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Symmetric Formulation of the Kalman-Yakubovich-Popov Lemma and Exact Losslessness Condition

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Abstract— **This paper presents a new algebraic framework for robust stability analysis of linear time invariant systems with an emphasis on symmetry. The main motivation for this work is to provide a unified theory to answer when the the KYP lemma provides an exact LMI test for robust stability. The notions of weak and strong mutual losslessness are introduced to characterize for lossless S-procedures and the KYP lemma. The new framework has sufficient flexibility to unify some recent extensions of the KYP lemma, including the Generalized KYP lemma for finite frequency analysis, the KYP lemma for nD systems, and the diagonal KYP lemma for positive systems. Finally, we show that the new theory also suggests that the structured singular value of internally positive systems with arbitrary number of scalar uncertainties can be exactly computed.**

I. Introduction

The symmetry we emphasize throughout this paper can be seen in the following illustrative example. Consider a discrete time linear model

$$G: \begin{array}{l} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) + Du(k) \end{array} .$$

For the sake of simplicity, we assume both A and D are real square matrices throughout this paper. From the Kalman-Yakubovich-Popov (KYP) lemma, we know that the above system is dissipative with respect to the supply rate $s(u,y)$ with a positive definite storage function $V(x)$, where

$$s(u,y) = y^T Qy - u^T Qu \; ; \; V(x) = x^T Px$$

if and only if the LMI

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \Theta \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^* \Pi \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} < 0 \qquad (1)$$

$$\Theta = \begin{bmatrix} P & 0 \\ 0 & -P \end{bmatrix}, \; \Pi = \begin{bmatrix} Q & 0 \\ 0 & -Q \end{bmatrix}$$

is feasible. Now, notice an apparent symmetry holding between Θ and Π in (1). Namely, if we define a new system

$$\tilde{G}: \begin{array}{l} x(k+1) = Dx(k) + Cu(k) \\ y(k) = Bx(k) + Au(k) \end{array}$$

then the above condition is equivalent to saying that $\tilde{G}$ is dissipative with respect to the supply rate $\tilde{s}(u,y)$ with a storage function $\tilde{V}(x)$ given by

$$\tilde{s}(u,y) = y^T Py - u^T Pu \; ; \; \tilde{V}(x) = x^T Qx.$$

The KYP Lemma plays a central role in dissipativity theory [1] and IQC theory [2]. Among the voluminous literature related to the KYP lemma, the aforementioned symmetry seems to be implied by not many, but some papers. The most explicit reference is given by [3], which points out the "duality" between the H_∞ norm conditions and the parameter dependent Lyapunov conditions in the context of well-posedness analysis of uncertain LTI systems. Another indicator of this symmetry in the literature is the fact that the notion of S-procedure is used for both purposes of frequency domain specification and uncertainty specification. For example, the Generalized KYP lemma [4] utilizes a sophisticated lossless S-procedure to derive a tractable algorithm for nontrivial frequency domain test, while the role of the S-procedure in the IQC framework is well known [5]. However, to the best of our knowledge, there is no literature discussing the KYP lemma with an explicit emphasis on this symmetry.

Our own interest in formally studying this symmetry was sprung by recent results on the "diagonal" Bounded Real lemma for Metzler systems [6]. Indeed, while an algebraic property known as *rank-one separability* (defined in [4]) plays a fundamental role in many modern proofs of losslessness of various types of KYP like lemmas, it appears that the diagonal Bounded Real Lemma can be proved without resorting to a rank-one separability argument. (By losslessness, we mean that the S-procedure or the KYP lemma converts a system theoretic condition into computable condition such as LMIs without introducing conservatisms). This suggests a need for a more general and essential notion to indicate the losslessness of the KYP lemma.

One benefit of highlighting the symmetry is that it clarifies the point that the losslessness of the S-procedure and the KYP lemma should be naturally discussed as a matter of the relationship between two Hermitian sets (the *primal* set and the *dual* set in the sequel). In this way, we propose a new notion, which we call *mutual losslessness* to characterize the exact condition for losslessness. We first propose a symmetric formulation of the S-procedure. Then we show that an algebraic condition called *weak mutual losslessness* between the primal and the dual set is the exact condition for the S-procedure to be lossless. Based on this consideration, it is also shown that *strong mutual losslessness* between the primal and the dual set is the exact condition for the KYP lemma to be lossless. Although these new algebraic conditions remain difficult to check, we believe that this approach opens new perspectives for understanding and proving the lossless S-procedure and the KYP lemma.

Our symmetric formulation of the KYP lemma naturally suggests to use a matrix valued frequency variable, just

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as the uncertainty block is naturally modeled as a matrix in traditional robust control. This matrix valued frequency turns out to have sufficient flexibility to describe various types of frequency regions including continuous time models, discrete time models, finite frequency regions, and nD (multi dimensional) systems. As a result, the notion of mutual losslessness can explain the lossy and lossless properties of various types of KYP lemmas in a single framework. Finally, within the proposed framework, we point out a new lossless KYP lemma that was heretofore absent from the literature, suggesting that the structured singular value μ for internally positive systems with an arbitrary number of scalar blocks can be efficiently computed.

In short, the contributions of this paper lie in: (1) Introduction of the symmetric S-procedure, which has its own beauty and novelty, (2) The notion of mutual losslessness, which is a new way of characterizing the losslessness of KYP lemma, (3) A framework for matrix valued frequency variables, which enables a unified description of various types of the KYP lemma, (4) A proof of losslessness of μ-analysis for internally positive systems.

Notations: $\bar{\mathbb{C}}_+$: closed right half plane, $\bar{D}$: closed unit disc, $\mathbb{S}^n$: unit sphere in $\mathbb{C}^n$, $\mathbb{H}_n$: $n\times n$ Hermitian matrices.

II. Preliminaries

A. Well-posedness of an algebraic loop

In this paper, multi-input multi-output, linear time invariant, rational transfer functions are expressed in the following form:

$$G(\Lambda)=C(I-\Lambda A)^{-1}\Lambda B+D, \tag{2}$$

where the matrix valued parameter $\Lambda\in\mathbb{C}^{n\times n}$ is considered as the "frequency variable." Notice the relationship between the expression (2) and the standard expression using the Laplace variable $s\in\mathbb{C}$

$$\hat{G}(s)=C(sI-A)^{-1}B+D. \tag{3}$$

If $\Lambda=\lambda I$, $\lambda\in\mathbb{C}$, and the relation $\lambda=s^{-1}$ holds, the two complex functions (2) and (3) are identical everywhere on the Riemann sphere except at $\{0\}$ and $\{\infty\}$.

The transfer function (2) can be viewed as a linear fractional transform (LFT) of the system matrix

$$M=\begin{bmatrix} A & B\\ C & D\end{bmatrix}\in\mathbb{R}^{(n+m)\times(n+m)}$$

by the frequency variable Λ provided that the interconnection is well-posed in the following sense (Fig.1).

Definition 1: (Well-posedness)

Let a matrix $A\in\mathbb{C}^{n\times n}$ and a subset $\mathbf{\Lambda}\subseteq\mathbb{C}^{n\times n}$ be given. The interconnection $[A,\mathbf{\Lambda}]$ defined by

$$z=Aw+v;\ w=\Lambda z+u,\ \Lambda\in\mathbf{\Lambda} \tag{4}$$

is said to be well-posed if the following conditions hold.

