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# An Exact and Uniformly Convergent Arbitrary Order Differentiator

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***Abstract*—An arbitrary order differentiator that, in absence of noise, converges to the true derivatives of the signal after a finite time independent of the initial differentiator error is presented. The only assumption on a signal to be differentiated $(n-1)$-times is that its $n$-th derivative is uniformly bounded by a known constant. The new differentiator is obtained by combining the HOSM differentiator with an additional part that converges uniformly with respect to the initial conditions.**

***Index Terms*—differentiator; robustness; sliding-mode control.**

## I. Introduction

Real-time differentiators are used in a wide variety of problems; from classical PID regulators, to the observer design or fault diagnosis for switching systems, e.g. [1], [2], [3], [4].

For a given signal with known frequency content, the internal model principle can be used to trivially design an exact differentiator by simply using a linear observer. However, if the same differentiator is to be used on a different signal, it would have to compensate an intrinsic disturbance due to frequency mismatch in the new input signal. In theory (i.e. in absence of measurement noises), the effect of such disturbance can be totally removed by using discontinuous injections in the differentiator [5], showing that discontinuous differentiators can be exact (theoretically at least) for a wider class of input signals than any continuous differentiator.

The High-Order Sliding Mode (HOSM) differentiator by Levant [6] is a popular example of an arbitrary order discontinuous differentiator. Since its introduction, it has been extensively used to construct unknown input observers beyond the relative degree one condition, for linear [7], nonlinear [4], [3] and hybrid (or switched) systems [1], [2], [8].

In this note, the application of the HOSM differentiator to the latter kind of systems is considered, namely, for systems with some type of (strictly positive) dwell-time. This kind of behavior naturally arises in hybrid or switching systems, but is not limited to them. A "dwell-time" is also present in nonlinear dynamics with escape to infinity in finite-time, in the gain-scheduling approach to control nonlinear systems and, in general, in any model whose validity is limited to certain bounded time interval. If the HOSM differentiator is to be applied in any of the situations described above, it must provide an estimate of the required derivatives *during* the dwell-time of the system.

This last condition is possible, in general, only by increasing the gain of the HOSM differentiator as the initial condition of the system increases. Unfortunately, it is not possible to implement this last strategy since the HOSM differentiator is precisely used to construct an observer; a problem that has as basic hypothesis that the initial condition of the system is unknown. Therefore, the only possibility for applying the HOSM differentiator is to assume a known bound on the initial condition, see, e.g., [1], [2], [9]. To solve this problem without making additional assumptions, it is necessary that the convergence time of the differentiator is uniform with respect to the initial condition, i.e. its convergence time from an arbitrary initial condition is uniformly bounded. An exact *first* order differentiator with this property has been recently designed in [10] by modifying [5] using a Lyapunov based analysis.

The contribution of this paper is the construction of an exact *arbitrary order* differentiator with uniform convergence with respect to the initial condition. Due to the absence of Lyapunov functions for the general HOSM differentiator, we are lead to use homogeneity properties in the differentiator design, similarly to [11]. As first point, we extend the definition of uniform convergence given in [10] by distinguishing two important cases: when the uniform convergence is to a compact (practical uniform convergence) and when the convergence is towards the origin (uniform exact). In both definitions we consider systems with inputs that will represent the disturbance in the differentiator. We will show that, by simply reversing the homogeneity degree in [11] that characterize finite-time convergent HOSM controllers, practical uniform convergence is obtained. This consideration immediately allows the design of a practically uniform convergent differentiator that later is combined with the standard HOSM differentiator. Unlike [10], the stability properties of the practically uniform convergent part are deduced by applying a simple quadratic Lyapunov function on a linear system, and then using the properties of homogeneity and continuity of the uniform convergent part, in the same spirit than [12]. In this sense, the newly designed nonlinear part is totally designed by using linear methods.

The remainder of the paper is organized as follows. Section II introduces the problem statement and recalls some properties about the HOSM differentiator. Uniform convergence

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is presented and analyzed in Section III. Section IV presents the stability analysis of the new uniform convergent part of the differentiator. Section V presents a simulation example and, finally, Section VI summarizes the paper.

## II. Problem Statement and Preliminaries

Given a signal $\sigma(t) : [0,\infty) \to \mathbb{R}$, the real-time differentiation problem consists in obtaining an estimate of its successive derivatives $\sigma^{(i)}(t)$, $i = 1,\ldots,n-1$. The only assumption on the signal to be differentiated is that $|\sigma^{(n)}(t)| \leq L$ for any $t$, with $L$ a known constant.

Defining $x_1 := \sigma$, $x_2 := \dot\sigma$, $\ldots$, $x_n := \sigma^{(n-1)}$ yields

$$\dot x_1 = x_2, \quad \dot x_2 = x_3, \quad \cdots \quad \dot x_n = \sigma^{(n)}, \tag{1}$$

and the problem of constructing an $(n-1)$-th order differentiator for $\sigma(t)$ has been transformed into the construction of an observer for system (1), based on the measured output $x_1 = \sigma$, despite the bounded disturbance $\sigma^{(n)}$. Let $\hat{\boldsymbol{x}}(t)$ denote the estimate of $\boldsymbol{x}(t)$, and $\tilde{\boldsymbol{x}} := \boldsymbol{x} - \hat{\boldsymbol{x}}$ its observation error. There will be two the main properties of the differentiator: its finite-time exactness (i.e. its observation error converges in finite-time despite the disturbance) and its uniform convergence with respect to initial condition. The latter means that, despite the initial observation error, the convergence time of the differentiator will be uniformly bounded by a constant.

