Fig. 2:

$$\mathbb{C} \setminus \mathbb{C}_{\nu 0}^+ = \mathcal{D}_{s1} \cup \mathcal{D}_{s2} \cup \mathcal{D}_{s3}. \quad (47)$$

Similarly, domain $\mathbb{C} \setminus \mathbb{C}_{\nu 0}^-$ is the union of $\mathcal{D}_{s4}$, $\mathcal{D}_{s5}$ and $\mathcal{D}_{s3}$ with $\varphi_4 = -\varphi_1$ and $\varphi_5 = -\varphi_2$ as shown in Fig. 2:

$$\mathbb{C} \setminus \mathbb{C}_{\nu 0}^- = \mathcal{D}_{s4} \cup \mathcal{D}_{s5} \cup \mathcal{D}_{s3}. \quad (48)$$

Therefore, showing first that H_γ eigenvalues are not in $\mathbb{C} \setminus \mathbb{C}_{\nu 0}^+$, then that they are not in $\mathbb{C} \setminus \mathbb{C}_{\nu 0}^-$ will prove that FOS (1) H_∞ norm is bounded by γ.

Each $\mathcal{D}_{si} = \left\{z \in \mathbb{C} : Re\left(z e^{j\varphi_i}\right) < 0\right\}$, $\forall i \in \{1 \cdots 5\}$ also writes

$$\mathcal{D}_{si} = \left\{ z \in \mathbb{C} : \exists w_i \in \mathbb{R}^+ \text{s.t. } e^{j\varphi_i z w_i} + e^{-j\varphi_i \bar{z}\bar{w}_i} < 0 \right\}, \quad (49)$$

and is thus a GLMI region of the form (46) with $m = 1$, $\theta_1 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$, $\psi_1 = \begin{bmatrix} e^{j\varphi_i} & 0 \\ 0 & 0 \end{bmatrix}$, $H_1 = 1$ and $J_1 = -1$.

As proved in [1], the union of m GLMI regions written

$$\mathcal{D}_k = \{ z \in \mathbb{C} : f_k(z) = \alpha_k + \beta_k z + \beta_k^* \bar{z} < 0 \}, \forall k \in \{1 \cdots m\}, \quad (50)$$

is a GLMI region of the form (46) with order $l = m+1$ and

$$\theta_k = \frac{1}{2}\begin{bmatrix} \Theta_k & \mathbf{0}_{1\times m} \\ \mathbf{0}_{m\times 1} & -\varepsilon_k^m \end{bmatrix}, \quad (51)$$

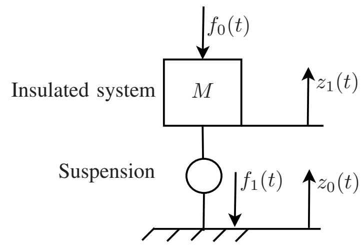

Fig. 3. One degree of freedom general model of a car suspension

$$\psi_k = \begin{bmatrix} \Psi_k & \mathbf{0}_{1\times m} \\ \mathbf{0}_{m\times 1} & \mathbf{0}_m \end{bmatrix}; H_k = -J_k = \varepsilon_{k+1}^{m+1}, \tag{52}$$

where ε_i^j are square matrices of size j defined as follows: $\varepsilon_i^j(\rho,\sigma) = 1$ if $\rho = \sigma = i$, $\varepsilon_i^j(\rho,\sigma) = 0$ else, and:

$$\Theta_k = \begin{bmatrix} \alpha_k & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}; \Psi_k = \begin{bmatrix} \beta_k & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}; \forall k \in \{1,\ldots,m\}. \tag{53}$$

Each region $\mathcal{D}_{si}$ can be described by relation (50) with $\alpha_k = 0$ and $\beta_k = e^{j\varphi_k}$. Therefore domains $\mathbb{C} \setminus \mathbb{C}_{\nu 0}^{+}$ and $\mathbb{C} \setminus \mathbb{C}_{\nu 0}^{-}$ are GLMI regions of form (46).

LMIs (44) and (45) are then found using the extension of the following $\mathcal{D}$-stability definition to GLMI regions thanks to lemma 1 from [10] with $\mathcal{D} = \mathbb{C} \setminus \mathbb{C}_{\nu 0}^{+}$ then $\mathcal{D} = \mathbb{C} \setminus \mathbb{C}_{\nu 0}^{-}$.

