

# Control of underactuated systems with viability constraints

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**Abstract**—This paper addresses the control design for a class of nonholonomic systems which are subject to inequality state constraints defining a constrained (viability) set  $K$ . Based on concepts from viability theory, the necessary conditions for selecting viable controls for a nonholonomic system are given. Furthermore, a class of nonholonomic control solutions are redesigned by means of switching control, so that system trajectories are viable in  $K$  and converge to a goal set  $G$  in  $K$ . The motion control for an underactuated marine vehicle in a constrained configuration set  $K$  is treated as a case study. The set  $K$  essentially describes the limited sensing area of a vision-based sensor system, and viable control laws which establish convergence to a goal set  $G$  in  $K$  are constructed. The efficacy of the methodology is demonstrated through simulation results.

## I. INTRODUCTION

Control of systems which are subject to both nonholonomic and state (configuration) constraints is a challenging problem, often encountered within the fields of robotics and multi-agent systems. Nonholonomic constraints apply to various robotic systems, either due to the rolling without slipping condition, or due to underactuation. Furthermore, control systems in real-world applications are additionally subject to hard state constraints, imposed for safety or performance issues. Indicative examples of such constraints often arise due to limited sensing; consider, for instance, an underactuated robotic vehicle equipped with sensors (e.g. cameras) of limited range and angle-of-view, which has to track or move with respect to (w.r.t.) a target of interest. The requirement of always having the target in the camera field-of-view (f.o.v.) imposes a set of inequality state constraints which should never be violated. This problem is often termed as maintaining visibility and applies in leader-follower formations [1]–[3], in landmark-based navigation [4]–[6], in autonomous inspections [7], or in visibility-based pursuit-evasion problems, see [8] and the references therein. Similar state constraints apply in maintaining connectivity problems involving  $n$  nonholonomic agents with limited sensing and/or communication capabilities, that have to accomplish a task while always staying connected [9].

This paper proposes a control design methodology for a class of nonholonomic systems subject to hard state constraints. The state constraints are realized as nonlinear inequalities w.r.t. the state variables, which constitute a closed subset  $K$  of the state space  $\mathcal{Q}$ . The set  $K$  is thus the subset of state space in which the system trajectories should evolve  $\forall t \geq 0$ . Therefore, the control objective is to find a (possibly switching) state feedback control law, so that

system trajectories starting in  $K$  converge to a goal set  $G$  in  $K$  without ever leaving  $K$ .

The proposed approach combines concepts and tools from viability theory [10] and results from our work on feedback control for nonholonomic systems with kinematic Pfaffian constraints [11] (Section II). In the sequel, following [10], state constraints are called viability constraints, the set  $K$  is called the viability set of the system, and system trajectories that remain in  $K \forall t \geq 0$  are called viable (see Section III). We adopt the concept of tangency to a set  $K$  defined by inequality constraints [10], and provide the necessary conditions under which the admissible velocities of a kinematic nonholonomic system are viable in  $K$ , as well as the necessary conditions for selecting viable controls (Section IV). As a case study, we consider the motion planning for an underactuated marine vehicle subject to configuration constraints due to limited sensing. The task is to control the vehicle so that it converges to a desired configuration w.r.t. a target of interest, while this target is always visible in the f.o.v. of the onboard camera. The visibility maintenance requirement imposes a set of configuration constraints that define a viability set  $K$ . Given the control solutions in [11], we propose a way of redesigning them so that the resulting trajectories are viable in  $K$  and furthermore converge to a desired set  $G \subset K$ , along with simulation results that demonstrate the efficacy of our methodology (Section V). Our conclusions and plans for future extensions are summarized in Section VI.

The problem formulation is similar to the characterization of viable capture basins of a target set  $C$  in a constrained set  $K$  [12], which is based on the Frankowska method. However, in this paper we rather address the problem in terms of set invariance [13], where the objective is to render the viability set  $K$  a positively invariant set, and the goal set  $G$  the largest invariant set of the system (if possible) by means of state feedback control. Compared to our prior work [7], here we do not adopt an optimal control formulation, and propose viable solutions that also converge to a goal set  $G \subset K$ . Finally, we extend our control design method in [11] to the case of nonholonomic systems with dynamic (second order) Pfaffian constraints.

## II. NONHOLONOMIC CONTROL DESIGN

This section gives a brief introduction on our methodology [11] on the feedback control of drift-free, kinematic nonholonomic systems of the form

$$\dot{\mathbf{q}} = \sum_{i=1}^m \mathbf{g}_i(\mathbf{q})u_i, \quad (1)$$

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which are subject to  $\kappa < n$  kinematic Pfaffian constraints

$$\mathbf{A}(\mathbf{q})\dot{\mathbf{q}} = \mathbf{0}, \quad (2)$$

where  $\mathbf{A}(\mathbf{q}) \in \mathbb{R}^{\kappa \times n}$ , the state vector  $\mathbf{q} \in \mathbb{R}^n$  includes the system generalized coordinates,  $\mathbf{g}_i(\mathbf{q})$  are the control vector fields,  $u_i$  are the control inputs.<sup>1</sup> The main idea is that one can define a smooth N-dimensional reference vector field  $\mathbf{F}(\cdot)$  for (1), given as

$$\mathbf{F}(\mathbf{x}) = \lambda_f (\mathbf{p}^\top \mathbf{x}) \mathbf{x} - \mathbf{p} (\mathbf{x}^\top \mathbf{x}), \quad (3)$$