The $(n-1)$-th order HOSM differentiator [6], guarantees only the first property: the finite-time exact estimate of the derivatives despite the disturbance $\sigma^{(n)}$. It takes the following (non-recursive) form

$$\begin{aligned}\dot{\hat x}_i &= -k_i\lceil\tilde x_1\rfloor^{\frac{n-i}{n}} + \hat x_{i+1}, \quad i = 1,\ldots,n-1,\\ \dot{\hat x}_n &= -k_n \operatorname{sign}(\tilde x_1),\end{aligned} \tag{2}$$

where $\lceil x\rfloor^p := |x|^p \operatorname{sign}(x)$. The set of gains $\{k_i\}_{i=1}^n$ can be selected based on the well-known gains for the recursive form of the HOSM differentiator [6] (see also the example in Section V).

The HOSM differentiator was designed based on its homogeneity properties [11]. We briefly recall three basic concepts about homogeneity (see e.g. [13] and [11]):

*Definition 1:* a) The family of dilations $\Lambda = \Lambda_\lambda^{\boldsymbol{r}}$, associated with the "weight vector" $\boldsymbol{r} \in \mathbb{R}_+^n$, is the linear map

$$\Lambda : (x_1, x_2, \ldots, x_n) \mapsto (\lambda^{r_1}x_1, \lambda^{r_2}x_2, \ldots, \lambda^{r_n}x_n),$$

for $\lambda > 0$, where $r_i$ is the weight (or degree) of $x_i$ and is denoted as $\deg x_i := r_i$.

b) A vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is homogeneous of degree $p$ if

$$f(\Lambda\boldsymbol{x}) = \lambda^p \Lambda f(\boldsymbol{x}), \quad \forall\lambda > 0, \tag{3}$$

and is denoted as $\deg f := p$.

c) A differential equation $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$, is said to be homogeneous[1] with degree $p$ if $\deg f = -p$.

In [11], it was shown that asymptotically stable systems with negative homogeneity degree (i.e. $\deg f > 0$) converges in finite-time.

The structure of the HOSM differentiator (2) can be easily derived based on the homogeneity restrictions, as we now show. Set the homogeneity degree as $\deg t = -\alpha$, or equivalently $\deg f = -\alpha$. Due to the structure of the chain of integrators, the weights of every coordinate are fixed once a initial weight for one variable is selected. Let us select $\deg\tilde x_1 = n$, this yields

$$\deg\tilde x_1 = n, \quad \deg\tilde x_2 = n+\alpha, \quad \deg\tilde x_i = n + \alpha(i-1).$$

Consider now a general form for the observer

$$\dot{\hat x}_i = -f_i(\tilde x_1) + \hat x_{i+1}, \quad i = 1,\ldots,n,$$

and restrict the functions $f_i$ to be the simplest homogeneous functions $f_i(\tilde x_1) = k_i\lceil\tilde x_1\rfloor^{p_i}$, $k_i, p_i \in \mathbb{R}$. This, together with the homogeneity restrictions on the system, produce the following structure for an homogeneous differentiation error

$$\begin{aligned}\dot{\tilde x}_1 &= -k_1\lceil\tilde x_1\rfloor^{\frac{n+\alpha}{n}} + \tilde x_2,\\ \dot{\tilde x}_2 &= -k_2\lceil\tilde x_1\rfloor^{\frac{n+2\alpha}{n}} + \tilde x_3,\\ &\vdots\\ \dot{\tilde x}_{n-1} &= -k_{n-1}\lceil\tilde x_1\rfloor^{\frac{n+(n-1)\alpha}{n}} + \tilde x_n,\\ \dot{\tilde x}_n &= -k_n\lceil\tilde x_1\rfloor^{1+\alpha} + \sigma^{(n)}(t),\end{aligned} \tag{4}$$

which means that the observer (differentiator) has to take form

$$\begin{aligned}\dot{\hat x}_i &= -k_i\lceil\tilde x_1\rfloor^{\frac{n+\alpha i}{n}} + \hat x_{i+1}, \quad i = 1,\ldots,n-1,\\ \dot{\hat x}_n &= -k_n\lceil\tilde x_1\rfloor^{1+\alpha},\end{aligned} \tag{5}$$

where $\{k_i\}_{i=1}^n$ are parameters to be selected.

The HOSM differentiator (2) is obtained from (5) by setting $\alpha = -1$. Its homogeneity degree $\alpha$ is negative (so $\deg f$ is positive) to guarantee finite-time stability and, more important, is fixed to one ensuring its exactness against the bounded disturbance $\sigma^{(n)}$, see [11]. Any other choice of $\alpha < 0$ results in a continuous observer that is not able to completely remove a bounded disturbance.

