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Stability and Stabilization of T-S Fuzzy Systems with Time-Varying Delay: An Input-Output Approach

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Abstract— **An input-output approach to the stability and stabilization of Takagi-Sugeno (T-S) fuzzy system with time-varying delay is proposed in this paper. A novel method is employed to approximate the time-varying delay. Thus the considered system can be formulated into a feedback interconnection form, which contains a constant time-delayed forward subsystem. Then based on the scaled small gain theorem, the problem reduces to studies of the bounded real property and H_∞ control problem of the scaled forward subsystem. Some effective techniques used to be applied to the constant time delay systems are now utilized to solve the time-varying case. By virtue of this conversion, less-conservative stability criteria and stabilization methods via Parallel Distributed Compensation (PDC) scheme are obtained. Numerical experiments are performed to illustrate the advantage of the proposed techniques.**

I. Introduction

Recent years have witnessed growing interests and extensive studies of the Takagi-Sugeno (T-S) fuzzy systems [13]. The T-S fuzzy modeling approach has been widely accepted as 1) powerful tool for approximating nonlinear systems and 2) a flexible framework to fully take advantage of the advances in linear system theories. On another aspect, it's well know that time-delay can be source of instability or performance degradation and it is still a challenging problem when facing engineering and communications applications. In view of these considerations, T-S fuzzy modeling approach has been extended to tackle the analysis and synthesis of nonlinear systems with time delay.

Since that time, a systematical investigation of the time-delayed T-S fuzzy system has been conducted by virtue of the optimality of linear matrix inequalities (LMIs) technique (see [7] and the references therein). Generally, most of the approaches involve a simple Lyapunov Krasovskii Functional (LKF) [4], [6], [8], [11], and directly apply some more or less tight techniques, such as Moon's inequality [9], free-weighting matrix [15], or use Jensen's inequality [2], to derive the stability criteria for the time-delayed T-S fuzzy system. These choices of over-bounding techniques are the origin of conservatism. Among all the simple LKF based articles, a more effective technique is the delay partitioning [18].

In this paper, an *indirect* approach, namely, the *input-output* (IO) approach is introduced to deal with the stability analysis and control design of the T-S fuzzy systems with time-varying delay. In retrospect, this approach initially prevailed in the robust stability analysis field [19], then it has been employed to study the linear time-invariant (LTI) delay systems in [16] and further to time-varying delay systems in [2] to cope with the delay "uncertainty". Recently, [5] extended the approach to deal with the discrete-time systems with time-varying delay. However, to the best of our knowledge, it hasn't been found any literature concerning the delayed T-S fuzzy systems via this approach.

The main procedures of the IO approach involve a model transformation of the original system into feedback interconnection formulation, which contains a constant time-delayed forward subsystem and a "delay uncertainty" feedback subsystem. Then by applying the scaled small gain theorem, only the forward subsystem needs to be considered to ensure the stability of the original systems. An essential problem in the above procedures which is directly related to the conservatism is to find a proper approximation for $x(t-d(t))$, such that the approximation error is as small as possible, where $d(t)$ is the time-varying delay but the delay rate information may be unavailable. Specifically, both the two cases $\dot{d}(t) < 1$ (*slowly varying delay*) and $\dot{d}(t) = \infty$ (*fast varying delay*) can be handled by the proposed methods. Moreover, the two-term approximation method is used to give the approximation for $x(t-d(t))$, which has been briefly discussed in [3] for LTI delay systems and proved to be more precise than other approximation methods. Besides, some useful techniques like fuzzy weighting-dependent LKF and delay partitioning are also incorporated. As a result, the new method essentially improves the existing results for T-S fuzzy system with time-varying delay.