where  $N \leq n$ ,  $\lambda_f \geq 2$ ,  $\mathbf{x} \in \mathbb{R}^N$  is a (particular) subvector of the state vector and  $\mathbf{p} \in \mathbb{R}^N$  is a vector that “generates” the vector field  $\mathbf{F}(\cdot)$ ; the vectors  $\mathbf{x}$ ,  $\mathbf{p}$  are determined by the form of the constraint matrix  $\mathbf{A}(\mathbf{q})$ , see [11]. The vector field  $\mathbf{F}(\cdot)$  serves as a velocity reference for (1); i.e. one can use the available control authority to first “align” the system vector field  $\dot{\mathbf{q}}$  with  $\mathbf{F}$ , and “flow” in the direction of the reference vector field on its way to  $\mathbf{q} = \mathbf{0}$ . These two objectives dictate the choice of particular Lyapunov-like functions  $V$ . Thus, one can find a smooth function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  of compact level sets, and a feedback control law  $\gamma(\cdot) = (\gamma_1(\cdot), \dots, \gamma_m(\cdot)) : \mathbb{R}^n \rightarrow \mathbb{R}^m$  such that

$$\dot{V} \leq 0 \Leftrightarrow \nabla V \dot{\mathbf{q}} = \nabla V \sum_{i=1}^m \mathbf{g}_i(\mathbf{q}) \gamma_i(\cdot) \leq 0, \quad (4)$$

where  $\nabla V \triangleq \left[ \frac{\partial V}{\partial q_1} \quad \dots \quad \frac{\partial V}{\partial q_n} \right]$  the gradient of  $V$  at  $\mathbf{q}$ . Using standard techniques, one can establish that the system trajectories  $\mathbf{q}(t)$  asymptotically converge to the origin.<sup>2</sup>

### III. TOOLS FROM VIABILITY THEORY

This section gives a brief description of concepts and tools from viability theory [10], [14] that are used in the paper.

Consider a system described by a (single-valued) map  $f : \Omega \mapsto X$ , where  $X$  is a finite dimensional vector space and  $\Omega$  an open subset of  $X$ , and the initial value problem:

$$\forall t \in [0, T], \quad \dot{x}(t) = f(x(t)), \quad x(0) = x_0. \quad (5)$$

*Definition 1: (Viable Functions)* Let  $K$  be a subset of  $X$ . A function  $x(\cdot)$  from  $[0, T]$  to  $X$  is viable in  $K$  on  $[0, T]$ , if  $x(t) \in K, \forall t \in [0, T]$ .

The characterization of viable sets  $K$  under  $f$  is based on the concept of tangency: A subset  $K$  is viable under  $f$  if at each state  $x$  of  $K$  the velocity  $f(x)$  is “tangent” to  $K$  at  $x$ , for bringing back a solution to (5) inside  $K$ . This concept of tangency is realized via the contingent cone.

*Definition 2: (Contingent Cone at a Fréchet differentiable point)* Consider the continuous real-valued map  $g = (g_1, g_2, \dots, g_p) : X \rightarrow \mathbb{R}^p$  and the subset  $K$  of  $X$  defined

$$K = \{x \in X \mid g_i(x) \geq 0, \quad i = 1, 2, \dots, p\}, \quad (6)$$

where  $g_i(\cdot)$  are Fréchet differentiable at  $x$ . For  $x \in K$ , denote

$$I(x) = \{i = 1, 2, \dots, p \mid g_i(x) = 0\} \quad (7)$$

<sup>1</sup>It is shown in Section V that the method can be extended to the control design of underactuated systems with dynamic Pfaffian constraints.

<sup>2</sup>Convergence to the origin is “almost” global, in the sense that  $\gamma(\mathbf{q})$  is undefined on particular singularity subsets  $\mathcal{A}$ .

the subset of active constraints. The contingent cone  $T_K(x)$  to  $K$  is  $T_K(x) = X$  whenever  $I(x) = \emptyset$ , otherwise

$$T_K(x) = \{v \in X \mid \forall i \in I(x), \langle g'_i(x), v \rangle \geq 0\},$$

where  $g'_i(x) \in X^*$  is the gradient of  $g_i$  at  $x$ ,  $X^*$  is the dual space of  $X$  and  $\langle \cdot, \cdot \rangle$  stands for the duality pairing.

*Definition 3: (Viability Domain)* The subset  $K$  of  $\Omega$  is a viability domain of  $f : \Omega \mapsto X$  if  $\forall x \in K, f(x) \in T_K(x)$ .

*Definition 4:* Consider a control system  $(U, f)$ :

$$\dot{x}(t) = f(x(t), u(t)), \quad \text{where } u(t) \in U(x(t)),$$

where  $U : X \rightsquigarrow Z$  a feedback set-valued map,  $X$  the state space,  $Z$  the control space, and  $f : \text{Graph}(U) \rightarrow X$ . The map  $R_K := K \rightsquigarrow Z$  of viable controls  $u$  is defined as

$$\forall x \in K, \quad R_K(x) := \{u \in U(x) \mid f(x, u) \in T_K(x)\}.$$

If the subset  $K$  is given by (6), the set of active constraints is as in (7), and for every  $x \in K, \exists v_0 \in X$  such that  $\forall i \in I(x), \langle g'_i(x), v_0 \rangle \geq 0$ , then the regulation map  $R_K(x)$  is

$$R_K(x) := \{u \in U(x) \mid \forall i \in I(x), \langle g'_i(x), f(x, u) \rangle \geq 0\}.$$

### IV. VIABLE NONHOLONOMIC CONTROLS

Consider a nonholonomic system of the form (1) subject to  $\lambda$  inequality state constraints determining the viability set

$$K := \{\mathbf{q} \in Q \mid c_j(\mathbf{q}) \leq 0, \quad j = 1, 2, \dots, \lambda\}, \quad (8)$$

where  $c_j(\cdot) : Q \rightarrow \mathbb{R}$  are continuously differentiable maps.