The main concern of this paper is on the second desired property of the differentiator: its uniform convergence with respect to the initial condition. This will be done by designing a new observer whose error converges uniformly in the initial condition to a compact set, even in the presence of the disturbance. Its design will be also based on homogeneity properties, and thus it looks also like (5): we will show that it is just necessary to reverse the homogeneity degree (from

[1] This is the standard notation for homogeneous systems, c.f. [11]. The homogeneity degree $p$ of the differential equation is opposite to the homogeneity degree of the vector field, since it is associated to the weight of time in the following way: $\deg t = -p$, i.e. $\frac{d}{dt}\Lambda x = \lambda^{-p} f(\Lambda x)$; see also Lemma 3 in the Appendix.

negative to positive) to obtain uniform convergence instead of finite-time convergence. In this sense, its design is closely related to the HOSM differentiator. The two properties of the new differentiator are stated precisely below.

*Definition 2:* A differentiator $\hat{\boldsymbol{x}}$ is said to be uniformly finite-time exact if there exists time $T$ such that $\hat{\boldsymbol{x}}(t) \equiv \boldsymbol{x}(t), \forall t \geq T$, for all initial differentiation errors $\tilde{\boldsymbol{x}}(0)$; i.e. $T$ is independent of $\tilde{\boldsymbol{x}}(0)$.

The next section formally introduces the concept of uniform convergence together with its characterization using homogeneity.

## III. Uniform Convergence

Consider

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}) + g(\boldsymbol{x}, \boldsymbol{w}), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0, \tag{6}$$

with $\boldsymbol{x}(t) \in \mathbb{R}^n$, $\boldsymbol{w}(t) \in \mathbb{R}^m$, the state and input of the system, respectively. The vector field $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is not necessarily continuous in $\boldsymbol{x}$ nor $\boldsymbol{w}$, and it is used as a disturbance to the nominal part $f$. The variable $\boldsymbol{w}$ is used to represent an input (known or unknown) to the system and is assumed to belong to certain (abstract) class of functions that we denote as $\mathscr{W}$.

We can identify (6) with (4) using $f_i(\boldsymbol{x}) = -k_i \lceil \tilde{x}_1 \rfloor^{\frac{n+\alpha i}{n}} + \tilde{x}_{i+1}$ , $i = 1, \ldots, n$, and $g_n(\boldsymbol{x}, \boldsymbol{w}) = w = \sigma^{(n)}$. Therefore, in (4), $\mathscr{W}$ is the class of functions uniformly bounded by a constant.

The main concepts used along the paper are introduced below:

*Definition 3:* System (6) is said to be (with respect to the initial condition):

i) practical uniform convergent if for any $\boldsymbol{w} \in \mathscr{W}$, there exists $T_{\boldsymbol{w}} \geq 0$ and $r_{\boldsymbol{w}} \geq 0$ such that $\forall \boldsymbol{x}_0 \in \mathbb{R}^n$
$$\|\boldsymbol{x}(t)\| \leq r_{\boldsymbol{w}} \quad \text{if} \quad t \geq T_{\boldsymbol{w}};$$
ii) uniformly exact convergent if for any $r > 0$, there exists $T_r \geq 0$ such that $\forall (\boldsymbol{x}_0 \in \mathbb{R}^n, \boldsymbol{w} \in \mathscr{W})$
$$\|\boldsymbol{x}(t)\| \leq r \quad \text{if} \quad t \geq T_r;$$
iii) uniformly finite-time exact convergent if there exists $T \geq 0$ such that $\forall (\boldsymbol{x}_0 \in \mathbb{R}^n, \boldsymbol{w} \in \mathscr{W})$
$$\boldsymbol{x}(t) \equiv 0 \quad \text{if} \quad t \geq T.$$

It is possible to interpret point (ii) in the last definition as uniformity with respect to the initial condition together with uniformity respect to the input $\boldsymbol{w}$.

For the particular case of a undisturbed system, i.e. $g \equiv 0$, with continuous $f$, uniform convergence has been characterized in [14] based on its positive homogeneity degree. Practical uniform convergence can be characterized for general disturbed systems of the form (6), by the properties that the disturbance $g$ need to satisfy with respect to the nominal part $f$, as shown below.

*Theorem 1:* System (6) is practically uniform convergent if:

i) when $g \equiv 0$, its origin is globally asymptotically stable;
ii) $f$ is a continuous vector field and $\deg f = p < 0$;
iii) $\|f(\boldsymbol{x})\| > \|g(\boldsymbol{x}, \boldsymbol{w})\|$ as $\|\boldsymbol{x}\| \to \infty$ for all $\boldsymbol{w} \in \mathscr{W}$.

*Proof:* See the Appendix. ■

Therefore, the general homogeneous observer (5) is practical uniform convergent if $\alpha > 0$ and $\{k_i\}_{i=1}^n$ is selected to ensure asymptotic stability when $\sigma^{(n)} \equiv 0$ (since the bounded disturbance gets eventually dominated by the function $\lceil \tilde{x}_1 \rfloor^{\alpha+1}$, satisfying Theorem 1). A method to determine these parameters is presented in Section IV.

To obtain a uniform exact finite-time convergent differentiator, the HOSM differentiator can be combined with the practical uniform convergent differentiator. This combination can be made in several ways:

- start with the practical uniform convergent differentiator and switch to the finite-time convergent differentiator after a fixed amount of time $T$. However, in this case, the switching is made in "open-loop";
- use of a state norm observer (see e.g. [15]) to switch *only once*, from the practical uniform convergent differentiator to the finite-time differentiator, when the estimated norm of the observation error enters, for the first time, into some predefined region around the origin of the state space.