Notations: $\mathbb{R}^n$, $\mathbb{R}^{m\times n}$, and $\mathbb{S}^n$ represent the set of real n-vector, $m \times n$ matrices, and $n \times n$ symmetric positive definite matrices, respectively. $\mathbf{G}_1 \circ \mathbf{G}_2$ represents the series connection of mapping $\mathbf{G}_1$ and $\mathbf{G}_2$. I_n denotes an identity matrix with dimension n and $0_{m,n}$ denotes an $m \times n$ dimension zero matrix. We use "$*$" to denote the symmetric terms in a block matrix P, sym(P) to abbreviate $P + P^T$ and diag$\{\cdots\}$ to express a block-diagonal matrix. $\|G\|_\infty$ denotes the l_2-induced norm of a transfer function matrix or a general operator. $\mathcal{R}$ denotes the set $\{1, 2 \ldots r\}$. Finally, $\mathcal{C}^1$ denotes the class of continuously differentiable functions.

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II. Preliminaries

Consider a nonlinear system represented by a delayed T-S fuzzy model:

Plant Rule i: **IF** $\theta_1(t)$ is M_{i1} and $\theta_2(t)$ is M_{i2} and ...and $\theta_p(t)$ is M_{ip}, **THEN**

$$\dot{x}(t) = A_i x(t) + A_{di} x(t - d(t)) + B_i u(t), \qquad (1)$$

with the vector-valued initial condition $\phi(t) \in \mathbb{R}^n$, for all $t \in [-h_2, 0]$ and $i \in \mathcal{R}$. The matrices A_i, A_{di}, B_i are known constant matrices with appropriate dimensions. M_{ij} is the fuzzy set, r is the number of IF-THEN rules, and $\theta(t) = [\theta_1(t), \theta_2(t), \cdots \theta_p(t)]$ are the premise variables which do not depend on the input $u(t) \in \mathbb{R}^{n_u}$. The time varying delay $d(t)$ satisfies $0 < h_1 \leq d(t) \leq h_2$. Denoting $\tau = \dot{d}(t)$, the cases of $\tau < 1$ and $\tau = \infty$ will be considered.

The overall fuzzy system is inferred as

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + A_d(t)x(t - d(t)) + B(t)u(t) \\ &= \sum_{i=1}^{r} \lambda_i(t) \left[A_i x(t) + A_{di} x(t - d(t)) + B_i u(t)\right], \qquad (2)\end{aligned}$$

where $\sum_{i=1}^{r} \lambda_i(t) = 1$, $\lambda_i(t) = \omega_i(\theta(t)) / \sum_{i=1}^{r} \omega_i(\theta(t)) \geq 0$ and $\omega_i(\theta(t)) = \prod_{i=1}^{r} M_{ij}(\theta_j(t))$ with $M_{ij}(\theta_j(t))$ representing the grade of membership of $\theta_j(t)$ in M_{ij}.

Note that in this paper, we simplify the traditional denotation of $\lambda_i(\theta(t))$ as $\lambda_i(t)$, where no confusion should be so caused.

The stabilization problem is investigated under the PDC scheme, where the controller rule shares the same fuzzy sets with the T-S model, that is,

Controller Rule i: **IF** $\theta_1(t)$ is M_{i1} and $\theta_2(t)$ is M_{i2} and ...and $\theta_p(t)$ is M_{ip}, **THEN**

$$u(t) = K_i x(t) + \frac{1}{2} K_{\underline{d}i} x(t - h_1) + \frac{1}{2} K_{\bar{d}i} x(t - h_2), i \in \mathcal{R} \qquad (3)$$

The state feedback control law inferred is

$$\begin{aligned}u(t) &= \sum_{i=1}^{r} \lambda_i(t) [K_i x(t) + \frac{1}{2} K_{\underline{d}}^i x(t - h_1) + \frac{1}{2} K_{\bar{d}}^i x(t - h_2)] \\ &\triangleq K(t)x(t) + \frac{1}{2} K_{\underline{d}}(t) x(t - h_1) + \frac{1}{2} K_{\bar{d}}(t) x(t - h_2),\end{aligned}$$

It is worth noting that the proposed control law takes advantages of the lower and upper bounds of the time varying delay $d(t)$. Thus it can be applied to the case where the time delay is not on-line measurable. Moreover, it covers the special cases of the memoryless control where $K_{\underline{d}}^i = K_{\bar{d}}^i = 0$ and the purely delayed control where $K_i = 0$.