Assume that  $I(\mathbf{q}) = \emptyset$ , i.e. none of the constraints is active; then  $\mathbf{q} \in \text{Int}(K)$ , and the contingent cone of  $K$  at  $\mathbf{q}$  coincides with the state space  $Q$ ,  $T_K(\mathbf{q}) = Q$ .<sup>3</sup> Thus the system can evolve along any direction  $\dot{\mathbf{q}} \in T_{\mathbf{q}}Q$  without violating the viability constraints. For a nonholonomic system (1) with Pfaffian constraints (2), the admissible velocities  $\dot{\mathbf{q}} \in T_{\mathbf{q}}Q$  belong into the null space of the constraint matrix  $\mathbf{A}(\mathbf{q})$ , an  $(n - \kappa)$  dimensional subspace of the tangent space  $T_{\mathbf{q}}Q$ . Thus, for  $\mathbf{q} \in \text{Int}(K)$ , the viable admissible velocities  $\dot{\mathbf{q}}$  of (1) are tangent to an  $(n - \kappa)$  dimensional subspace of the contingent cone  $T_K(\mathbf{q})$ .

Assume now that, for some  $j \in \{1, 2, \dots, \lambda\}$ , at least one of the  $j$ -th constraint becomes active:  $c_j(\mathbf{z}) = 0$ , where  $\mathbf{z} \in \partial K$ . The contingent cone  $T_K(\mathbf{z})$  is now a subset (not necessarily a vector space but rather a cone) of the tangent space  $T_{\mathbf{z}}Q$ . Thus, an admissible velocity for a nonholonomic system (1) is viable at  $\mathbf{z}$  if and only if

$$\dot{\mathbf{z}} \in \left( \text{Null}(\mathbf{A}(\mathbf{z})) \cap T_K(\mathbf{z}) \right) \neq \emptyset.$$

Based on these, we are able to characterize the conditions for selecting viable controls (if any) for the system (1).

For  $\mathbf{q} \in \text{Int}(K)$ , an admissible control  $\mathbf{u} = (u_1, \dots, u_m) : \mathbb{R}^n \rightarrow \mathbb{R}^m$  for (1) is viable at  $\mathbf{q}$  if and only if

$$\mathbf{u} \in U(\mathbf{q}), \quad \dot{\mathbf{q}} = \sum_{i=1}^m \mathbf{g}_i(\mathbf{q}) u_i \in T_K(\mathbf{q}) \triangleq T_{\mathbf{q}}Q,$$

<sup>3</sup>If  $K$  is a differentiable manifold, then the contingent cone  $T_K(\mathbf{q})$  coincides with the tangent space to  $K$  at  $\mathbf{q}$ .

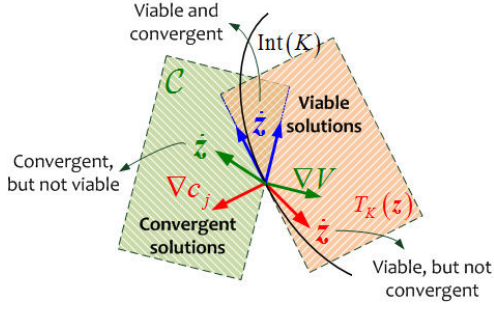


Fig. 1. Any control law  $\gamma(\cdot) = (\gamma_1(\cdot), \dots, \gamma_m(\cdot)) : \mathbb{R}^n \rightarrow \mathbb{R}^m$  such that  $\gamma(z) \in \mathbf{U}(z)$ ,  $\dot{z} = \sum_{i=1}^m \mathbf{g}_i(z)\gamma_i(\cdot) \in (\mathcal{C} \cap T_K(z))$  is also viable at  $z \in \partial K$ , bringing the system trajectories into the interior of  $K$ .

where  $\mathbf{U}(q)$  is the subset of feasible controls.

For  $z \in \partial K$  such that a single constraint is active:  $c_j(z) = 0$ , for some  $j \in \{1, \dots, \lambda\}$ , the map of viable controls at  $z$  reads  $R_K(z) = \{\mathbf{u} \in \mathbf{U}(z) \mid \nabla c_j \sum_{i=1}^m \mathbf{g}_i(z)u_i \leq 0\}$ , where  $\nabla c_j = \begin{bmatrix} \frac{\partial c_j}{\partial q_1} & \dots & \frac{\partial c_j}{\partial q_n} \end{bmatrix}$  at  $z \in \partial K$ . An admissible control  $\mathbf{u} = (u_1, \dots, u_m) : \mathbb{R}^n \rightarrow \mathbb{R}^m$  for (1) is thus viable at  $z \in \partial K$  if and only if

$$\mathbf{u}(z) \in \mathbf{U}(z), \quad \begin{bmatrix} \frac{\partial c_j}{\partial q_1} & \dots & \frac{\partial c_j}{\partial q_n} \end{bmatrix} \sum_{i=1}^m \mathbf{g}_i(z)u_i \leq 0. \quad (9)$$

If  $1 \leq \lambda_\alpha \leq \lambda$  constraints  $c_j(q) : Q \rightarrow \mathbb{R}$  are simultaneously active at some  $z \in \partial K$ , then a control law  $\mathbf{u}(\cdot)$  is viable at  $z$  if (9) holds for each one of the active constraints. If *all*  $\lambda$  constraints are active at  $z$ , the conditions can be written as

$$\mathbf{u}(z) \in \mathbf{U}(z), \quad \mathbf{J}_c(z) \sum_{i=1}^m \mathbf{g}_i(z)u_i \leq \mathbf{0}, \quad (10)$$

$\mathbf{J}_c(z)$  is the Jacobian of the map  $\mathbf{c} = (c_1(\cdot), \dots, c_\lambda(\cdot)) : Q \rightarrow \mathbb{R}^\lambda$  at  $z$ .