With both options, *no further stability analysis is required.*

*Remark 1:* After extensive simulations, it was noticed that the practical uniform convergent differentiator and the finite-time differentiator cooperate with each other without the need of any switching between them. Unfortunately, no proof of this fact is available for the moment.

## IV. Stability analysis of the uniform part

The practical uniform convergence of the differentiation error (4) needs to be shown, by selecting appropriately the parameters $\alpha$ and $\{k_i\}_{i=1}^n$. As discussed in the previous section, using Theorem 1, this is equivalent to determine the parameters $\{k_i\}_{i=1}^n$ such that the system

$$\begin{aligned} \dot{\tilde{x}}_1 &= -k_1 \lceil \tilde{x}_1 \rfloor^{\frac{n+\alpha}{n}} + \tilde{x}_2, \\ \dot{\tilde{x}}_2 &= -k_2 \lceil \tilde{x}_1 \rfloor^{\frac{n+2\alpha}{n}} + \tilde{x}_3, \\ &\vdots \\ \dot{\tilde{x}}_{n-1} &= -k_{n-1} \lceil \tilde{x}_1 \rfloor^{\frac{n+(n-1)\alpha}{n}} + \tilde{x}_n, \\ \dot{\tilde{x}}_n &= -k_n \lceil \tilde{x}_1 \rfloor^{1+\alpha}, \end{aligned} \tag{7}$$

is asymptotically stable for $\alpha > 0$.

When $\alpha = 0$, the nonlinear system (7) is reduced to a linear one, whose stability is completely determined by the stability of the matrix

$$\boldsymbol{A} := \begin{bmatrix} -k_1 & 1 & 0 & \cdots & 0 \\ -k_2 & 0 & 1 & \cdots & 0 \\ \vdots & & & \vdots & \\ -k_n & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

This fact can be easily proved using a quadratic Lyapunov function; nevertheless, in such case the compensator does not converge uniformly since $\alpha$ is not positive. To analyze the stability when $\alpha > 0$, the idea is to use the information of the same quadratic Lyapunov function on the nonlinear system together with the continuity of its derivative with respect to the parameter $\alpha$. This classical idea can be found in [12].

*Theorem 2:* If $\alpha > 0$ is sufficiently small and the parameters $\{k_i\}_{i=1}^n$ are selected such that the $\boldsymbol{A}$ matrix is Hurwitz, then the differentiation error (7) is asymptotically stable.

*Proof:* Consider $V(\tilde{\boldsymbol{x}}) = \tilde{\boldsymbol{x}}^T P \tilde{\boldsymbol{x}}$ with $\mathbf{P}$ positive definite and solution to $\mathbf{A^T P} + \mathbf{PA} \prec 0$, then

$$\dot{V}(0, \tilde{\boldsymbol{x}}) < 0, \quad \forall \tilde{\boldsymbol{x}} \in S,$$

for any set $S \subseteq \mathbb{R}^n \setminus \{0\}$, since it indeed is a Lyapunov function for the linear system obtained when $\alpha = 0$. Moreover, note that $\dot{V}(\alpha, \tilde{\boldsymbol{x}})$ is continuous in both of its arguments, $\alpha$ and $\tilde{\boldsymbol{x}}$. In particular, if $S$ is compact, $\dot{V}(\alpha, \tilde{\boldsymbol{x}})$ is uniformly continuous in the set $\{(\alpha, \tilde{\boldsymbol{x}}) \in \mathbb{R} \times \mathbb{R}^n | \alpha = 0, \tilde{\boldsymbol{x}} \in S\}$. This means that there exists vicinities $\mathcal{N}_S$ and $\mathcal{N}_{\alpha=0}$ such that

$$\dot{V}(\alpha, \tilde{\boldsymbol{x}}) < 0, \quad \forall (\alpha, \tilde{\boldsymbol{x}}) \in \mathcal{N}_{\alpha=0} \times \mathcal{N}_S.$$

Now, let us choose $S$ as an arbitrary level curve of $V$, i.e. $S = V^{-1}(\delta), \delta > 0$. Therefore, there exists $\alpha > 0$ and a vicinity $\mathcal{N}_S$ (i.e. a ring around the level curve), such that the trajectories of the system cross from the outside of the ring to its inside. By the homogeneity of the system, this also occurs on any dilated version of such ring. Moreover, since $\mathbb{R}^n \setminus \{0\}$ can be covered by dilated rings, this ensures that the system is globally asymptotically stable; c.f. the proof for Theorem 4 in the Appendix. ∎

In the case of a second order differentiator ($n = 3$), we have the following corollary to select the gains:

*Corollary 1:* For system (7) with $n = 3$ and $\alpha > 0$ small enough, the gains selected as

$$k_1 > 0, \quad k_3 > 0, \quad k_2 > k_3/k_1,$$

are sufficient conditions for asymptotic stability.