Then the closed loop system is obtained as

$$\begin{aligned}\dot{x}(t) \quad = \quad & \bar{A}(t)x(t) + A_d(t)x(t - d(t)) + \frac{1}{2} B(t) \\ & \times [K_{\underline{d}}(t) x(t - h_1) + K_{\bar{d}}(t) x(t - h_2)], \qquad (4)\end{aligned}$$

where $\bar{A}(t) = A(t) + B(t)K(t)$.

The objective of this paper is to determine the stability and stabilization conditions for system (4) via an IO approach. This approach actually utilizes the Scaled Small Gain theorem (SSG). To apply this theorem, the original system is first converted into feedback interconnection formulation, where the ”delay uncertainty” is pulled out to form the feedback loop. Then we prove that the obtained feedback subsystem satisfies a certain SSG condition. Therefore, according to SSG theorem, the rest of the work is to derive the sufficient conditions for the forward subsystem to satisfy the complementary SSG condition.

So the most elemental notion on SSG theorem is briefly recalled here and we refer readers to Chapter 8 of [2] for more information.

Consider an interconnected system consisting of two subsystems:

$$(\mathcal{S}_1): \ z(t) = \mathbf{G}\omega(t), \ (\mathcal{S}_2): \ \omega(t) = \Delta z(t), \qquad (5)$$

where the forward $\mathcal{S}_1$ is a known linear time-invariant system (LTI) with operator $\mathbf{G}$ mapping ω to z, the feedback $\mathcal{S}_2$ is an unknown linear time-varying one with operator $\Delta \in \mathcal{D} \triangleq \{\Delta\colon \|\Delta\|_\infty \leq 1\}$ and $z(t) \in \mathbb{R}^z, \omega(t) \in \mathbb{R}^\omega$. As a direct result of the small gain theorem [19], a sufficient condition for the robustly asymptotic stability of the interconnection in (5) is given as follows.

Lemma 1 (SSG Theorem): Consider (5) and assume $\mathcal{S}_1$ is internally stable. The closed-loop system formed by $\mathcal{S}_1$ and $\mathcal{S}_2$ is robustly asymptotically stable for all $\Delta \in \mathcal{D}$ if there exist matrices $\{T_\omega, T_z\} \in \mathbf{T}$ with

$$\begin{aligned}\mathbf{T} \quad \triangleq \quad & \Big\{\{T_\omega, T_z\} \in \mathbb{R}^{\omega \times \omega} \times \mathbb{R}^{z \times z}: \\ & T_w, T_z \text{ nonsingular}; \left\|T_\omega \circ \Delta \circ T_z^{-1}\right\|_\infty \leq 1\Big\},\end{aligned}$$

such that the following condition holds:

$$\left\|T_z \circ \mathbf{G} \circ T_\omega^{-1}\right\|_\infty < 1. \qquad (6)$$

In the above formulation, one critical issue closely related to the reduction of the conservatism is how to pull out the “delay uncertainty”, or more specifically, to what degree of precision can one estimate the uncertain delay $d(t)$. The recent work of [3] proposes a two-term approximation method which gives better estimation of the time varying delay $d(t)$. And this method constitutes our main technique to achieve less-conservative conditions. Besides, other effective ways to further reduce the conservatism are also employed, one of which is to select the fuzzy weighting-dependent Lyapunov function (FWLF)

$$P(t) = \sum_{i=1}^{r} \lambda_i(t) P_i > 0, \qquad (7)$$

other than the traditional quadratic Lyapunov function. Some other fuzzy weighting-dependent matrices have also been introduced to further reduce the conservatism in the sequent derivation. Since the time-derivative of (7) requires that of $\lambda_i(t)$, the following assumption is made:

($\mathcal{A}$) :Assume that $\lambda_i(t) \in \mathcal{C}^1, i = 1, 2 \ldots r.$ and $|\dot{\lambda}_i(t)| \leq \beta_i$ with $\beta_i \geq 0$.

For the fuzzy models constructed using the sector nonlinearity approach [13], the assumption $\lambda_i(t) \in \mathcal{C}^1$ is met.