Consequently, the state feedback control laws  $\gamma(\cdot) = (\gamma_1(\cdot), \dots, \gamma_m(\cdot)) : \mathbb{R}^n \rightarrow \mathbb{R}^m$  given [11] are viable at  $z \in \partial K$  if and only if

$$\gamma(z) \in \mathbf{U}(z), \quad \begin{bmatrix} \frac{\partial c_j}{\partial q_1} & \dots & \frac{\partial c_j}{\partial q_n} \end{bmatrix} \sum_{i=1}^m \mathbf{g}_i(z)\gamma_i(\cdot) \leq 0, \quad (11)$$

for each one of the active constraints  $c_j(z) = 0$  at  $z \in \partial K$ , where  $\mathbf{U}(z) \subseteq \mathbb{R}^m$  is the subset of feasible controls at  $z$ .

To illustrate this, consider that a single constraint is active,  $c_j(z) = 0$  for some  $z \in \partial K$  (Fig. 1). The viable system velocities  $\dot{z}$  belong into the contingent cone  $T_K(z)$  at  $z$ . The system velocities that establish asymptotic convergence to the origin [11] define the subset  $\mathcal{C} = \{\dot{z} \in T_z Q \mid \nabla V \dot{z} \leq 0\}$ . Thus, a convergent control law  $\gamma(\cdot)$  is viable at  $z \in \partial K$  if and only if  $\gamma(z) \in \mathbf{U}(z)$  and the system velocity  $\dot{z} = \sum_{i=1}^m \mathbf{g}_i(z)\gamma_i(\cdot)$  belongs into the intersection  $(\mathcal{C} \cap T_K(z))$ ; if this intersection is empty, then  $\gamma(\cdot)$  steers the system trajectories out of  $K$ .

## V. A MARINE VEHICLE WITH LIMITED SENSING

As a case study, we consider the motion control on the horizontal plane for an underactuated marine vehicle subject to configuration constraints, which arise due to limited

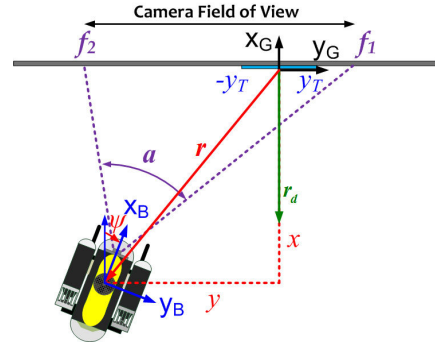


Fig. 2. Modeling of the state constraints imposed by the sensor system

sensing. The sensor suite includes an onboard camera, and provides the vehicle's position and orientation (pose) vector  $\boldsymbol{\eta} = [x \ y \ \psi]^T$  w.r.t. a global coordinate frame  $\mathcal{G}$ , which lies on the center of a target on a vertical surface (Fig. 2). The target is tracked using computer vision algorithms, and this information is used for estimating  $\boldsymbol{\eta}$ . Thus the target should always be visible in the camera f.o.v., for the sensor system to be effective. This requirement imposes a set of viability constraints w.r.t.  $\boldsymbol{\eta}$ .

### A. Mathematical Modeling

The marine vehicle has two back thrusters for moving along the surge and the yaw degree of freedom (d.o.f.), but no side (lateral) thruster for moving along the sway d.o.f.. Following [15], the kinematic and dynamic equations of motion are analytically written as:

$$\dot{x} = u \cos \psi - v \sin \psi, \quad (12a)$$

$$\dot{y} = u \sin \psi + v \cos \psi, \quad (12b)$$

$$\dot{\psi} = r, \quad (12c)$$

$$m_{11}\dot{u} = m_{22}vr + X_u u + X_{u|u}|u|u + \tau_u, \quad (12d)$$

$$m_{22}\dot{v} = -m_{11}ur + Y_v v + Y_{v|v}|v|v, \quad (12e)$$

$$m_{33}\dot{r} = (m_{11} - m_{22})uv + N_r r + N_{r|r}|r|r + \tau_r, \quad (12f)$$

where  $\boldsymbol{\eta} = [x \ y \ \psi]^T$  is the pose vector w.r.t. the global frame  $\mathcal{G}$ ,  $\boldsymbol{\nu} = [u \ v \ r]^T$  is the vector of linear and angular velocities in the body-fixed coordinate frame  $\mathcal{B}$ ,  $m_{11}$ ,  $m_{22}$ ,  $m_{33}$  are the inertia matrix terms (including the "added mass" effect) along the axes of the body-fixed frame,  $X_u$ ,  $Y_v$ ,  $N_r$  are linear drag terms,  $X_{u|u}$ ,  $Y_{v|v}$ ,  $N_{r|r}$  are nonlinear drag terms, and  $\tau_u$ ,  $\tau_r$  are the control inputs along the surge and yaw d.o.f.. Furthermore, the thrust allocation implies that  $\tau_u = F_p + F_{st}$  and  $\tau_r = d(F_p - F_{st})$ , where  $F_p \in [-f_p, f_p]$ ,  $F_{st} \in [-f_{st}, f_{st}]$  are the port and starboard thrust forces, respectively,  $f_p$ ,  $f_{st} > 0$  are the bounds of the thrust forces and  $2d$  is the distance between the two thrusters.

### B. Nonholonomic control design

The system (12) falls into the class of control affine underactuated mechanical systems with drift,

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \sum_{i=1}^m \mathbf{g}_i(\mathbf{x})u_i, \quad (13)$$

where  $\mathbf{x} = [x \ y \ \psi \ u \ v \ r]^T$  is the state vector, and  $u_1 = \tau_u$ ,  $u_2 = \tau_r$  are the control inputs. The dynamics of the sway d.o.f. (12e) serve as a second-order (*dynamic*) nonholonomic constraint. Since the constraint equation is not of the form  $\mathbf{a}^T(\boldsymbol{\eta})\dot{\boldsymbol{\eta}} = 0$ , where  $\boldsymbol{\eta}$  a vector of generalized coordinates, the approach in [11] can not be directly applied.