## V. Simulation Example

A second order uniform differentiator was tested using the signal $\sigma(t) = 5t + \sin(t) + 0.01\cos(10t)$. For $n = 3$ the uniform compensator takes the form

$$\begin{aligned} \dot{\hat{x}}_1 &= -k_1 \lceil \hat{x}_1 - \sigma \rfloor^{1+\frac{\alpha}{3}} + \hat{x}_2, \\ \dot{\hat{x}}_2 &= -k_2 \lceil \hat{x}_1 - \sigma \rfloor^{1+\frac{2\alpha}{3}} + \hat{x}_3, \\ \dot{\hat{x}}_3 &= -k_3 \lceil \hat{x}_1 - \sigma \rfloor^{1+\alpha}, \end{aligned} \tag{8}$$

while the HOSM differentiator has the following form

$$\begin{aligned} \dot{\hat{x}}_1 &= -\kappa_1 \lceil \hat{x}_1 - \sigma \rfloor^{\frac{2}{3}} + \hat{x}_2, \\ \dot{\hat{x}}_2 &= -\kappa_2 \lceil \hat{x}_1 - \sigma \rfloor^{\frac{1}{3}} + \hat{x}_3, \\ \dot{\hat{x}}_3 &= -\kappa_3 \operatorname{sign}(\hat{x}_1 - \sigma). \end{aligned} \tag{9}$$

Its gains $\{\kappa_i\}_{i=1}^3$ can be selected based on the gains for the recursive HOSM differentiator as

$$\kappa_1 = \theta_2 L^{1/3}, \quad \kappa_2 = \theta_1 \theta_2^{1/2} L^{1/2+1/6}, \quad \kappa_3 = \theta_0 L,$$

with $\theta_0 = 1.1, \theta_1 = 1.5, \theta_2 = 2$ and $L$ such that $|\sigma^{(3)}(t)| \le L$, see [6]. The initial condition was selected as $\hat{x}(0) = (100, 200, 300)$ and the gains as $k_1 = 7, k_2 = 1/7 + 2, k_3 = 1, L = 30$. Theorem 2, ensures the existence of a small enough parameter $\alpha > 0$ that guarantees uniform convergence with these gains. Let us show, additionally, that it is possible to compute the explicit value for parameter $\alpha > 0$.

Using the proof of Theorem 2, one picks a level curve $S = V^{-1}(\delta)$, and checks if $\dot{V}(\alpha, \tilde{\boldsymbol{x}}) < 0$ on that level curve for the given $\alpha$. The largest value of $\alpha$ that satisfies the last condition can be computed iteratively, starting with $\alpha = 0$ and increasing its value while[2] $V^{-1}(\delta) \cap \dot{V}_\alpha^{-1}(0) = \emptyset$. When $\alpha = 0$, i.e. a linear system, there is no intersection for any $\delta > 0$, since $V^{-1}(\delta)$ is an ellipsoid and $\dot{V}_0^{-1}(0)$ is just the point $\tilde{\boldsymbol{x}} = 0$.

Let us illustrate the procedure described above for our particular example. First, selecting the matrix $\boldsymbol{P}$ as a solution to $\boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} = \boldsymbol{Q} = -\boldsymbol{I}$ yields

$$\boldsymbol{P} = \begin{bmatrix} 71/196 & 57/28 & 1/2 \\ 57/28 & 9965/686 & 757/196 \\ 1/2 & 757/196 & 87/28 \end{bmatrix}.$$

The surface $\dot{V}_\alpha^{-1}(0)$ for $\alpha = 0.06$ is shown in Fig. 1; it consist of two parts: a central lobe and two exterior hyperboloids. As $\alpha$ tends to zero, the central lobe shrinks and the exterior hyperboloids tend to retreat to infinity; in the limit $\alpha = 0$, it consists of only one point $\tilde{\boldsymbol{x}} = 0$. The ellipsoids $V^{-1}(\delta)$ turned out to be aligned horizontally, irrespectively of the selection of $\boldsymbol{Q}$ in the Lyapunov equation. For $\alpha = 0.06$, the ellipsoid $V^{-1}(10)$ does not intersect neither the interior lobes, nor the exterior hyperboloids of $\dot{V}_{0.06}^{-1}(0)$, see Fig. 2. According to the discussion above, this shows that the system is stable for $\alpha = 0.06$ and the selected gains $\{k_i\}_{i=1}^3$, and, therefore, the combined differentiator is uniform exact convergent. In Figure 3, the convergence of the trajectories of the differentiator is shown.

To confirm the uniform convergence property of the differentiator, several initial conditions $\boldsymbol{x}_0$ were tested and the convergence time of the differentiator measured. The initial conditions $\boldsymbol{x}_0$ were chosen along along the subspace $(1, 2, 3)$ as $\boldsymbol{x}_{0,i+1} := 10\boldsymbol{x}_{0,i}$ with $\boldsymbol{x}_{0,0} = (0.1, 0.2, 0.3)^T$. Figure 4 presents a graph between the convergence time of the differentiator versus the initial condition. As expected, it shows the presence of asymptote in the convergence time as the initial condition increase.

## VI. Conclusions

We have presented an arbitrary differentiator that converges to the true derivatives of the signal after finite-time

[2]We use the notation $\dot{V}_\alpha^{-1}(0) := \{\tilde{\boldsymbol{x}} \in \mathbb{R}^n | \dot{V}(\alpha, \tilde{\boldsymbol{x}}) = 0\}$.