(1). For each $\Lambda\in\mathbf{\Lambda}$ and $(u,v)\in\mathbb{C}^n\times\mathbb{C}^n$, there exist a unique pair of vectors $(z,w)\in\mathbb{C}^n\times\mathbb{C}^n$ such that (4) holds.

(2). There exists $\gamma>0$ such that

$$\left\|\begin{bmatrix} z\\ w\end{bmatrix}\right\|\le\gamma\left\|\begin{bmatrix} u\\ v\end{bmatrix}\right\|,\ \forall\begin{bmatrix} u\\ v\end{bmatrix}\in\mathbb{C}^{2n},\forall\Lambda\in\mathbf{\Lambda}.$$

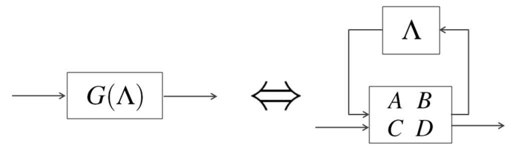

Fig. 1. LFT representation of a transfer function

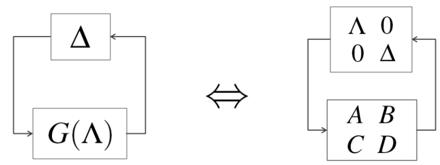

Fig. 2. Generalized main loop theorem

Lemma 1: The following statements hold.

(a) The interconnection $[A,\mathbf{\Lambda}]$ is well-posed if and only if

$$\exists\beta>0 \text{ s.t. } \left\|(I-\Lambda A)^{-1}\begin{bmatrix}\Lambda & I\end{bmatrix}\right\|\le\beta,\ \forall\Lambda\in\mathbf{\Lambda}. \tag{5}$$

(b) The interconnection $[A,\mathbf{\Lambda}]$ is ill-posed if and only if

$$\exists\begin{bmatrix} z\\ w\end{bmatrix}\in\mathbb{S}^{2n} \text{ s.t } \inf_{\Lambda\in\mathbf{\Lambda}}\left\|\begin{bmatrix} I & -A\\ -\Lambda & I\end{bmatrix}\begin{bmatrix} z\\ w\end{bmatrix}\right\|=0. \tag{6}$$

(c) If $[A,\mathbf{\Lambda}]$ is well-posed then

$$det(I-\Lambda A)\neq 0,\ \forall\Lambda\in\mathbf{\Lambda} \tag{7}$$

The proof is straightforward and omitted for reason of space. Consider the interconnected system of Fig.2 (left) in which the nominal transfer function $G(\Lambda)$ is interconnected to an unknown matrix $\Delta\in\mathbb{C}^{m\times m}$. Since the transfer function $G(\Lambda)$ can be represented as the LFT of Fig. 1, the original system Fig.2 (left) can be converted to an algebraic interconnection of the system matrix M and two external matrices Λ and Δ in Fig.2 (right). Hence the well-posedness of the original system can be analyzed through the well-posedness of Fig.2 (right) where $(\Lambda,\Delta)\in(\mathbf{\Lambda},\mathbf{\Delta})\subseteq\mathbb{C}^{n\times n}\times\mathbb{C}^{m\times m}$. This operation can be thought of as a generalization of the main loop theorem [3]. As we will see in the sequel, the set $\mathbf{\Lambda}$ is typically a user specified frequency region and the set $\mathbf{\Delta}$ is the user specified uncertainly region. In this formulation, the algebraic roles of the two matrices Λ and Δ are symmetric, even though the physical meanings of these two objects are originally different. This observation provides us with the intuition that the techniques used today to characterize system uncertainties can also be used to characterize frequency regions, and vice versa.

Remark 1: Frequently we need to analyze the well-posedness of the interconnection of a transfer function $G(\Lambda)$ and the uncertainly Δ, with Λ and Δ varying in subsets $\mathbf{\Lambda}\in\mathbb{C}^{n\times n}$ and $\mathbf{\Delta}\in\mathbb{C}^{m\times m}$, respectively. As Fig.2 shows, the generalized main loop theorem converts the original question into the well-posedness of $[M,\mathbf{\Omega}]$ where

$$M=\begin{bmatrix} A & B\\ C & D\end{bmatrix},\quad \mathbf{\Omega}=\begin{bmatrix}\mathbf{\Lambda} & 0\\ 0 & \mathbf{\Delta}\end{bmatrix}.$$

Hence in the sequel, the expression $[G(\mathbf{\Lambda}),\mathbf{\Delta}]$ is understood to mean $[M,\mathbf{\Omega}]$ as long as $G(\Lambda)$ is well-defined on $\mathbf{\Lambda}$. Similarly, $[\tilde{G}(\mathbf{\Delta}),\mathbf{\Lambda}]$ is understood to mean $[M,\mathbf{\Omega}]$ as long as $\tilde{G}(\Delta):=B(I-\Delta D)^{-1}\Delta C+A$ is well-defined on $\mathbf{\Delta}$.

Table I. Examples of frequency regions $\mathscr{R}(\mathbf{\Theta})$

	$\mathbf{\Theta}$	Frequency region $\mathscr{R}(\mathbf{\Theta})$	Applications
(a)	$\begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} : P > 0$	$\Lambda = \lambda I : \lambda \in \bar{\mathbb{C}}_+$	Frequency domain for continuous time stable systems
(b)	$\begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} : P > 0$ is diagonal	$\Lambda = diag(\lambda_1, \cdots, \lambda_n)$: $\lambda_i \in \bar{\mathbb{C}}_+ \; \forall i = 1, \cdots, n$	Frequency domain for stable Metzler systems
(c)	$\begin{bmatrix} Q & 0 \\ 0 & -Q \end{bmatrix} : Q > 0$	$\Lambda = \lambda I, \lambda \in \bar{\mathbb{D}}$	Frequency domain for discrete time stable systems
(d)	$\begin{bmatrix} Q & 0 \\ 0 & -Q \end{bmatrix} : \begin{array}{l} Q = diag(Q_h, Q_v) \\ Q_h \in \mathbb{H}_{n_h}, Q_v \in \mathbb{H}_{n_v} \end{array}$	$\Lambda = diag(e^{j\omega_h} I_{n_h}, e^{j\omega_h} I_{n_v})$	Frequency domain for 2D Roesser Model
(e)	$\begin{bmatrix} Q & P \\ P & -\omega_0^2 Q \end{bmatrix} : Q > 0, P \in \mathbb{H}_n$	$\Lambda = (j/\omega) I : \lvert\omega\rvert \geq \omega_0$	High frequency range for continuous time systems

Table II. Examples of uncertainty regions $\mathscr{R}(\mathbf{\Pi})$

	$\mathbf{\Pi}$	Uncertainty region $\mathscr{R}(\mathbf{\Pi})$	Applications
(f)	$\begin{bmatrix} \tau I & 0 \\ 0 & -\tau I \end{bmatrix} : \tau > 0$	$\Delta \in \mathbb{C}^{n \times n} : \lVert\Delta\rVert \leq 1$	Small gain uncertainties
(g)	$\begin{bmatrix} \Pi_d & 0 \\ 0 & -\Pi_d \end{bmatrix} : \begin{array}{l} \Pi_d = diag(\tau_1 I, \cdots, \tau_r I) \\ \tau_i > 0 \; \forall i = 1, \cdots, r \end{array}$	$\Delta = diag(\Delta_1, \cdots, \Delta_r)$: $\lVert\Delta_i\rVert \leq 1 \; \forall i = 1, \cdots, r$	Structured uncertainties
(h)	$\begin{bmatrix} 0 & \tau I \\ \tau I & 0 \end{bmatrix} : \tau > 0$	$\Delta \in \mathbb{C}^{n \times n}, \Delta + \Delta^* \geq 0$	Positive real uncertainties

B. Frequency regions in $\mathbb{C}^{n\times n}$

Using Lemma 1, it can be shown that a continuous time transfer function (2) is stable if and only if (A,B,C,D) is the minimal realization and the interconnection $[A, \mathbf{\Lambda}]$ is well-posed, where $\mathbf{\Lambda} = \{\lambda I : \lambda \in \bar{\mathbb{C}}_+\}$. Likewise, a discrete time transfer function (2) is stable if and only if (A,B,C,D) is the minimal realization and the interconnection $[A, \mathbf{\Lambda}]$ is well-posed where $\mathbf{\Lambda} = \{\lambda I : \lambda \in \bar{D}\}$. A matrix valued frequency variable Λ can represent other types of frequency regions in the same framework.