However, the bounds of $\dot{\lambda}_i(t)$ may not be directly acquired in the practical application. One way to avoid using this information is to construct another kind of FWLF with a special structure in each P_i [12].

III. Main Results

In this section, we first transform the delayed T-S fuzzy system (2) into the interconnection of two subsystems as in (5) and then analyzing the SSG of $\mathcal{S}_1$. By virtue of the two-term approximation method, the obtained $\mathcal{S}_1$ is a constant time-delay T-S fuzzy system which can be treated in many ways. So delay partitioning technique is further employed to analyze the SSG of $\mathcal{S}_1$. The stability and the stabilization problems are elaborated sequentially.

A. Model Transformation

Considering system (2) with input vector $u(t) = 0$, we now estimate the time varying delay $d(t)$ using its lower and upper bounds. The two term approximation $\frac{1}{2}[x(t-h_1) + x(t-h_2)]$ results in the estimation error

$$\begin{aligned} \omega_0(t) &= x(t-d(t)) - \frac{1}{2}[x(t-h_1) + x(t-h_2)] \\ &= \frac{1}{2}\int_{-h_2}^{-d(t)} \dot{x}(t+\zeta)d\zeta - \frac{1}{2}\int_{-d(t)}^{-h_1} \dot{x}(t+\zeta)d\zeta \\ &\triangleq \frac{1}{2}\int_{-h_2}^{-h_1} k(\zeta)z(t+\zeta)d\zeta, \end{aligned} \tag{8}$$

where $z(t) = \dot{x}(t) = A(t)x(t) + A_d(t)x(t-d(t))$ and

$$k(\xi) = \begin{cases} 1, & \zeta \le -d(t); \\ -1, & \zeta > -d(t). \end{cases}$$

Then system (2) may be written as a feedback system with $\mathcal{S}_1$and $\mathcal{S}_2$ are

$$(\mathcal{S}_1): \begin{cases} \dot{x}(t) = A(t)x(t) + \frac{1}{2}A_d(t)[x(t-h_1) + x(t-h_2)] \\ \qquad + \frac{h_{12}}{2}A_d(t)\omega(t), \\ z(t) = \dot{x}(t), \end{cases}$$

$$(\mathcal{S}_2): \omega(t) = \Delta_d z(t) = \frac{1}{h_{12}}\int_{-h_2}^{-h_1} k(\zeta)z(t+\zeta)d\zeta,$$

respectively, where $h_{12} \triangleq h_2 - h_1$ and operator $\Delta_d : z \mapsto \omega$ are defined. Note that in the above formulation, $\omega(t)$ is the normalization of $\omega_0(t)$ by multiplying $\frac{2}{h_{12}}$. Then the following result can be concluded which, in the meantime, provides a possible choice of the scaling matrices $\{T_\omega, T_z\} \in \mathbf{T}$.

Lemma 2: X is a general invertible matrix, and then the operator $\Delta_d : z \mapsto \omega$ satisfies $\left\|X \circ \Delta_d \circ X^{-1}\right\|_\infty \le 1$.

Proof: Denote

$$\begin{aligned} I &= \int_0^t \omega^T(\zeta)S\omega(\zeta)d\zeta \\ &= \left(\frac{1}{h_{12}}\right)^2 \int_0^t \left[\int_{-h_2}^{-h_1} k(\alpha)z(\zeta+\alpha)d\alpha\right]^T \\ &\quad \times S\left[\int_{-h_2}^{-h_1} k(\alpha)z(\zeta+\alpha)d\alpha\right]d\zeta, \end{aligned}$$

where $S = X^TX \in \mathbb{S}^n$. Then using Jensen inequality (Appendix B.6 in [2]), considering zero initial condition and exchanging the order of integration, it follows