Definition 4: A matrix A is $\mathcal{D}$-stable iff its eigenvalues are strictly located in region $\mathcal{D}$ of the complex plane.

Lemma 1 (Chilali, 1996[10]): Let $A \in \mathbb{C}^{n\times n}$ and $\mathcal{D}$ a GLMI region. A is $\mathcal{D}$-stable iff $\exists$ m matrices $X_k \in \mathbb{C}^{n\times n}$ s.t.

$$\sum_{k=1}^{m} \left(\theta_k \otimes X_k + \theta_k^* \otimes X_k^* + \psi_k \otimes (AX_k) + \psi_k^* \otimes (AX_k)^*\right) \prec 0 \tag{54}$$

$$\sum_{k=1}^{m} (H_k \otimes X_k + J_k \otimes X_k^*) = \mathbf{0}_{nl\times nl}. \tag{55}$$

That concludes the proof. ■

Remark 4: H_∞ norm can be computed using section IV-B iterative algorithm replacing H_γ eigenvalues location test of step 2 by feasibility of LMIs (44) and (45) of theorem 3.

V. APPLICATION

Consider the general one degree of freedom model of a passive car suspension presented in Fig. 3 where $M = 300$ kg is the car quarter mass. The profile of the road $z_0(t)$ and efforts $f_0(t)$ applied on the suspension are respectively viewed as disturbances at the output and input of the system whose transfer function is given by:

$$G(s) = \frac{1}{Ms^2}. \tag{56}$$

$f_1(t)$ is the force generated by the suspension and $z_1(t)$ is the vertical movement of the mass which is being insulated. The suspension deflection $z_{10}(t) = z_1(t) - z_0(t)$ is thus regulated around a null set point as shown in Fig. 4. The controller $C(s)$ used in that application is defined by the transfer function:

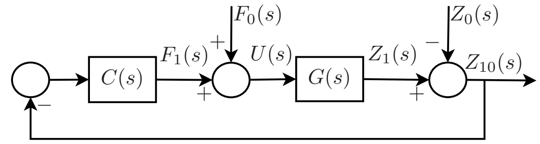

Fig. 4. One degree of freedom car suspension general model block diagram

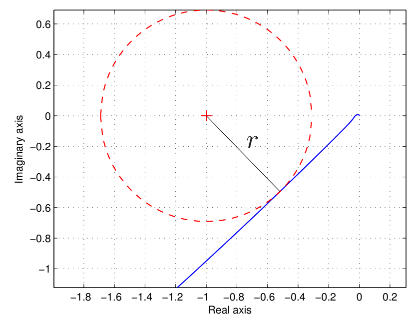

Fig. 5. Intersection of $G(s)C(s)$ Nyquist curve (plain line) and circle centered in -1 with radius r (dotted line).

$$C(s) = C_0 \frac{1 + \left(\frac{s}{\omega_b}\right)^{1.5}}{\frac{s}{\omega_b}\left(1 + \left(\frac{s}{\omega_h}\right)^{1.5}\right)} \tag{57}$$

where $C_0 = 100$ N.m^{-1} and $\omega_b = 0.08$ rad.s^{-1} and $\omega_h = 20$ rad.s^{-1} are respectively the low and high corner frequencies of the controller. Synthesis was performed in frequency domain in order to minimize disturbances effect on closed-loop system and to ensure a phase margin of about 45°. Such a controller is used in CRONE suspension to ensure the robustness of the deflection overshoot to sprung mass variations [18].

The methodology proposed in the previous section is used to evaluate the modulus margin of the closed-loop system presented in Fig. 4. Modulus margin r of a single input single output LTI system is the shortest distance between the open loop Nyquist curve and critical point -1. In this case, series interconnection $G(s)C(s)$ is a commensurate FOS of order $\nu = 1.5$ with pseudo state-space matrices:

$$\begin{aligned} A &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\omega_h^{1.5} \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \\ C &= \frac{C_0\omega_b\omega_h^{1.5}}{M}\begin{pmatrix} 1 & \frac{1}{\omega_b^{1.5}} & 0 \end{pmatrix} \qquad D = 0. \end{aligned} \tag{58}$$

and pseudo state vector $x = \left(z_{10} \quad \left(z_{10}^{(1.5)}\right) \quad \left(z_{10}^{(3)}\right)\right)^T$. The shortest distance between $G(s)C(s)$ Nyquist curve and critical point -1 is shown on Fig. 5 and is equal to $r = 0.69$.