Example 1: (Finite frequency analysis) Let $\hat{G}(s) \in \mathscr{RH}_\infty$ be given. Let γ be the largest gain of the frequency response over a finite frequency range $0 < \omega_1 \leq |\omega| \leq \omega_2$, i.e.,

$$\|\hat{G}(j\omega)\| < \gamma, \; \forall \omega \in [\omega_1, \omega_2].$$

The value γ can be efficiently analyzed by considering the well-posedness of the interconnection Fig.2 where

$$\mathbf{\Lambda} = \{\frac{j}{\omega} I : \omega_1 \leq |\omega| \leq \omega_2\}; \; \mathbf{\Delta} = \{\Delta \in \mathbb{C}^{m\times m} : \|\Delta\| \leq 1/\gamma\}.$$

The Generalized KYP lemma [4] provides a lossless LMI test for this type of well-posedness analysis. For a complete discussion, readers are referred to the original report [4].

Example 2: (2-D Discrete Roesser Model)

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = A \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + Bu(i,j)$$
$$y(i,j) = C \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + Du(i,j)$$

The positive realness of this system can be analyzed through the well-posedness of the interconnection Fig.2 with

$$\mathbf{\Lambda} = \left\{ \begin{bmatrix} e^{-j\omega_h I} & 0 \\ 0 & e^{-j\omega_v I} \end{bmatrix} : \omega_h, \omega_v \in \mathbb{R} \right\}$$
$$\mathbf{\Delta} = \{\Delta \in \mathbb{C}^{m\times m} : \Delta + \Delta^* \geq 0\}$$

A version of the KYP lemma is known for the analysis of the 2-D Roesser model [7]. However, the KYP lemma for 2-D system is generally lossy – the LMI test provided by the lemma is only a sufficient condition for the well-posedness. Example 1 and 2 will be revisited in Section IV. It is interesting to notice that the KYP lemma is lossless in Example 1 whereas it is lossy in Example 2. In Section III, we will see that the losslessness of the KYP lemma is determined by the relative relationship between $\mathbf{\Lambda}$ and $\mathbf{\Delta}$.

C. Quadratic characterization of $\mathbf{\Lambda}$ and $\mathbf{\Delta}$

In many cases, robustness analysis problems can be stated as well-posedness problems. In order to obtain computationally efficient algorithms for well-posedness analysis, we consider the characterization of frequency regions $\mathbf{\Lambda}$ and uncertainty regions $\mathbf{\Delta}$ using quadratic forms. This makes the S-procedure applicable and converts the original well-posedness analysis into semidefinite programming (SDP) problems. Namely, we specify a frequency region by $\mathbf{\Lambda} = \mathscr{R}(\mathbf{\Theta}) \subseteq \mathbb{C}^{n\times n}$ with the following expression using a set $\mathbf{\Theta}$ of Hermitian matrices:

$$\mathscr{R}(\mathbf{\Theta}) := \left\{ \Lambda \in \mathbb{C}^{n\times n} : \begin{bmatrix} I \\ \Lambda \end{bmatrix}^* \Theta \begin{bmatrix} I \\ \Lambda \end{bmatrix} \geq 0 \;\; \forall \Theta \in \mathbf{\Theta} \right\}.$$

For example, it is straightforward to show that a Hermitian set $\mathbf{\Theta} = \left\{ \Theta \in \mathbb{H}_{2n} : \Theta = \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix}, P > 0 \right\}$, yields $\mathscr{R}(\mathbf{\Theta}) = \left\{ \Lambda = \lambda I : \lambda \in \bar{\mathbb{C}}_+ \right\}$, which is the frequency region for well-posedness analysis for continuous time systems.

Many practically useful frequency regions and uncertainty regions can be generated with proper choices of $\mathbf{\Theta}$ and $\mathbf{\Pi}$. A few examples are summarized in Table I and II. Notice that (a) through (e) show frequency regions that are used for the analyses of various type of systems, while (f) (g) and (h) show regions that are frequently used to represent various types of system uncertainties. In the former cases, variables P and Q are conventionally interpreted as the Lyapunov functions, while in the latter cases $\mathbf{\Pi}$ can be viewed as a set of IQCs. Defining the frequency variables in $\mathbb{C}^{n\times n}$ enables us to express these different objects in a single framework, and it allows for a symmetric formulation of the KYP lemma.

III. Main Result

A. Symmetric S-procedure and mutual losslessness property

A strong connection between the KYP lemma and the S-procedure is well known. Let $\mathbf{\Psi}$ be a convex cone in $\mathbb{H}_n$ and $\Phi \in \mathbb{H}_n$. The S-procedure concerns the relationship between the following conditions.

(I) $\exists \Psi \in \mathbf{\Psi}$ such that $\Psi + \Phi < 0$.

(II) For every nonzero complex vector ζ,

$$(\zeta^*\Psi\zeta \geq 0 \; \forall \Psi \in \mathbf{\Psi}) \Rightarrow \zeta^*\Phi\zeta < 0.$$

It is easy to see the implication (I)⇒(II) holds in general. If the other direction holds as well, the S-procedure is said to be *lossless*. Conditions for the S-procedure to be lossless have been studied in many literatures. A popular condition, which has been a basis of modern proofs of the KYP lemma, is called *rank-one separability* [8][4].

Definition 2: $\mathbf{\Psi} \subset \mathbb{H}_n$ is said to be *rank-one separable* if $S(\mathbf{\Psi})$ is equal to the convex hull of $S_1(\mathbf{\Psi})$ where

$$S(\mathbf{\Psi}) := \{X \in \mathbb{H}_n : X \geq 0, X \neq 0, tr\Psi X \geq 0 \; \forall \Psi \in \mathbf{\Psi}\}$$
$$S_1(\mathbf{\Psi}) := \{\zeta\zeta^* \in \mathbb{H}_n : \zeta \in \mathbb{C}^n, \zeta \neq 0, \zeta^*\Psi\zeta \geq 0 \; \forall \Psi \in \mathbf{\Psi}\}.$$

In [4], it is proven that the rank-one separability of the set $\mathbf{\Psi}$ is the exact condition for the S-procedure to be lossless when Φ is an arbitrary Hermitian matrix. However, it is important to notice that the exact condition can be relaxed if Φ is known to belong to a restricted class of matrices. For example, the authors of [9] have introduced the notion of *one-vector-lossless sets*, which is weaker than rank-one separability, but sufficient to prove that the S-procedure is lossless for matrices $\Phi \leq 0$.