$$\begin{aligned} I &\le \frac{1}{h_{12}^2}\int_0^t h_{12}\int_{-h_2}^{-h_1} z^T(\zeta+\alpha)Sz(\zeta+\alpha)d\alpha d\zeta \\ &= \frac{1}{h_{12}}\int_{-h_2}^{-h_1}\int_0^t z^T(\zeta+\alpha)Sz(\zeta+\alpha)d\zeta d\alpha \\ &= \frac{1}{h_{12}}\int_{-h_2}^{-h_1}\int_\alpha^{t+\alpha} z^T(\zeta)Sz(\zeta)d\zeta d\alpha \\ &\le \frac{1}{h_{12}}\int_{-h_2}^{-h_1}\int_0^t z^T(\zeta)Sz(\zeta)d\zeta d\alpha \\ &= \int_0^t z^T(\zeta)Sz(\zeta)d\zeta, \end{aligned}$$

which implies $\left\|X \circ \Delta_d \circ X^{-1}\right\|_\infty \le 1$. ■

Remark 1: Note that $\{X, X\} \in \mathbf{T}$ is the scaling matrix in the SSG analysis. Then according to lemma 1, to ensure that the system (2) is input-output stable, the main task is to verify that $\mathcal{S}_1$ is internally stable and there exists X such that the SSG condition $\left\|X \circ \mathbf{G}_{z\omega} \circ X^{-1}\right\|_\infty < 1$ holds. This leads to the new bounded real lemma (BRL) for T-S fuzzy systems with two constant time delays.

B. Stability Analysis via BRL Condition

Considering the subsystem $\mathcal{S}_1$ subject to the scaling manipulation, we have

$$(\mathcal{S}_x): \begin{cases} \dot{x}(t) = A(t)x(t) + \frac{1}{2}A_d(t)[x(t-h_1) + x(t-h_2)] \\ \qquad + \frac{h_{12}}{2}A_d(t)X^{-1}\tilde{\omega}(t), \\ \tilde{z}(t) = X\dot{x}(t), \end{cases}$$

where $\tilde{z}(t) = Xz(t), \tilde{\omega}(t) = X\omega(t)$.

Let $V(t)$ be a LKF, which guarantees the stability of the forward subsystem of $\mathcal{S}_x$. Then it's well-known that the following condition along $(\mathcal{S}_x)$:

$$\begin{aligned} \mathcal{W} &\triangleq \dot{V}(t) + \tilde{z}^T(t)\tilde{z}(t) - \tilde{\omega}^T(t)\tilde{\omega}(t) \\ &< -\varepsilon(\|x(t)\|^2 + \|\tilde{\omega}(t)\|^2),\ \varepsilon > 0, \end{aligned} \tag{9}$$

guarantees that the H_∞ norm of $\mathcal{S}_x$ is less than 1. Therefore, (9) is a sufficient condition for the BRL problem.

The following results related to the stability criterion of (2) is presented first.

Theorem 1: Consider the scaled subsystem $(\mathcal{S}_x)$ converted from (2) and assume $(\mathcal{A})$ holds. Given an integer $m \ge 1$ and scalars $h_2 > h_1 > 0$, $\tau < 1$, the original time-delayed T-S fuzzy system (2) with $u(t) = 0$ is asymptotically stable if there exist matrices S, P_i, T, W, E_i, R, $Q \in \mathbb{S}^n$ and $U_i \in \mathbb{S}^{mn}$ such that

$$P_i > P_r, \quad i = 1, 2 \ldots r-1 \tag{10}$$

$$\psi_{iilk} < 0, \quad i,\ l,\ k \in \mathcal{R} \tag{11}$$

$$\psi_{ijlk} + \psi_{jilk} < 0, \quad i < j, \quad i,\ j,\ l,\ k \in \mathcal{R} \tag{12}$$