Another method to measure the modulus margin is to determine the largest additive disturbance $\overline{\Delta}(j\omega)$ among all possible disturbances $\Delta(j\omega)$ which guarantee Fig. 6 closed-loop block diagram stability. Modulus margin is thus given

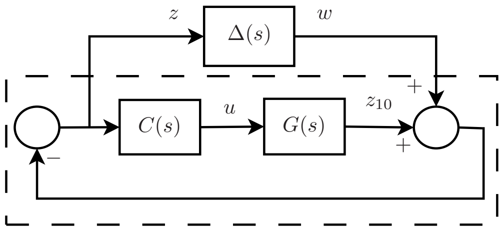

Fig. 6. Disturbed closed-loop with an additive disturbance Δ

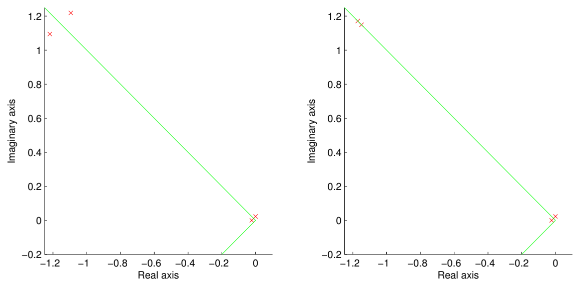

Fig. 7. Eigenvalues of H_γ for $\gamma = 1.45$ (left) and $\gamma = 1.4478$ (right)

by $r = \left\|\overline{\Delta}(j\omega)\right\|_\infty$. According to small gain theorem [11], modulus margin is thus the inverse of the H_∞ norm of transfer $T_{w\to z}$:

$$r = \frac{1}{\|T_{w\to z}\|_\infty}. \tag{59}$$

$T_{w\to z}$ pseudo state-space representation matrices are given by: $A_\Delta = A - BC$, $B_\Delta = -B$, $C_\Delta = -C$ and $D_\Delta = -1$. According to remark 4, an iterative procedure involving theorem 3 LMIs leads to:

$$\|T_{w\to z}\|_\infty = 1.4479 \tag{60}$$

and thus the modulus margin

$$r = \frac{1}{1.4479} = 0.6907 \tag{61}$$

is exactly retrieved.

Fig. 7 shows $\mathbb{C}_{\nu 0}$ and some eigenvalues of H_γ for $\gamma = 1.45 > \|T_{w\to z}\|_\infty$ and $\gamma = 1.4478 < \|T_{w\to z}\|_\infty$. For $\gamma > \|T_{w\to z}\|_\infty$, no eigenvalue belongs to $\mathbb{C}_{\nu 0}$. For $\gamma < \|T_{w\to z}\|_\infty$, eigenvalues close to $-1.1 + 1.1i$ now belong to $\mathbb{C}_{\nu 0}$. These graphical considerations thus coincide with theorem 3 result.

VI. CONCLUSIONS AND FUTURE WORKS

The major contributions of this paper are

- Hamiltonian matrix definition for a FOS ;
- two methods based on this Hamiltonian for FOS H_∞ norm computation.

The first one is based on the inspection of the Hamiltonian matrix eigenvalues location using a dichotomy algorithm. In the second one, the eigenvalues location inspection is replaced by two LMI conditions. These two LMIs result from the introduction of the GLMI concept in the definition of the area of the complex plane in which the Hamiltonian matrix eigenvalues must not appear to ensure a FOS H_∞ norm less than predefined value. This last method has been used to measure accurately the modulus margin of an passive car suspension whose deflection is regulated by a fractional controller. Our future works aim now to find a linearizing change of variable in the proposed LMI conditions that will permit to find directly, without an iterative process, the FOS H_∞ norm.

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