Therefore, the losslessness property of the S-procedure should be discussed as a matter of the relationship between the set $\mathbf{\Psi}$ and the set $\mathbf{\Phi}$ to which Φ belongs. To clarify this point, it is natural to introduce the following generalization of the S-procedure, which has a symmetric structure:

(I) $\exists (\Psi, \Phi) \in (\mathbf{\Psi}, \mathbf{\Phi})$ such that $\Psi + \Phi < 0$.

(II) For every nonzero complex vector ζ,

$$(\zeta^*\Psi\zeta \geq 0 \; \forall \Psi \in \mathbf{\Psi}) \Rightarrow (\exists \Phi \in \mathbf{\Phi} \text{ such that } \zeta^*\Phi\zeta < 0).$$

(III) For every nonzero complex vector ζ,

$$(\zeta^*\Phi\zeta \geq 0 \; \forall \Phi \in \mathbf{\Phi}) \Rightarrow (\exists \Psi \in \mathbf{\Psi} \text{ such that } \zeta^*\Psi\zeta < 0).$$

It is easy to check that (I)⇒(II), (I)⇒(III) in general. We refer to the relaxation from (II) to (I) as the *primal* S-procedure, and (III) to (I) as the *dual* S-procedure. Accordingly, we call $\mathbf{\Psi}$ the primal set and $\mathbf{\Phi}$ the dual set. In the usual form of S-procedures, the dual set is a singleton (up to its positive multiples). In what follows, the primal set and the dual set are assumed to be convex cones.

In order to define a relationship between $\mathbf{\Psi}$ and $\mathbf{\Phi}$ that suffices to ensure a lossless S-procedure, it is crucial to observe that conditions (II) and (III) are contrapositive to each other. Thus, the primal S-procedure is lossless if and only if the dual S-procedure is lossless. This suggests that the condition characterizing lossless S-procedure should be a symmetric relationship between $\mathbf{\Psi}$ and $\mathbf{\Phi}$. Now consider the following two conditions.

(A) There exists a nonzero $X \geq 0$ such that

$$tr\Psi X \geq 0 \; \forall \Psi \in \mathbf{\Psi} \text{ and } tr\Phi X \geq 0 \; \forall \Phi \in \mathbf{\Phi} \tag{8}$$

(B) There exists a nonzero vector ζ such that

$$\zeta^*\Psi\zeta \geq 0 \; \forall \Psi \in \mathbf{\Psi} \text{ and } \zeta^*\Phi\zeta \geq 0 \; \forall \Phi \in \mathbf{\Phi} \tag{9}$$

Notice that implication (A)⇐(B) holds in general.

Definition 3: $\mathbf{\Psi}$ and $\mathbf{\Phi}$ are said to be *weakly mutually lossless* (denoted by $\mathbf{\Psi} \lhd\!\rhd \mathbf{\Phi}$) if (A)⇔(B).

Proposition 1: The primal and the dual S-procedure are lossless if and only if $\mathbf{\Psi} \lhd\!\rhd \mathbf{\Phi}$.

Proof: It is easy to see that (B) is the negation of (II) and (III). Also, it is possible to show that (A) is the negation of (I) using the Hahn-Banach separation theorem as follows. Consider the following convex cones in $\mathbb{H}_n$.

$$\mathscr{N} = \{N \in \mathbb{H}_n : N < 0\}; \;\; \mathscr{M} = \{\Psi + \Phi : \Psi \in \mathbf{\Psi}, \Phi \in \mathbf{\Phi}\}.$$

Since $\mathscr{N}$ is open in the standard topology on $\mathbb{H}_n$, and the negation of (I) means that $\mathscr{N}$ and $\mathscr{M}$ are disjoint, there exists a nonzero Hermitian matrix X and $r \in \mathbb{R}$ such that

$$trNX < r \leq trMX \; \forall (N, M) \in (\mathscr{N}, \mathscr{M}).$$

Since $\mathscr{N}$ and $\mathscr{M}$ are cones, we can take $r = 0$ without loss of generality. Then the above inequalities read that there exists a nonzero $X \geq 0$ such that

$$tr(\Psi + \Phi)X \geq 0 \; \forall (\Psi, \Phi) \in (\mathbf{\Psi}, \mathbf{\Phi}).$$

This implies condition (A) since $\mathbf{\Psi}$ and $\mathbf{\Phi}$ are cones. The other direction (A)⇒ ¬(I) is easy to see. Thus (A)⇔ ¬(I). Hence (I)(II) and (III) are all equivalent if and only if $\mathbf{\Psi} \lhd\!\rhd \mathbf{\Phi}$. ■

For example, if $\mathbf{\Psi}$ is rank-one separable [4],

$$\mathbf{\Psi} \lhd\!\rhd \{k\Phi_0 : k > 0\} \text{ for any given } \Phi_0 \in \mathbb{H}_n. \tag{10}$$

If $\mathbf{\Psi}$ is one-vector-lossless [9],

$$\mathbf{\Psi} \lhd\!\rhd \{k\Phi_0 : k > 0\} \text{ for any given } \Phi_0 \leq 0. \tag{11}$$

B. KYP lemma and exact losslessness condition

In the context of system theoretic analysis, $\mathbf{\Psi}$ and $\mathbf{\Phi}$ in (8) and (9) are often chosen of the form

$$\mathbf{\Psi} = \left\{ \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \Theta \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} : \Theta \in \mathbf{\Theta} \right\} \tag{12}$$

$$\mathbf{\Phi} = \left\{ \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^* \Pi \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} : \Pi \in \mathbf{\Pi} \right\} \tag{13}$$

where $\mathbf{\Theta} \in \mathbb{H}_{2n}, \mathbf{\Pi} \in \mathbb{H}_{2m}$ are convex cones. For our formulation of the KYP lemma to be lossless, weak mutual losslessness between $\mathbf{\Psi}$ and $\mathbf{\Phi}$ (i.e, (A)⇔(B)) is necessary but not sufficient. Consider the following stronger condition in addition to (A) and (B).

(C) The interconnection $[M,\mathbf{\Omega}]$ is ill-posed where

$$M=\begin{bmatrix} A & B \\ C & D \end{bmatrix},\ \mathbf{\Omega}=\begin{bmatrix} \mathscr{R}(\mathbf{\Theta}) & 0 \\ 0 & \mathscr{R}(\mathbf{\Pi}) \end{bmatrix}.$$

Proposition 2: (C)$\Rightarrow$(B).