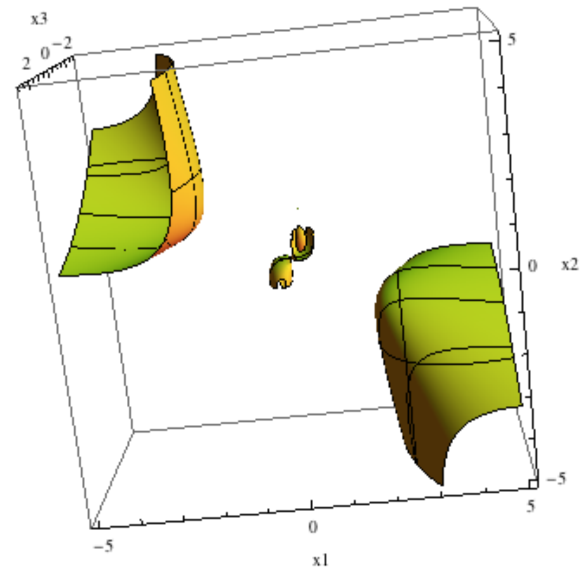


Fig. 1. The surface $\dot{V}_{\alpha}^{-1}(0)$ for $\alpha = 0.06$.

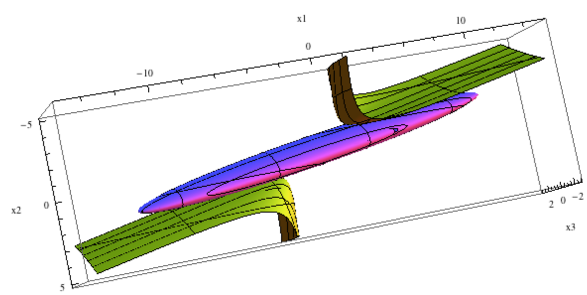


Fig. 2. $\dot{V} = 0$ for $\alpha = 0.06$ (yellow) and the ellipsoid $V = 10$ (magenta). There is no intersection of $\dot{V} = 0$ with $V = 10$, therefore $\dot{V} < 0$ in $V = 10$.

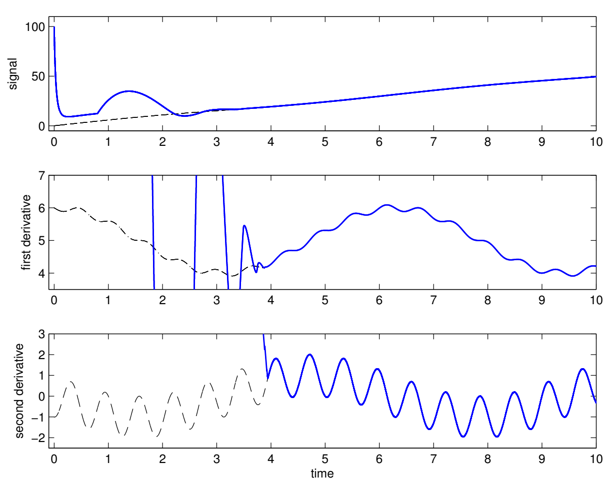


Fig. 3. True derivatives: black, estimated derivatives: blue.

independent of the initial differentiation error. The differentiator was constructed by combining the HOSM differentiator

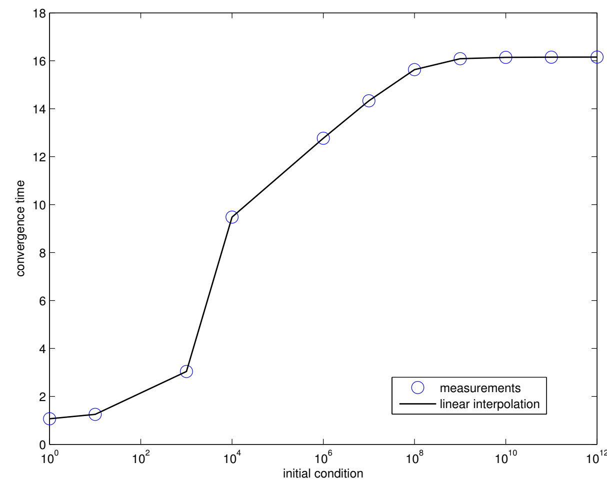


Fig. 4. Convergence time $vs$ initial condition.

with an observer that is practical uniform convergent. For this, uniform convergence was characterized in terms of the homogeneity of the vector fields, and latter shown to be robust (in the sense of practical uniform convergence) to any disturbance that gets eventually dominated by the nominal part. The stability of the new uniform part of the differentiator was analyzed by using a quadratic "Lyapunov function", together with the continuity and homogeneity of the differentiation error. Using this analysis, we illustrated in the example a simple method to compute all the required parameters of this new part of the differentiator.


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## APPENDIX

We will denote by $\boldsymbol{x}(t, \boldsymbol{x}_0)$ the solution of (6) with initial condition $\boldsymbol{x}_0$ at time $t$.