where

$$
\begin{aligned}
\psi_{ijlk} &= \begin{bmatrix} \tilde{\Pi}_{ijlk} & \left\{\tilde{W}_{Pi}\right\}_1^T S \\ * & -S \end{bmatrix}, \bar{P}_i = \begin{bmatrix} 0 & P_i \\ P_i & 0 \end{bmatrix}, \\
\tilde{\Pi}_{ijlk} &= \tilde{W}_{Pi}^T \bar{P}_j \tilde{W}_{Pi} + \tilde{W}_{Ri}^T \bar{R} \tilde{W}_{Rj} + \bar{W}_U^T \bar{U}_{ilk} \bar{W}_U + \bar{\Lambda}, \\
\tilde{W}_{Pi} &= \begin{bmatrix} A_i & 0_{n,(m-1)n} \ \frac{1}{2}A_{di} & \frac{1}{2}A_{di} \ \frac{h_{12}}{2}A_{di} \\ I_n & 0_{n,(m+2)n} \end{bmatrix}, \\
\tilde{W}_{Ri} &= \begin{bmatrix} A_i & 0_{n,(m-1)n} & \frac{1}{2}A_{di} \ \frac{1}{2}A_{di} & \frac{h_{12}}{2}A_{di} \\ I_n & -I_n & 0_{n,(m+1)n} & \\ I_n & 0_{n,mn} & -I_n & 0_n \\ 0_{n,mn} & I_n & -I_n & 0_n \\ 0_{n,mn} & \frac{I_n}{2} & \frac{I_n}{2} & \frac{h_{12}}{2}I_n \end{bmatrix}, \\
\bar{W}_U &= \begin{bmatrix} W_U & 0_{2n(m+1),n} \end{bmatrix}, \bar{\Lambda} = \text{diag}\left\{0_{n(m+2)}, -S\right\}, \\
W_U &= \begin{bmatrix} I_{mn} & 0_{mn,2n} \\ 0_{mn,n} & I_{mn} \ \ 0_{mn,n} \\ I_n & 0_{n,(m+1)n} \\ 0_{n,(m+1)n} & I_n \end{bmatrix}, \\
\bar{R} &= \text{diag}\Big\{(\frac{h_1}{m})^2 R + (h_2)^2 Q + (h_{12})^2 W, -R, \\
&\qquad -Q,\ -W,\ -(1-\tau)T\Big\}, \\
\bar{U}_{ilk} &= \text{diag}\left\{U_i,\ -U_l,\ E_i + \Theta,\ -E_k\right\}, \\
\Theta &= \sum_{\rho=1}^{r-1} \beta_\rho (P_\rho - P_r) + T. \qquad (13)
\end{aligned}
$$

Proof: Omitted here for the length limitation of the paper. Interested readers could refer to [17] for detailed derivation. ■

Remark 2: Note that the current proposed theorems deal with the case where the delay rate $\tau < 1$. However, it can be easily extended to fit the $\tau = \infty$ case. To do so, simply set $\tau \geq 1$ in conditions (11)-(12), then the associated matrix T will be automatically optimized to be close to zero matrix, that is, the "T" related items have no effect on the final results of the LMI conditions within the numerical precision of calculations.

For systems which are asymptotically stable under zero time-delay, the simple LKF based stability criterion in Theorem 1 is valid. However, if the system is unstable under zero time-delay, this method is not applicable. The alternative method to test the latter situation is to employ the "Complete LKF" [2]. Besides, it is also possible to enhance the stability or stabilize this kind of systems via a feedback mechanism. The next subsection discusses this issue.

C. Stabilization via H_∞ control

This section is dedicated to the state-feedback controller design problem. In order to obtain a tractable solution of the controller gain matrices, some additional matrix variables are introduced. Then by using the Finsler's Lemma, decoupling between the system matrices and the FWLKF matrices has been achieved. Finsler's Lemma is stated as follows:

Lemma 3 (Finsler's Lemma[1]): Let $\omega \in \mathbb{R}^s$, $\mathcal{Q} \in \mathbb{S}^s$ and $\mathcal{B} \in \mathbb{R}^{q\times s}$, such that $rank(\mathcal{B}) < s$. The following statements are equivalent

1) $\omega' \mathcal{Q}\omega < 0$, $\forall \mathcal{B}\omega = 0$, $\omega \neq 0$
2) $\exists X \in \mathbb{R}^{s\times q} : \mathcal{Q} + X\mathcal{B} + \mathcal{B}' X' < 0$

In the previous control literature, the Finsler's Lemma has been mainly used to eliminate variables in certain matrix inequalities. It is closely related to the S-procedure. In recent literature [10], this lemma has been extended to prove the equivalence between several techniques which are used to achieve the decoupling effect in the context of fuzzy Lyapunov functions. Thus decoupling is the converse application of Finsler's Lemma as opposed to its elimination usage. And we follow the work of [10] to derive the existence condition of the state-feedback controllers with the form of (3).