Proof: By Lemma 1 (b), (C) implies

$$\exists \begin{bmatrix} \nu \\ \zeta \end{bmatrix} \in \mathbb{S}^{2n} \text{ s.t } \inf_{\Omega\in\mathbf{\Omega}} \left\| \begin{bmatrix} I & -M \\ -\Omega & I \end{bmatrix} \begin{bmatrix} \nu \\ \zeta \end{bmatrix} \right\| = 0.$$

This implies $\exists\zeta\neq 0$ s.t $\inf_{\Omega\in\mathbf{\Omega}}\|(I-\Omega M)\zeta\|=0$. Therefore, there exist $\zeta=[\xi^T\ \eta^T]^T\neq 0$ and sequences $\{\Lambda_k\}\in\mathbf{\Lambda}$, $\{\Delta_k\}\in\mathbf{\Delta}$ such that

$$\left(I-\begin{bmatrix} \Lambda_k & 0 \\ 0 & \Delta_k \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right)\begin{bmatrix} \xi \\ \eta \end{bmatrix}=\begin{bmatrix} u_k \\ v_k \end{bmatrix}$$

and $\lim_{k\to\infty} u_k=0$, $\lim_{k\to\infty} v_k=0$. Since $\Lambda_k\in\mathscr{R}(\mathbf{\Theta})$ for each k,

$$\lim_{k\to\infty} w^*\begin{bmatrix} I \\ \Lambda_k \end{bmatrix}^*\Theta\begin{bmatrix} I \\ \Lambda_k \end{bmatrix} w\geq 0,\ \forall\Theta\in\mathbf{\Theta},\forall w.$$

Thus by taking $w=A\xi+B\eta$, and noticing that $\Lambda_k(A\xi+B\eta)=\xi-u_k$, we have

$$\lim_{k\to\infty} w^*\begin{bmatrix} I \\ \Lambda_k \end{bmatrix}^*\Theta\begin{bmatrix} I \\ \Lambda_k \end{bmatrix} w=\lim_{k\to\infty}\begin{bmatrix} A\xi+B\eta \\ \xi-u_k \end{bmatrix}^*\Theta\begin{bmatrix} A\xi+B\eta \\ \xi-u_k \end{bmatrix}$$
$$=\zeta^*\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^*\Theta\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}\zeta\geq 0\ \forall\Theta\in\mathbf{\Theta}.$$

Therefore we obtained the first condition of (B). Similarly, noticing that $\Delta_k\in\mathscr{R}(\mathbf{\Pi})$ for each k and using the equality $\Delta_k(C\xi+D\eta)=\eta-v_k$, we obtain the second condition of (B). ■

Definition 4: $\mathbf{\Psi}$ and $\mathbf{\Phi}$ are said to be *strongly mutually lossless* (denoted by $\mathbf{\Psi} \blacktriangleleft\!\blacktriangleright \mathbf{\Phi}$) if (A)$\Leftrightarrow$(C).

To summarize, the implication (A)$\Leftarrow$(B)$\Leftarrow$(C) holds in general, (A)$\Leftrightarrow$(B) holds if and only if $\mathbf{\Psi} \lhd\!\rhd \mathbf{\Phi}$, and (A)$\Leftrightarrow$(B)$\Leftrightarrow$(C) holds if and only if $\mathbf{\Psi} \blacktriangleleft\!\blacktriangleright \mathbf{\Phi}$. As a result $\mathbf{\Psi} \blacktriangleleft\!\blacktriangleright \mathbf{\Phi} \Rightarrow \mathbf{\Psi} \lhd\!\rhd \mathbf{\Phi}$ holds.

Unfortunately, for a given combination of Hermitian sets, there is no general method for checking the losslessness conditions introduced so far. Hence in many cases, we have to rely on individual techniques from the existing literature to verify these conditions. We are now ready to state the main theorem.

Theorem 1: (KYP lemma) Let the frequency region and the uncertainty region be defined by $\mathbf{\Lambda}=\mathscr{R}(\mathbf{\Theta})$ and $\mathbf{\Delta}=\mathscr{R}(\mathbf{\Pi})$. Suppose $[A,\Lambda]$ and $[D,\Delta]$ are well-posed. Then (I)$\Rightarrow$(II) and (I)$\Rightarrow$(III) hold in general, where

(I) There exists $\Theta\in\mathbf{\Theta}$ and $\Pi\in\mathbf{\Pi}$ such that

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^*\Theta\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}+\begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^*\Pi\begin{bmatrix} C & D \\ 0 & I \end{bmatrix}<0. \quad (14)$$

(II) The interconnection $[G(\mathbf{\Lambda}),\mathbf{\Delta}]$ is well-posed.

(III) The interconnection $[\tilde{G}(\mathbf{\Delta}),\mathbf{\Lambda}]$ is well-posed.

Moreover, (I)$\Leftrightarrow$(II)$\Leftrightarrow$(III) holds if and only if $\mathbf{\Psi} \blacktriangleleft\!\blacktriangleright \mathbf{\Phi}$.

Proof: As we saw in Proposition 1, (I)$\Leftrightarrow$ $\neg$(A). Also, $\neg$(A)$\Leftrightarrow$ $\neg$(C) in general and $\neg$(A)$\Leftrightarrow$ $\neg$(C) if and only if $\mathbf{\Psi} \blacktriangleleft\!\blacktriangleright \mathbf{\Phi}$. Since $G(\Lambda)$ and $\tilde{G}(\Delta)$ are well-defined on $\mathbf{\Lambda}$ and $\mathbf{\Delta}$ respectively by assumption, $\neg$(C) (well-posedness of $[M,\mathbf{\Omega}]$) is equivalent to (II) and (III). ■

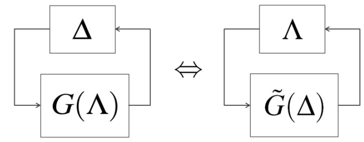

Fig. 3. Two different ways to interpret the LMI condition (14)

In many literatures (e.g., [8][4]), the KYP lemma is stated as an equivalence between a matrix inequality condition such as (14) and a frequency domain inequality (FDI). Here we employ Theorem 1, which does not involve an FDI, as the standard form of the KYP lemma since in many robust stability and performance analyses, the desired system theoretic property is really the well-posedness condition.

Theorem 1 provides two different ways to interpret the LMI condition (14). Assuming that $G(\Lambda)$ and $\tilde{G}(\Delta)$ are well-defined on $\mathscr{R}(\mathbf{\Theta})$ and $\mathscr{R}(\mathbf{\Pi})$ respectively, the LMI (14) means that the interconnection of $G(\Lambda)$ and Δ is well-posed, and at the same time, the interconnection of $\tilde{G}(\Delta)$ and Λ is well-posed. This implies that $\mathscr{R}(\mathbf{\Theta})$ and $\mathscr{R}(\mathbf{\Pi})$ play symmetric roles, although conventionally the former is considered as the frequency domain and the latter is considered as the space of uncertainties. With this symmetry in mind, consider the congruence transformation $T^*\Theta_c T=\Theta_d$ applied to the quadratic forms in $\mathbf{\Theta}$ where $T=\frac{1}{\sqrt{2}}\begin{bmatrix} I & -I \\ I & I \end{bmatrix}$. This operations amounts to the Möbius transformation $s=\frac{z-1}{z+1}$ between frequency variables, and converts the continuous-time frequency domain into the discrete-time one ((a) and (c) in Table I). The same transformation $T^*\Pi_s T=\Pi_p$ can be viewed as a conversion of the small gain IQC into the passivity IQC ((f) and (h) in Table II). The meaning of these two operations are flipped when the LMI (14) is interpreted through system $\tilde{G}$ instead of G (Fig.3).