Nevertheless, considering for a moment the kinematic subsystem only, one can easily verify that (12a), (12b) can be combined into  $-\dot{x} \sin \psi + \dot{y} \cos \psi = v \Rightarrow$

$$\underbrace{\begin{bmatrix} -\sin \psi & \cos \psi & 0 \end{bmatrix}}_{\mathbf{a}^T(\boldsymbol{\eta})} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\psi} \end{bmatrix} = v \Rightarrow \mathbf{a}^T(\boldsymbol{\eta})\dot{\boldsymbol{\eta}} = v, \quad (14)$$

which for  $v \neq 0$  can be seen as a violation of the constraint equation of the unicycle. The ‘‘constraint’’ equation (14) implies that  $\boldsymbol{\eta} = \mathbf{0}$  is an equilibrium point if and only if  $v|_{\boldsymbol{\eta}=\mathbf{0}} = 0$ . With this insight, one can try to steer the kinematic subsystem augmented with the second order constraint (12e) to the origin  $\boldsymbol{\eta} = \mathbf{0}$ , using the velocities  $u, r$  as virtual control inputs, while ensuring that the velocity  $v$  vanishes at  $\boldsymbol{\eta} = \mathbf{0}$ . Following [11], we define an  $N = 2$  dimensional reference vector field  $\mathbf{F}(\cdot) = F_x \frac{\partial}{\partial x} + F_y \frac{\partial}{\partial y}$ , given by (3), where  $\mathbf{x} = [x \ y]^T$ . For  $\lambda_f = 3$  and  $\mathbf{p} = [1 \ 0]^T$ , the vector field components  $F_x, F_y$  read

$$F_x = 2x^2 - y^2, \quad F_y = 3xy. \quad (15)$$

The vector field (15) is non-vanishing everywhere in  $\mathbb{R}^2$  except for  $\mathbf{x} = \mathbf{0}$ , and has integral curves that all converge to  $\mathbf{x} = \mathbf{0}$  with direction  $\phi \rightarrow 0$ . Thus, the idea for the kinematic control design is that the vehicle can be controlled so that it ‘‘aligns with’’ the direction and flows along the integral curves of the vector field  $\mathbf{F}(\cdot)$ , until it converges to  $\boldsymbol{\eta} = \mathbf{0}$ .

In order to design a feedback control law  $r = \gamma_2(\psi)$  for stabilizing the orientation error  $e = \psi - \phi$  to zero, one can require that  $\dot{e} = -k_2 e$ , where  $k_2 > 0$ , that reads

$$\dot{\psi} - \dot{\phi} = -k_2(\psi - \phi) \stackrel{(12c)}{\Rightarrow} r = -k_2(\psi - \phi) + \dot{\phi}. \quad (16)$$

Then, one can take a function  $V$  in terms of the states  $x, y$  and the orientation error  $e = \psi - \phi$  as

$$V = \frac{1}{2}(x^2 + y^2 + e^2) = \frac{1}{2}(x^2 + y^2 + (\psi - \phi)^2),$$

which is positive definite w.r.t.  $[x \ y \ e]^T$  and radially unbounded, and take its time derivative as

$$\dot{V} \stackrel{(16)}{=} [x \ y] \begin{bmatrix} \cos \psi \\ \sin \psi \end{bmatrix} u + [x \ y] \begin{bmatrix} -\sin \psi \\ \cos \psi \end{bmatrix} v - k_2 e^2. \quad (17)$$

The behavior of  $\dot{V}$  depends on the velocity  $v$ . If  $v$  can be seen as a bounded perturbation that vanishes at  $[x \ y \ e]^T = \mathbf{0}$ , then this point is an equilibrium of the kinematic subsystem (in the sense that, for  $x = y = 0$ , one has  $e = 0 \Rightarrow \psi = \phi|_{x=y=0} = 0$ ) and therefore it is meaningful to analyze its (asymptotic) stability. Moreover, since  $v$  comes from the control input  $\zeta = ur$ , one should study its evolution in an input-to-state stability (ISS) framework. With this insight, consider the candidate ISS-Lyapunov function  $V_v = \frac{1}{2}v^2$  and take its

time derivative  $\dot{V}_v = -\frac{m_{11}}{m_{22}}v(ur) - \left( \frac{|Y_v|}{m_{22}}v^2 + \frac{|Y_{v|v}|}{m_{22}}|v|v^2 \right)$ , where by definition  $Y_v, Y_{v|v} < 0$ , and  $w(v) = \frac{|Y_v|}{m_{22}}v^2 + \frac{|Y_{v|v}|}{m_{22}}|v|v^2$  is a continuous, positive definite function. Take  $0 < \theta < 1$ , then  $\dot{V}_v = -\frac{m_{11}}{m_{22}}v(ur) - (1-\theta)w(v) - \theta w(v) \Rightarrow$

$$\dot{V}_v \leq -(1-\theta)w(v), \quad \forall v : -\frac{m_{11}}{m_{22}}v(ur) - \theta w(v) < 0.$$

If the control input  $\zeta = ur$  is bounded,  $|\zeta| \leq \zeta_b$ , then

$$\dot{V}_v \leq -(1-\theta)w(v), \quad \forall |v| : |Y_v||v| + |Y_{v|v}||v|^2 > \frac{m_{11}}{\theta}\zeta_b.$$

Then, the subsystem (12e) is ISS [16, Thm 4.19]. Thus, for any bounded input  $\zeta = ur$ , the linear velocity  $v(t)$  will be ultimately bounded by a class  $\mathcal{K}$  function of  $\sup_{t>0} |\zeta(t)|$ . If furthermore  $\zeta(t) = u(t)r(t)$  converges to zero as  $t \rightarrow \infty$ , then  $v(t)$  converges to zero as well.