Consider the closed-loop system (4). Use the two-term approximation method again, yielding the following scaled forward subsystem,

$$
(\mathcal{S}_c): \begin{cases} \dot{x}(t) = \bar{A}(t)x(t) + \frac{1}{2}\overline{A}_{\underline{d}}(t)x(t-h_1) \\ \qquad + \frac{1}{2}\overline{A}_{\bar{d}}(t)x(t-h_2) + \frac{h_{12}}{2}A_d(t)X^{-1}\tilde{\omega}(t), \\ \tilde{z}(t) = X\dot{x}(t), \end{cases}
$$

where $\overline{A}_{\underline{d}}(t) = A_d(t) + B(t)K_{\underline{d}}(t)$ and $\overline{A}_{\bar{d}}(t) = A_d(t) + B(t)K_{\bar{d}}(t)$.

According to the SSG theorem in lemma 1, to stabilize the closed loop system (4), we only need to consider the standard H_∞ control problem of $(\mathcal{S}_c)$ with the performance index $\gamma = 1$. The latter is much easier to deal with, since $(\mathcal{S}_c)$ only includes two constant time delays.

Theorem 2: Consider the scaled subsystem $(\mathcal{S}_c)$ converted from (4) and assume $(\mathcal{A})$ holds. Given an integer $m \geq 1$ and scalars $h_2 > h_1 > 0$, $\delta \neq 0$, the original closed-loop time-delayed T-S fuzzy system (4) using the control law (3) is asymptotically stable if there exist matrices S, P_Y^i, W_Y, E_Y^i, R_Y, $Q_Y \in \mathbb{S}^n$, $\bar{Y} \in \mathbb{R}^{n\times n}$, $U_Y^i \in \mathbb{S}^{mn}$ and general matrices $\bar{K}_i$, $\bar{K}_{\underline{d}}^i$, $\bar{K}_{\bar{d}}^i \in \mathbb{R}^{n_u\times n}$ for $i \in \mathcal{R}$ such that

$$
\begin{aligned}
P_Y^i &> P_Y^r, \quad i = 1, 2 \ldots r-1 \qquad (14) \\
\bar{\psi}_{iilk} &< 0, \quad i,\ l,\ k \in \mathcal{R} \qquad (15) \\
\bar{\psi}_{ijlk} + \bar{\psi}_{jilk} &< 0, \quad i < j, \quad i,\ j,\ l,\ k \in \mathcal{R} \qquad (16)
\end{aligned}
$$

where

$$
\begin{aligned}
\bar{\psi}_{ijlk} &= \begin{bmatrix} \bar{\Psi}_{ijlk} & N^T S \\ * & -S \end{bmatrix}, \\
N &= \left[0_{n,(m+2)n}\ I_n\ 0_n\right], \\
\bar{\Psi}_{ijlk} &= \hat{W}_P^T \tilde{P}_i \hat{W}_P + \hat{W}_R^T \tilde{R} \hat{W}_R + \hat{W}_U^T \tilde{U}_{ilk} \hat{W}_U \\
&\quad + \text{sym}(\bar{C}\bar{D}_{ij}) + \text{diag}\left\{0_{(m+3)n}, -S\right\}, \\
\tilde{U}_{ilk} &= \text{diag}\left\{U_Y^i,\ -U_Y^l,\ E_Y^i + \Theta_Y,\ -E_Y^k\right\}, \\
\tilde{R} &= \text{diag}\Big\{(\frac{h_1}{m})^2 R_Y + h_2^2 Q_Y + h_{12}^2 W_Y, \\
&\quad -R_Y,\ -Q_Y,\ -W_Y,\Big\}, \\
\bar{D}_{ij} &= \Big[\ A_i\bar{Y} + B_i\bar{K}_j \quad 0_{n,(m-1)n} \quad \tfrac{1}{2}(A_{di}\bar{Y} + B_i\bar{K}_{\underline{d}}^j) \\
&\qquad \tfrac{1}{2}(A_{di}\bar{Y} + B_i\bar{K}_{\bar{d}}^j) \quad -\bar{Y} \quad \tfrac{h_{12}}{2}A_{di}\ \Big], \\
\bar{C} &= \Big[I_n \ \overbrace{0_n \ \cdots \ 0_n}^{m+1} \ \delta I_n \ \ 0_n\ \Big] \\
\hat{W}_P &= \begin{bmatrix} 0_{n,(m+2)n} & I_n & 0_n \\ I_n & & 0_{n,(m+3)n} \end{bmatrix}, \\
\tilde{P}_i &= \begin{bmatrix} 0 & P_Y^i \\ P_Y^i & 0 \end{bmatrix},
\end{aligned}
$$