C. Connection with Rank-One Separability

Rank-one separability is a useful notion for a lossless S-procedure because of (10). However, rank-one separability of the primal set $\mathbf{\Psi}$ is neither necessary nor sufficient for the S-procedure and the KYP lemma to be lossless. An example is already shown in (11). In the context of system analysis, we usually have a good knowledge of the dual set $\mathbf{\Phi}$. For example, $\mathbf{\Phi}$ of the form (13) with $\mathbf{\Pi}=\{\tau\Pi_0:\tau>0\}$ can be used for the following analysis.

$$\begin{aligned}
\Pi_0&=\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}: H_\infty \text{ analysis} \\
\Pi_0&=\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}: \text{Passivity analysis} \qquad (15) \\
\Pi_0&=\begin{bmatrix} \Pi_d & 0 \\ 0 & -\Pi_d \end{bmatrix}: \mu\text{-analysis}
\end{aligned}$$

where Π_d is a block diagonal scaling matrix. Hence in practice, requiring $\mathbf{\Psi}$ to be rank-one separable is asking for too much since the weaker condition $\mathbf{\Psi} \lhd\!\rhd \mathbf{\Phi}$ is sufficient for a lossless S-procedure where $\mathbf{\Phi}$ is a specific dual set under the consideration. Also, the rank-one separability of the primal set $\mathbf{\Psi}$ does not guarantee losslessness when the dual set is not of the form $\mathbf{\Phi} = \{k\Phi_0 : k > 0\}$. This is the case when, for example, multiple IQCs are used to characterize uncertainties. In the general μ-analysis (Section IV-D), $\mathbf{\Psi} \lhd\rhd \mathbf{\Phi}$ does not hold in general (let alone $\mathbf{\Psi} \blacktriangleleft\blacktriangleright \mathbf{\Phi}$) even if $\mathbf{\Psi}$ is rank-one separable.

The next example demonstrates the fact that mutual losslessness can be established without relying on rank-one separability.

Lemma 2: Let A be a Metzler matrix (all off-diagonal entries are nonnegative) and B, C, D be entry-wise nonnegative. Then $\mathbf{\Psi} \lhd\rhd \mathbf{\Phi}$ holds between

$$\mathbf{\Psi}=\left\{\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \Theta \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} : \Theta \in \mathbf{\Theta}\right\}, \mathbf{\Theta}=\left\{\begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} : \begin{array}{c} P > 0 \text{ is} \\ \text{diagonal} \end{array}\right\}$$
$$\mathbf{\Phi}=\left\{\begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^* \Pi \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} : \Pi \in \mathbf{\Pi}\right\}, \mathbf{\Pi}=\left\{\begin{bmatrix} Q & 0 \\ 0 & -Q \end{bmatrix} : \begin{array}{c} Q > 0 \text{ is} \\ \text{diagonal} \end{array}\right\}. \tag{16}$$

In Section IV-D, it will be shown that the above combination of $\mathbf{\Psi}$ and $\mathbf{\Phi}$ is even *strongly* mutually lossless.

Proof: (Outline only) It suffices to prove (A)$\Leftrightarrow$(B) holds between (8) and (9). Suppose (A) holds and let $X = WW^* \geq 0$ be any factorization of X. Then we claim that a choice $\zeta = [\|w_{,1}\| \ \cdots \ \|w_{,n}\|]^T \neq 0$ satisfies (9) where $w_{,i}$ denotes the i-th row of W. This can be shown using an algebraic technique used in [10] (also used in [6]). ■

The mutual lossless property between these sets will be used to develop the *diagonal* KYP lemma for internally positive systems in Section IV-B. The proof of the diagonal KYP lemma is not based on the rank-one separability of $\mathbf{\Psi}$ in (16). In fact, to the authors' knowledge, it is not known whether $\mathbf{\Psi}$ in (16) is rank-one separable or not. This indicates that rank-one separability is not a necessary condition to develop a lossless KYP lemma.

IV. Applications to Special Cases

In this section, we show that the notion of mutual losslessness can be used to capture, and sometimes extend, known well-posedness analysis condition.

A. Finite frequency KYP lemma

The Generalized KYP lemma [4][11] proposed an exact LMI test for well-posedness analysis over a new type of frequency regions. The frequency region can be captured by $\Lambda = \lambda I$ where λ moves on finite curves on the complex plane $\mathbb{C}$ that can be represented by a certain class of quadratic forms. The breakthrough result of [4][11] is that this class of quadratic forms allows lossless S-procedures. Most notably, the framework provides a lossless LMI test for the performance analysis over a finite frequency range. For example, a low frequency range $\Lambda = j\tau I, |\tau| \geq 1/\omega_0$ for continuous time systems can be specified by the following Hermitian set (see also (e) in Table II):

$$\mathbf{\Psi}=\left\{\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \Theta \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} : \Theta \in \mathbf{\Theta}\right\}, \mathbf{\Theta}=\left\{\begin{bmatrix} -Q & P \\ P & \omega_0^2 Q \end{bmatrix} : \begin{array}{c} P \in \mathbb{H}_n \\ Q > 0 \end{array}\right\} \tag{17}$$

Let Π_0 specify performance criteria such as (15) and define

$$\mathbf{\Phi}=\left\{\begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^* \Pi \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} : \Pi \in \mathbf{\Pi}\right\}, \mathbf{\Pi}=\{\kappa\Pi_0 : \kappa > 0\}. \tag{18}$$

Then the Generalized KYP lemma can be essentially stated withing the framework of Theorem 1. For technical reason, we need an assumption that the uncertainty region can be expressed exactly by a single IQC parametrized by Π_0. Define

$$\mathscr{C}_1 = \left\{\begin{bmatrix} \xi \\ \eta \end{bmatrix} \in \mathbb{C}^{2m} : \begin{bmatrix} \xi \\ \eta \end{bmatrix}^* \Pi_0 \begin{bmatrix} \xi \\ \eta \end{bmatrix} \geq 0\right\}$$
$$\mathscr{C}_2 = \left\{\begin{bmatrix} \xi \\ \eta \end{bmatrix} \in \mathbb{C}^{2m} : \exists \Delta \in \mathbb{C}^{m\times m} \text{ s.t. } \eta = \Delta\xi, \begin{bmatrix} I \\ \Delta \end{bmatrix}^* \Pi_0 \begin{bmatrix} I \\ \Delta \end{bmatrix} \geq 0\right\}.$$

Assumption 1: The Hermitian matrix Π_0 satisfies $\mathscr{C}_1 = \mathscr{C}_2$.

Note that for a general Hermitian matrix Π_0, we have only $\mathscr{C}_1 \supset \mathscr{C}_2$. However, many practical situations including (15) satisfy Assumption 1.

Corollary 1: (Finite frequency KYP lemma) Let $\mathbf{\Lambda}$ denote the low frequency region

$$\mathbf{\Lambda} = \{j\tau I : |\tau| \geq 1/\omega_0\}$$

and the uncertainty region be given by $\mathbf{\Delta} = \mathscr{R}(\mathbf{\Pi})$ where $\mathbf{\Pi}$ is as in (18). Suppose Assumption 1 holds, and $I - \Lambda A$ is non-singular on $\mathbf{\Lambda}$. Then the following are equivalent:

(I) There exist $P, Q \in \mathbb{H}_n$ such that $Q > 0$ and
$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \begin{bmatrix} -Q & P \\ P & \omega_0^2 Q \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^* \Pi_0 \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} < 0$$

(II) The interconnection $[G(\mathbf{\Lambda}), \mathbf{\Delta}]$ is well-posed.

The proof is outlined as the following. Since the frequency region $\mathbf{\Lambda}$ can be expressed as $\mathbf{\Lambda} = \mathscr{R}(\mathbf{\Theta})$ where $\mathbf{\Theta}$ is given by (17), the above statement is a special case of Theorem 1. Therefore (I)$\Rightarrow$(II) holds in general. By Theorem 1, the other implication holds if and only if $\mathbf{\Psi} \blacktriangleleft\blacktriangleright \mathbf{\Phi}$ holds between (17) and (18). This can be shown by proving the rank-one separability of the primal set $\mathbf{\Psi}$ and Assumption 1. Namely, using the technique introduced in [11] and applying Lemma 1, one can prove that (A) implies (C) (conditions considered in Section III-B).