*Lemma 3:* Consider system (6). Assume that $g \equiv 0$ and that $\deg f = p$. Then

$$\Lambda \boldsymbol{x}(t, \boldsymbol{x}_0) = \boldsymbol{x}(\lambda^p t, \Lambda \boldsymbol{x}_0). \tag{10}$$

*Proof:* Without loss of generality, assume that the system has initial condition $\boldsymbol{x}(0) = \Lambda \boldsymbol{x}_0$. Applying the transformation $(t, \boldsymbol{x}) \mapsto (\tau = \lambda^{-p} t, \boldsymbol{z} = \Lambda^{-1} \boldsymbol{x})$ yields

$$\frac{d\boldsymbol{z}}{dt} = \Lambda^{-1} \dot{\boldsymbol{x}} = \Lambda^{-1} f(\boldsymbol{x}) = \Lambda^{-1} \lambda^p \Lambda f(\boldsymbol{z}).$$

Then we have

$$\begin{aligned} \frac{d\boldsymbol{x}}{dt} &= f(\boldsymbol{x}), \quad \boldsymbol{x}(0) = \Lambda \boldsymbol{x}_0, \\ \frac{d\boldsymbol{z}}{d\tau} &= f(\boldsymbol{z}), \quad \boldsymbol{z}(0) = \boldsymbol{x}_0. \end{aligned}$$

Their flows are $\boldsymbol{\varphi}(t, \Lambda \boldsymbol{x}_0)$ and $\boldsymbol{\varphi}(\tau, \boldsymbol{x}_0)$, for $\boldsymbol{x}$ and $\boldsymbol{z}$, respectively. Since $\boldsymbol{x} = \Lambda \boldsymbol{z}$, we have $\boldsymbol{\varphi}(t, \Lambda \boldsymbol{x}_0) = \Lambda \boldsymbol{\varphi}(\tau, \boldsymbol{x}_0)$.

Using $t = \lambda^p \tau$ in the left-hand side of the previous equality completes the claim of the lemma. ■

Using this last lemma we can prove the following preliminary result about the uniform convergence without inputs. It is an extension of the proof in [14] that does not require continuity of the vector field $f$.

*Theorem 4:* Consider (6) with $g \equiv 0$. Assume that its origin is asymptotically stable (AS) and that $\deg f = p < 0$. Then the system is practical uniformly convergent in the sense of Definition 3-(i).

*Proof:* Let $B_0$ be an arbitrary compact ball of $\mathbb{R}^n$. Set $\lambda > 1$ large enough and consider the dilations $B_1 = \Lambda B_0, B_2 = \Lambda B_1, \cdots, B_i = \Lambda B_{i-1}$ for all $i \geq 1$. Since $\lambda$ is positive and sufficiently large then $B_{i+1} \supset B_i$. Now, due to stability of the system, every trajectory starting in $B_1$ enters $B_0$ before certain time $T$; in symbols $\boldsymbol{x}(T, B_1) \in B_0$. Applying dilation to both sides of this last expression and using (10) yields

$$\Lambda \boldsymbol{x}(T, B_1) = \boldsymbol{x}(\lambda^p T, \Lambda B_1) = \boldsymbol{x}(\lambda^p T, B_2) \in \Lambda B_0 = B_1,$$

i.e., the trajectories starting in $B_2$ enter $B_1$ before time $\lambda^p T$. Applying the dilation again obtain

$$\Lambda \boldsymbol{x}(\lambda^p T, B_2) = \boldsymbol{x}(\lambda^{2p} T, B_3) \in \Lambda B_1 = B_2.$$

Repeating the same procedure for each $i \geq 1$ and summing each time interval, obtain that for any $\boldsymbol{x}_0 \in \mathbb{R}^n$ the trajectory of the system enters into $B_0$ after at most

$$t_{reach} = T \sum_{i=0}^{\infty} (\lambda^p)^i = T \frac{1}{1 - \lambda^p},$$

where the convergence of the geometric series occurs due to $\lambda^p < 1$, since $\lambda > 1$ but $p < 0$. Since $B_0$ was arbitrary, the claim of the Theorem is complete. ■

For the proof of Theorem 1, we will need a preliminary result from [13] that we state as follows:

*Theorem 5:* [13, Thm. 5.8] Let $f$ be a continuous vector field such that the origin of (6) with $g \equiv 0$ is locally asymptotically stable. Assume that $\deg f = p$ for some $\boldsymbol{r} \in (0, \infty)^n$. Then for any $s \in \mathbb{N}^*$ and any $m > s \cdot \max r_i$ there exists a strong Lypunov function $V \in C^p$ which is homogeneous with $\deg V = m$. Therefore $\dot{V} = \langle \nabla V, f \rangle$ is homogeneous of degree $m + p$.

Analogously to Definition 1, a function $V : \mathbb{R}^m \to \mathbb{R}$ is said to be homogeneous of degree $m$ if $V(\Lambda \boldsymbol{x}) = \lambda^m V(\boldsymbol{x}), \forall \lambda > 0$, and is written as $\deg V = m$. Now we are ready for the proof of Theorem 1:

*Proof of Theorem 1:* By Hypothesis (i)-(ii), and using Theorem 5, there exist a homogeneous function $V$ with $\deg V = m$ and $\langle \nabla V(\boldsymbol{x}), f(\boldsymbol{x}) \rangle < 0$. On the other hand, hypothesis (iii) means that if one takes the $n-1$ dimensional sphere $S$, then $\forall \epsilon > 0$ there exists $\lambda^*$ large enough such that

$$\|g(\Lambda S, \boldsymbol{w})\| \leq \epsilon \|f(\Lambda S)\|, \quad \forall \lambda \geq \lambda^*, \forall \boldsymbol{w} \in \mathscr{W}.$$