Going back to (17), set the control input  $u = \gamma_1(x, y)$  as

$$u = -k_1 \operatorname{sgn}(x \cos \psi + y \sin \psi) \tanh(k(x^2 + y^2)), \quad (18)$$

where  $k_1, k > 0$ ,  $\operatorname{sgn}(a) = 1$  if  $a \geq 0$ ,  $\operatorname{sgn}(a) = -1$  if  $a < 0$ , that yields

$$\dot{V} = -\left( k_1 \left| [x \ y] \begin{bmatrix} \cos \psi \\ \sin \psi \end{bmatrix} \right| \tanh(k(x^2 + y^2)) + k_2 e^2 \right) + [x \ y] \begin{bmatrix} \cos(\frac{\pi}{2}-\psi) \\ \sin(\frac{\pi}{2}-\psi) \end{bmatrix} v(t).$$

The control input (18) is bounded. Note also that the terms  $e, \dot{\phi}$  in (16) are also bounded; therefore  $v(t)$  is bounded,  $|v(t)| \leq b$ . Furthermore,  $r(t) \rightarrow 0$  as  $t \rightarrow \infty$ , since the error  $e(t) \rightarrow 0$  exponentially and  $\dot{\phi} \rightarrow 0$ . Therefore  $v(t) \rightarrow 0$  as  $t \rightarrow \infty$ . Then,  $\dot{V} \leq -V_1 + \|x\| b$ , where  $V_1 \geq 0$ , which implies that  $\dot{V}$  is negative semi-definite for suitable choice of the gains  $k_1, k_2$ . Furthermore,  $\dot{V}$  is uniformly continuous in time, since its derivative  $\ddot{V}$  is bounded. Then, according to Barbalat’s lemma,  $\lim_{t \rightarrow \infty} \dot{V} = 0$ . Given that  $e \rightarrow 0$  exponentially and that  $v(t) \rightarrow 0$  as  $t \rightarrow \infty$ , one has that the first term of  $\dot{V}$  converges asymptotically to zero as well, which implies that the system trajectories  $x(t), y(t)$  converge to zero as  $t \rightarrow \infty$ .<sup>4</sup>

Finally, for the design of the control inputs  $\tau_u, \tau_r$ , assuming that full state feedback  $\mathbf{x}$  is available, one can use a feedback linearization transformation for the dynamic subsystems (12d), (12f) given as  $\tau_u = m_{11}\alpha - m_{22}vr - X_u u - X_{u|u}|u|u$ ,  $\tau_r = m_{33}\beta - (m_{11} - m_{22})ur - N_r r - N_{r|r}|r|r$ , that yields  $\dot{u} = \alpha, \dot{r} = \beta$ , where  $\alpha, \beta$  are the new control inputs. Thus, the system should be controlled so that the velocities  $u, r$  track the virtual control inputs  $\gamma_1(\cdot), \gamma_2(\cdot)$ . To design the control laws  $\alpha(\cdot), \beta(\cdot)$ , consider the candidate Lyapunov function  $V_\tau = \frac{1}{2}(u - \gamma_1(\cdot))^2 + \frac{1}{2}(r - \gamma_2(\cdot))^2$  and take its time derivative as  $\dot{V}_\tau = (u - \gamma_1(\cdot))(\dot{u} - \frac{\partial \gamma_1}{\partial \boldsymbol{\eta}} \dot{\boldsymbol{\eta}}) + (r - \gamma_2(\cdot))(\dot{r} - \frac{\partial \gamma_2}{\partial \boldsymbol{\eta}} \dot{\boldsymbol{\eta}})$ . Then, under the control inputs  $\alpha = -k_u(u - \gamma_1(\cdot)) + \frac{\partial \gamma_1}{\partial \boldsymbol{\eta}} \dot{\boldsymbol{\eta}}$ ,  $\beta = -k_r(r - \gamma_2(\cdot)) + \frac{\partial \gamma_2}{\partial \boldsymbol{\eta}} \dot{\boldsymbol{\eta}}$ , where  $k_u, k_r > 0$ , one gets

<sup>4</sup>To verify that the case  $[x \ y] \begin{bmatrix} \cos \psi & \sin \psi \end{bmatrix}^T = 0$  does not affect the convergence of  $x(t), y(t)$  to zero, one can consider that  $\lim_{t \rightarrow \infty} V_1 = 0$  holds either at the origin  $x = y = 0, \psi = \phi|_{x=y=0} = 0$ , or in the set  $x = 0, \psi = \phi|_{x=0, y \in \mathbb{R}} = \pi$ . Nevertheless, in the latter case the control input  $u$  is non zero, and thus the system trajectories escape this set.

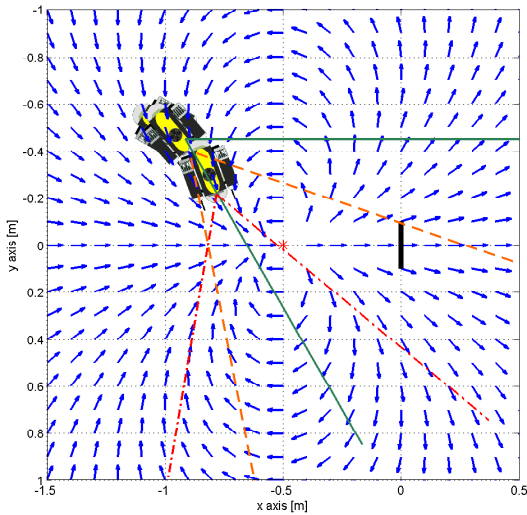


Fig. 3. A convergent solution  $\eta(t)$  given by the control law  $u = \gamma_1(\cdot)$ ,  $r = \gamma_2(\cdot)$  may violate viability during some (finite) time interval.