B. Diagonal KYP lemma for internally positive systems

A continuous time system $G(\Lambda) = C(I - \Lambda A)^{-1}\Lambda B + D$ is said to be *internally positive* if A is a Metzler matrix and B, C, D are entry-wise nonnegative matrices. For this class of systems, a diagonal quadratic storage function can be assumed without conservatism in the small gain test. We already established this fact in [6], but here the result is presented in a slightly different way by emphasizing the strong mutual losslessness between $\mathbf{\Psi}$ and $\hat{\mathbf{\Phi}}$ defined below:

$$\mathbf{\Psi}=\left\{\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \Theta \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} : \Theta \in \mathbf{\Theta}\right\}, \mathbf{\Theta}=\left\{\begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} : \begin{array}{c} P > 0 \text{ is} \\ \text{diagonal} \end{array}\right\} \tag{19}$$

$$\hat{\mathbf{\Phi}}=\left\{\begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^* \Pi \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} : \Pi \in \mathbf{\Pi}\right\}, \mathbf{\Pi}=\left\{\begin{bmatrix} \kappa I & 0 \\ 0 & -\kappa I \end{bmatrix} : \kappa > 0\right\}. \tag{20}$$

Corollary 2: (Diagonal KYP Lemma) Suppose $G(\Lambda)$ is internally positive and A is Hurwitz. Let the frequency region and the uncertainty region be given by

$$\begin{aligned}\mathbf{\Lambda} &= \{diag(\lambda_1, \cdots, \lambda_n) : \lambda_i \in \bar{\mathbb{C}}_+ \ \forall i = 1, \cdots, n\} \\ \mathbf{\Delta} &= \{\Delta \in \mathbb{C}^{n\times n} : \|\Delta\| \leq 1\}.\end{aligned}$$

Then the following are equivalent.

(I) There exists a diagonal matrix $P > 0$ such that

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^* \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} < 0$$

(II) The interconnection $[G(\mathbf{\Lambda}), \mathbf{\Delta}]$ is well-posed.

Remark 2: Since the frequency variable Λ can contain n independent complex numbers corresponding to each state, the above result can be seen as the KYP lemma for internally positive nD systems. In this sense the above result is a slight extension of our previous result [6]. In [6], it is shown that even if the frequency region is restricted to $\mathbf{\Lambda} = \{\lambda I : \lambda \in \bar{\mathbb{C}}_+\}$, the above LMI test is exact for the small gain analysis. In the language of [12], this observation is related to the fact that a diagonally stable system is D-stable as well.

Proof: Since A is Metzler and Hurwitz, it is D-stable, i.e., for any positive diagonal matrix E, EA is Hurwitz [12]. This implies that $I - \Lambda A$ is non-singular for all $\Lambda \in \mathbf{\Lambda}$. Thus a transfer function $G(\Lambda)$ is well-defined on $\mathbf{\Lambda}$. Referring to Table I (b) and Table II (f), frequency and uncertainty regions are expressed as $\mathbf{\Lambda} = \mathscr{R}(\mathbf{\Theta})$, $\mathbf{\Delta} = \mathscr{R}(\mathbf{\Pi})$ using $\mathbf{\Theta}$ in (19) and $\mathbf{\Pi}$ in (20). Thus Theorem 1 is applicable and the implication (I)$\Rightarrow$(II) holds in general. To prove (II)$\Rightarrow$(I), we need to prove $\mathbf{\Psi} \blacktriangleleft\!\blacktriangleright \hat{\mathbf{\Phi}}$ holds between (19) and (20). Assume (A). From Lemma 2 and $\hat{\mathbf{\Phi}} \subset \mathbf{\Phi}$, we have $\mathbf{\Psi} \lhd\!\rhd \hat{\mathbf{\Phi}}$. Thus there exists a nonzero vector $\zeta = [\xi^T \ \eta^T]^T$ such that

$$\zeta^* \Psi \zeta \geq 0 \ \forall \Psi \in \mathbf{\Psi} \text{ and } \zeta^* \Phi \zeta \geq 0 \ \forall \Phi \in \hat{\mathbf{\Phi}}. \qquad (21)$$

Assume that $\eta = 0$. Then the first condition of (21) implies that all diagonal entries of $A\xi\xi^*$ are nonnegative. Since A is Metzler and Hurwitz, the Barker-Berman-Plemmons result [10] implies $\xi = 0$. Since this contradicts $\zeta \neq 0$, we have that $\eta \neq 0$. Notice that it follows from the first condition of (21) that all diagonal entries of $\xi(A\xi + B\eta)^* + (A\xi + B\eta)\xi^*$ are nonnegative, i.e.,

$$\xi_i (A\xi + B\eta)_i^* + (A\xi + B\eta)_i \xi_i^* \geq 0 \ \forall i = 1, \cdots, n.$$

By Lemma 3 of [8], this implies that there exist complex numbers $\lambda_i \in \bar{\mathbb{C}}_+$ such that $\xi_i = \lambda_i (A\xi + B\eta)_i$ for all $i = 1, \cdots, n$. Define $\Lambda = diag(\lambda_1, \cdots, \lambda_n)$ then we have $\Lambda \in \mathbf{\Lambda}$ and

$$\xi = \Lambda(A\xi + B\eta). \qquad (22)$$

Next notice that $C\xi + D\eta \neq 0$ because otherwise the second condition of (21) implies $\eta = 0$. Define $\Delta = \eta(C\xi + D\eta)^*/\|C\xi + D\eta\|^2$ then we have $\Delta \in \mathbf{\Delta}$ and

$$\eta = \Delta(C\xi + D\eta). \qquad (23)$$

From (22) and (23), the interconnection $[G(\mathbf{\Lambda}), \mathbf{\Delta}]$ is ill-posed, and thus (A)$\Rightarrow$(C) was shown. ∎

Corollary 2 can be viewed as the Bounded Real Lemma for internally positive systems. The critical difference from the usual Bounded Real Lemma is that one can assume a *diagonal* solution P in the LMI condition (I). This result means that if an internally positive systems is dissipative with respect to the supply rate of the small gain type, then one can always find a diagonal storage function. This is a counterpart to the well-known fact that a linear stable autonomous positive systems always admit a diagonal Lyapunov function. From the viewpoint of decentralized controller and estimator synthesis, the fact that P can be chosen to be diagonal has significant merit. For details, the reader is referred to [6].

C. KYP lemma for nD-systems

In this subsection, we consider a case where Theorem 1 (KYP lemma) holds only in one direction, i.e., the LMI test is only a sufficient condition for the well-posedness. Such situations occur when $\mathbf{\Psi} \blacktriangleleft\!\blacktriangleright \mathbf{\Phi}$ does not hold.

In [7], the KYP lemma for discrete 2D Roesser model is proposed. It is different from the classical KYP lemma in that there are two frequency variables $e^{j\omega_h}$ and $e^{j\omega_v}$ corresponding to the horizontal state and the vertical state. Notice that Theorem 1 has a flexible form and it is possible to represent this situation by properly specifying $\mathbf{\Theta}$ (item (d) in Table I). However, if the primal set $\mathbf{\Psi}$ is defined by this $\mathbf{\Theta}$ and the dual set $\mathbf{\Phi}$ is defined by $\mathbf{\Pi} = \{\tau\Pi_0 : \tau > 0\}$ with a fixed Hermitian matrix Π_0, the relationship $\mathbf{\Psi} \blacktriangleleft\!\blacktriangleright \mathbf{\Phi}$ does not hold in general. This is essentially the reason why the LMI test proposed in [7] is only a sufficient condition.