Now compute the derivative of $V$ along the perturbed system $\dot{V}(\boldsymbol{x}) = \langle \nabla V(\boldsymbol{x}), f(\boldsymbol{x}) \rangle + \langle \nabla V(\boldsymbol{x}), g(\boldsymbol{x}, \boldsymbol{w}) \rangle$. Taking $\boldsymbol{x} \in S$ and fixing $0 < \epsilon < 1$, with its corresponding $\lambda^*$, yields

$$\begin{aligned}
\dot{V}(\Lambda \boldsymbol{x}) &= \langle \nabla V, f \rangle(\Lambda \boldsymbol{x}) + \langle \nabla V, g \rangle(\Lambda \boldsymbol{x}, \boldsymbol{w}) \\
&\leq -|\langle \nabla V, f \rangle|(\Lambda \boldsymbol{x}) + \|\nabla V\| \|g\|(\Lambda \boldsymbol{x}, \boldsymbol{w}) \\
&\leq -|\langle \nabla V, f \rangle|(\Lambda \boldsymbol{x}) + \epsilon \|\nabla V\| \|f\|(\Lambda \boldsymbol{x}), \\
&\leq -|\langle \nabla V, f \rangle|(\Lambda \boldsymbol{x}) + \epsilon |\langle \nabla V, f \rangle|(\Lambda \boldsymbol{x}) \\
&= -(1 - \epsilon)|\langle \nabla V, f \rangle|(\Lambda \boldsymbol{x}), \quad \forall \lambda \geq \lambda^*.
\end{aligned}$$

This means that if $\|\boldsymbol{x}\|$is taken large enough, then $\dot{V}|_{pert} = \text{const} \cdot \dot{V}|_{nopert}$. Thus the perturbed system inherits the behavior of the nominal system if $\|\boldsymbol{x}\|$ is large enough. By hypothesis (ii), $p < 0$ and using Theorem 4 obtain the claim of the theorem (or since $\deg V < 0$ and using Theorem 4). ■

## REFERENCES


[1] F. Bejarano and L. Fridman, "State exact reconstruction for switched linear systems via a super-twisting algorithm," *International Journal of Systems Science*, vol. 42, no. 5, pp. 717–724, 2011.

[2] J. Davila, A. Pisano, and E. Usai, "Continuous and discrete state reconstruction for nonlinear switched systems via high-order sliding-mode observers," *International Journal of Systems Science*, vol. 42, no. 5, pp. 725 – 735, 2011.

[3] D. Efimov, A. Zolghadri, and T. Raissi, "Actuator fault detection and compensation under feedback control," *Automatica*, vol. 47, no. 8, pp. 1699 – 1705, 2011.

[4] M. T. Angulo, L. Fridman, C. Moog, and J. Moreno, "Output feedback design for exact state stability of flat nonlinear systems," in *Variable Structure Systems (VSS), 2010 11th International Workshop on*, 2010, pp. 32–38.

[5] A. Levant, "Robust exact differentiation via sliding mode technique," *Automatica*, vol. 34, no. 3, pp. 379–384, 1998.

[6] ——, "High-order sliding modes: differentiation and output feedback control," *International Journal of Control*, vol. 76, no. 9-10, pp. 1924–041, nov 2003.

[7] F. J. Bejarano and L. Fridman, "High order sliding mode observer for linear systems with unbounded unknown inputs," *International Journal of Control*, vol. 83, no. 9, pp. 1920–1929, 2010.

[8] H. Saadaoui, N. Manamanni, M. Djemaï, J. Barbot, and T. Floquet, "Exact differentiation and sliding mode observers for switched lagrangian systems," *Nonlinear Analysis*, vol. 65, no. 5, pp. 1050 – 1069, 2006, hybrid Systems and Applications (4) - Hybrid Systems and Applications (4).

[9] A. Pisano and E. Usai, "Globally convergent real-time differentiation via second order sliding modes," *International Journal of Systems Science*, vol. 38, no. 10, pp. 833–844, 2007.

[10] E. Cruz-Zavala, J. Moreno, and L. Fridman, "Uniform second-order sliding mode observer for mechanical systems," in *Variable Structure Systems (VSS), 2010 11th International Workshop on*, 2010, pp. 14 –19.

[11] A. Levant, "Homogeneity approach to high-order sliding mode design," *Automatica*, vol. 41, no. 5, pp. 823–830, 2005.

[12] S. Bhat and D. Bernstein, "Geometric homogeneity with applications to finite-time stability," *Math. Control Signals Systems*, pp. 17: 101–127 DOI 10.1007/s00 498–005–0151–x, 2005.

[13] A. Bacciotti and L. Rosier, *Liapunov Functions and Stability in Control Theory*, 2nd ed. London: Springer, 2005.

[14] V. Andrieu, L. Praly, and A. Astolfi, "Homogeneous approximation, recursive observer design and output feedback," *SIAM Journal on Control and Optimization*, vol. 47, no. 4, pp. 1814–1850, 2008.

[15] A. J. Peixoto, T. R. Oliveira, and L. Hsu, "Global tracking output-feedback sliding mode control design via norm estimators and variable high gain observer," in *Proceedings of the 48th IEEE Conference on Decision and Control, CDC/CCC 2009*, 2009, pp. 6083 – 6088.