$\dot{V}_\tau = -k_u(u - \gamma_1(\cdot))^2 - k_r(r - \gamma_2(\cdot))^2$ , which verifies that the velocities  $u, r$  globally exponentially converge to  $\gamma_1(\cdot)$ ,  $\gamma_2(\cdot)$ , respectively.

### C. Viable nonholonomic control design

The visibility maintenance requirement imposes two nonlinear inequality constraints of the form  $c_j(x, y, \psi) \leq 0$ :

$$c_1 : y - x \tan(\psi - \alpha) + y_T \leq 0, \quad (19a)$$

$$c_2 : y_T - y + x \tan(\psi + \alpha) \leq 0. \quad (19b)$$

The control law (18), (16) yields solutions that converge to (any) desired configuration  $\eta_d \in K$ .<sup>5</sup> Nevertheless, the convergent trajectories  $\eta(t)$  may not be viable in  $K$ .

Such an example is shown in Fig. 3. The vehicle starts at  $\eta_0 \in K$ ; however, tracking the reference vector field  $\mathbf{F}(\cdot)$  under (18), (16) on its way to  $\eta_d = [-0.5 \ 0 \ 0]^T \in K$  implies that the convergent trajectories  $\eta(t)$  are driven out of  $K$  for some finite time. In particular, the system trajectories  $\eta(t)$  violate the constraint  $c_1(\eta)$  given by (19a). This constraint becomes active when the target lies on the left boundary of the f.o.v., where  $f_2 = -y_T$  (Fig. 3, dashed line). This condition defines a subset  $\mathcal{Z}_1 = \{z \in \partial K \mid c_1(\cdot) = y - x \tan(\psi - \alpha) + y_T = 0\}$  of the boundary of  $K$ . The viable system velocities at any  $z \in \mathcal{Z}_1$  satisfy  $\nabla c_1 \dot{z} \leq 0 \Rightarrow [-\tan(\psi - \alpha) \ 1 \ -x \sec^2(\psi - \alpha)] \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\psi} \end{bmatrix} \leq 0$ . Substituting the system equations yields

$$(-\tan(\psi - \alpha) \cos \psi + \sin \psi)u + (\tan(\psi - \alpha) \sin \psi + \cos \psi)v - x \sec^2(\psi - \alpha)r \leq 0. \quad (20)$$

The condition (20) verifies that the control inputs (18), (16) violate the constraint  $c_1(z) = 0$ : at  $z \in \mathcal{Z}_1$  the vehicle moves with  $u, r, \psi > 0$ , thus the first and third term are  $> 0$ ,

<sup>5</sup>For driving the system to  $\eta_d = [x_d \ y_d \ \psi_d]^T$  the reference vector field  $\mathbf{F}(x^*, y^*)$  is defined by a vector  $\mathbf{p} = [p_x \ p_y]^T$  such that  $\psi_d = \text{atan2}(p_y, p_x)$ , in terms of  $x^* = x - x_d$ ,  $y^* = y - y_d$ .

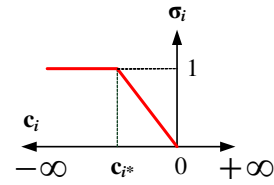


Fig. 4. The switching signal  $\sigma_i(c_i) : (-\infty, 0] \rightarrow [0, 1]$ .

whereas the velocity  $v$  is not negative enough to satisfy (20). Therefore, the control laws  $u = \gamma_1(\cdot)$ ,  $r = \gamma_2(\cdot)$  should be redesigned so that (20) holds  $\forall z \in \mathcal{Z}_1$ . To this end, note that (20) offers a way of selecting viable control inputs when  $z \in \mathcal{Z}_1$ . For picking a viable control input  $r(z)$ , one can choose to regulate the orientation  $\psi$  of the vehicle to the angle  $\phi_t = \text{atan2}(-y, -x)$ , which essentially is the orientation of the vector  $-\eta$  that connects the vehicle with the target; in that way, the system is controlled so that target is centered in the camera f.o.v., avoiding thus the left boundary.

Thus, for redesigning the control laws (18), (16) so that they are viable at  $z \in \mathcal{Z}_1$ , one can consider a continuous switch of the form

$$\sigma_1(c_1) = \begin{cases} \frac{c_1}{c_{1*}}, & \text{if } c_{1*} \leq c_1 \leq 0, \\ 1, & \text{if } c_1 < c_{1*}, \end{cases}$$

shown in Fig. 4, and use the control law

$$u = \sigma_1(c_1)u_{conv} + (1 - \sigma_1(c_1))u_{viab}, \quad (21a)$$

$$r = \sigma_1(c_1)r_{conv} + (1 - \sigma_1(c_1))r_{viab}, \quad (21b)$$

where  $u_{conv}$ ,  $r_{conv}$  are given by (18), (16), and  $u_{viab}$ ,  $r_{viab}$  are control inputs satisfying (20) at  $z \in \mathcal{Z}_1$ ; one can set  $r_{viab} = -k_2(\psi - \phi_t)$ , where  $\phi_t = \text{atan2}(-y, -x)$ ,  $u_{viab} \triangleq u_{conv}$ , given by (18), and select the control gains for  $u_{viab}$  so that (20) is satisfied. Then, if  $c_1(z) = 0$  one has  $\sigma_1(c_1) = 0$ , which ensures that the control laws at  $z \in \mathcal{Z}_1$  are viable.