Another type of 2D system is considered in [13], where there are two frequency variables corresponding to the temporal state and the spatial state. A sufficient LMI test is proposed for the H_∞ performance analysis of spatially interconnected systems. The necessity of the LMI condition does not hold since $\mathbf{\Psi} \blacktriangleleft\!\blacktriangleright \mathbf{\Phi}$ does not hold in general between

$$\mathbf{\Psi} = \left\{ \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} : \begin{array}{l} X = diag(X_T, X_S) \\ X_T > 0, X_S \in \mathbb{H} \end{array} \right\} \text{ and}$$

$$\mathbf{\Phi} = \left\{ \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^* \begin{bmatrix} \kappa I & 0 \\ 0 & -\kappa I \end{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} : \kappa > 0 \right\}.$$

D. μ-analysis

It is well known that computing the structured singular value μ of a complex matrix is intractable [14]. This can be understood with a combination of Hermitian sets $\mathbf{\Psi}$ and $\mathbf{\Phi}$ that fail to satisfy the strong mutual losslessness property.

The structured singular value of $G \in \mathbb{C}^{m\times m}$ is defined by

$$\mu_\Pi(G) = \frac{1}{\min\{|\tau| : det(I - \tau\Delta G) = 0, \Delta \in \mathscr{R}(\mathbf{\Pi})\}}.$$

Now consider the problem of determining whether

$$\sup_{\lambda \in \bar{\mathbb{C}}_+} \mu_\Pi(G(\lambda I)) < 1. \qquad (24)$$

Condition (24) implies that the interconnection $[G(\mathbf{\Lambda}), \mathbf{\Delta}]$ is well-posed for $\mathbf{\Lambda} = \mathscr{R}(\mathbf{\Theta})$ and $\mathbf{\Delta} = \mathscr{R}(\mathbf{\Pi})$ where $\mathbf{\Theta}$ and $\mathbf{\Pi}$ are specified by item (a) in Table I and item (g) in Table II. Thus it can be readily seen from Theorem 1 that the existence of $\Theta \in \mathbf{\Theta}$ and $\Pi \in \mathbf{\Pi}$ such that

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \Theta \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^* \Pi \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} < 0 \qquad (25)$$

is a sufficient condition. However, if $\boldsymbol{\Psi}$ and $\boldsymbol{\Phi}$ are defined by (12) and (13) using the above $\boldsymbol{\Theta}$ and $\boldsymbol{\Pi}$, the relationship $\boldsymbol{\Psi} \blacktriangleleft\!\blacktriangleright \boldsymbol{\Phi}$ does not hold in general. Hence (25) is only a sufficient condition for (24). This gives another way of looking at the computational difficulty of μ. In fact, the LMI test (25) only corresponds to computing a convex upper bound of μ using a diagonal scaling technique.

Interestingly, however, it is possible to perform exact μ-analysis of internally positive systems with an arbitrary number of scalar uncertainty blocks using LMI. This result is significant in that it breaks the well-known "$2s+f \leq 3$" rule [14] where s is the number of scaler blocks and f is the number of full blocks.

Corollary 3: (μ-analysis for internally positive systems) Suppose $G(\Lambda)$ is internally positive and A is Hurwitz. Let the frequency region and the uncertainty region be defined by

$$\boldsymbol{\Lambda} = \{diag(\lambda_1, \cdots, \lambda_n) : \lambda_i \in \bar{\mathbb{C}}_+ \ \forall i = 1, \cdots, n\}$$
$$\boldsymbol{\Delta} = \{diag(\delta_1, \cdots, \delta_n) : |\delta_i| \leq 1 \ \forall i = 1, \cdots, n\}$$

Then the following are equivalent.

(I) There exist diagonal matrices $P > 0$ and $Q > 0$ such that

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^* \begin{bmatrix} Q & 0 \\ 0 & -Q \end{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} < 0$$

(II) The interconnection $[G(\boldsymbol{\Lambda}), \boldsymbol{\Delta}]$ is well-posed.

Proof: For the same reasoning as in Corollary 2, the transfer function $G(\Lambda)$ is well-defined on $\boldsymbol{\Lambda}$. From Table I (b) and Table II (g), we see that the frequency and the uncertainty regions are expressed as $\boldsymbol{\Lambda} = \mathscr{R}(\boldsymbol{\Theta})$, $\boldsymbol{\Delta} = \mathscr{R}(\boldsymbol{\Pi})$ where $\boldsymbol{\Theta}$ and $\boldsymbol{\Pi}$ are defined in (16). Thus the result of Theorem 1 is applicable. To complete the proof, it suffices to show that $\boldsymbol{\Psi} \blacktriangleleft\!\blacktriangleright \boldsymbol{\Phi}$ holds. Note that the relationship $\boldsymbol{\Psi} \lhd\!\rhd \boldsymbol{\Phi}$ is already established in Lemma 2. Assume the condition (B) holds, i.e., there exists a nonzero vector $\zeta = [\xi^T \ \eta^T]^T$ satisfying (9). Following the same logic as in Corollary 2, we have that $\eta \neq 0$. Also, using the same construction of Λ, we obtain $\Lambda \in \boldsymbol{\Lambda}$ and

$$\xi = \Lambda(A\xi + B\eta). \tag{26}$$

On the other hand, define $\delta_i, i = 1, \cdots, n$ by

$$\delta_i = \begin{cases} 0 & \text{if } (C\xi + D\eta)_i = 0 \\ \eta_i/(C\xi + D\eta)_i & \text{if } (C\xi + D\eta)_i \neq 0. \end{cases}$$

Then $\Delta = diag(\delta_1, \cdots, \delta_n)$ satisfies $\Delta \in \boldsymbol{\Delta}$. Notice from the second condition of (9) that $(C\xi + D\eta)_i = 0$ implies $\eta_i = 0$. Thus Δ satisfies

$$\eta = \Delta(C\xi + D\eta). \tag{27}$$

From (26) and (27), the interconnection $[G(\boldsymbol{\Lambda}), \boldsymbol{\Delta}]$ is ill-posed. Hence we have shown (B)⇒(C). ■

V. Conclusion and Future Works

This paper proposed symmetric formulations of the S-procedure and the KYP lemma. The notions of weak and strong mutual losslessness were introduced to characterize the lossless S-procedures and the lossless KYP lemma. The proposed form of the KYP lemma was shown to have sufficient generality to unify some recent extensions of system analysis tools including the Generalized KYP lemma [4] [11], the diagonal KYP lemma for internally positive systems [6], and the KYP lemma for nD-systems [7][13].

However, there is so far no general method for proving the weak and strong mutual lossless property. Hence in many particular analyses, we need to rely on the existing individual techniques to prove the losslessness. Therefore, finding a practically useful combination of Hermitian sets that satisfies the mutual lossless properties remains an important future research direction. Nevertheless, the symmetry provides a new perspective on the existing system analysis tools. It is an interesting future work to consider how the new framework can unify the existing analysis tools and provides new intuitions for the future developments of system analysis tools.

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