Under this control setting, one has that if the system trajectories are such that  $c_1(\eta(t)) < c_{1*}(\eta(t)) \forall t \geq 0$ , i.e. if the viable controls are never activated, then the system is guaranteed to converge to  $\eta_d$ . On the other hand, if the switch  $\sigma_1(\cdot)$  is activated at some  $t \geq 0$ , it follows that the vehicle does not track the convergent to  $(x_d, y_d)$  integral curves of the vector field  $\mathbf{F}$  during the time interval that  $\sigma_1(c_1) \neq 1$ . If furthermore the control laws  $u_{viab}$ ,  $r_{viab}$  are not convergent to  $\eta_d$ , which is the general case, then the control law (21) does not any longer guarantee the convergence of the system trajectories to  $\eta_d$ . In this case, one can relax the requirement on convergence to a single point, and rather choose to establish convergence to a goal set  $G \subset K$  of desired configurations, given as

$$G = \{\eta_d \in K \mid x_d^2 + y_d^2 = d^2, \psi_d = \text{atan2}(-y_d, -x_d)\},$$

where  $d$  is a desired distance w.r.t. the target. The linear velocity controller is given as  $u = -k_1 \text{sgn}(x_1 \cos \psi + y_1 \sin \psi) \tanh(k(x_1^2 + y_1^2))$ , where  $\psi_d = \text{atan2}(-y, -x)$ ,  $x_1 = x - x_d$ ,  $x_d = d \cos \psi_d$ ,  $y_1 = y - y_d$ ,  $y_d = d \sin \psi_d$ .

Following the same ideas, one can define the switch  $\sigma_2(c_2)$  and viable control laws for the case that the constraint  $c_2(\eta)$

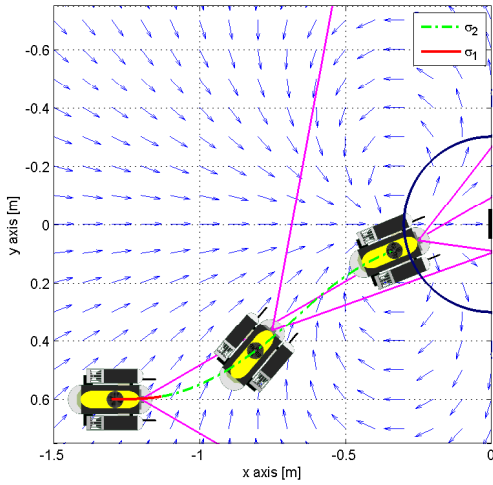


Fig. 5. The path  $x(t)$ ,  $y(t)$  under the proposed control scheme.

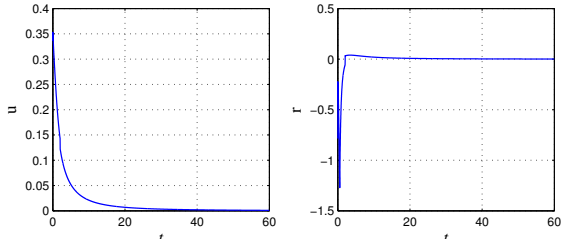


Fig. 6. The virtual control inputs  $u$ ,  $r$ .

becomes active, and synthesize the switching between the convergent and the viable control laws as

$$u = \sigma^* u_{conv} + (1 - \sigma^*) u_{viab}, \quad (22a)$$

$$r = \sigma^* r_{conv} + (1 - \sigma^*) r_{viab}, \quad (22b)$$

where the switching signal  $\sigma^*(\cdot, \cdot)$  can be defined as  $\sigma^* = \min(\sigma_1, \sigma_2)$ , or  $\sigma^* = \sigma_1 \sigma_2$ , i.e. as function that varies in  $[0, 1]$  such that its value is zero when at least one of the constraints is active.

Finally, note that the control gains  $k_1$ ,  $k_2$ ,  $k_u$ ,  $k_r$  can be properly tuned so that the “virtual” control inputs  $u$ ,  $r$  correspond to thrust forces  $F_p$ ,  $F_{st}$  that belong into the compact set  $\mathcal{U} = [-f_p, f_p] \times [-f_{st}, f_{st}]$ .

To evaluate the efficacy of the methodology, let us consider the scenario shown in Fig. 5, where the switching signal is chosen as  $\sigma^* = \min(\sigma_1, \sigma_2)$ . The vehicle initiates from a configuration  $\eta_0$  where the target lies near the left boundary of the f.o.v.; thus, the vehicle is controlled on its way to the goal set  $G$  including the “nominal” desired configuration  $\eta_d = [-0.3 \ 0 \ 0]^T$  under the control laws  $u$ ,  $r$  for  $\sigma_1(\cdot)$ . During the vehicle’s motion,  $c_2(\cdot)$  gets greater than  $c_1(\cdot)$ , which corresponds to the target being closer to the right boundary of the f.o.v., and thus  $\sigma_2(\cdot)$  is activated. The vehicle converges to a desired configuration in the goal set  $G$ . The evolution of the control inputs  $u$ ,  $r$  is shown in Fig. 6.

## VI. CONCLUSIONS

This paper presented a method for the control design of nonholonomic systems subject to state constraints defining

a viability set  $K$ . Using concepts and tools from viability theory, the necessary conditions for selecting viable control laws were given. Furthermore, a class of nonholonomic control solutions were redesigned in a switching control scheme, so that system trajectories starting in  $K$  converge to a goal set  $G$  in  $K$ , without ever leaving  $K$ . As a case study, the control design for an underactuated marine vehicle subject to configuration constraints due to limited visibility was treated. Viable control laws in the constrained set  $K$  which establish convergence to a goal set  $G \subset K$  were constructed. Our ideas and plans for future extensions include the consideration of perturbations, which typically arise in this class of problems, towards the solution of the overall control problem in a viability framework.

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