$$
\begin{aligned}
\hat{W}_R &= \begin{bmatrix} 0_{n,(m+2)n} & I_n & 0_n & \\ I_n & -I_n & 0_{n,(m+2)n} & \\ I_n & 0_{n,mn} & -I_n & 0_{n,2n} \\ 0_{n,mn} & I_n & -I_n & 0_{n,2n} \end{bmatrix}, \\
\hat{W}_U &= \begin{bmatrix} W_U & 0_{2n(m+1),2n} \end{bmatrix}, \\
\Theta_Y &= \sum_{\rho=1}^{r-1} \beta_\rho (P_Y^\rho - P_Y^r). \qquad (17)
\end{aligned}
$$

Moreover, if the above conditions have feasible solutions, the controller gain matrices in (3) are given by

$$K_i = \bar{K}_i \bar{Y}^{-1},\ K_{\underline{d}}^i = \bar{K}_{\underline{d}}^i \bar{Y}^{-1},\ K_{\bar{d}}^i = \bar{K}_{\bar{d}}^i \bar{Y}^{-1}.$$

Proof: Omitted here for the length limitation of the paper. Interested readers could refer to [17] for detailed derivation. ■

Remark 3: Make the linear transformation $r_a = \frac{h_1+h_2}{2}, r_d = \frac{h_2-h_1}{2}$, then r_a and r_d represent the center and the radius of the delay range respectively. Given $r_d > 0$ as a sufficiently small scalar and $r_a > r_d$, the proposed theorems can be applied to the constant time delay case. In this situation, two-term approximation technique contributes little to reduce the conservatism, while the delay partitioning technique mainly accounts for the conservatism reduction. Thus, the proposed methods work efficiently not only for time varying delay case but also for constant time delay case.

Remark 4: If the assumption $(\mathcal{A})$ cannot be met, that is , $\dot{\lambda}_i(t)$ is unavailable, the above theorems can also be adapted to be independent of $\dot{\lambda}_i(t)$. One only need to set $P_i = P$ in Theorem 1 or similarly $P_Y^i = P$ in Theorem 2 and at the same time, take the invalidate constraints (10) and (14) off the two theorems respectively.

IV. Simulation Results and Comparisons

In this section, numerical simulations of two examples are presented to illustrate the effectiveness of the proposed methods. And comparisons with existing results in recent publications are also demonstrated.

Example 1: (Stability Analysis) Consider the following fuzzy system with a time-varying delay, which has been used in many papers:

$$\dot{x}(t) = \sum_{i=1}^{2} \lambda_i(t) \left[A_i x(t) + A_{di} x(t-d(t))\right],$$

where A_i and B_i $(i = 1, 2)$ are given as

$$A_1 = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix},\ A_2 = \begin{bmatrix} -1 & 0.5 \\ 0.1 & -1 \end{bmatrix},$$

$$A_{d1} = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix},\ A_{d2} = \begin{bmatrix} -1 & 0 \\ 0.1 & -1 \end{bmatrix}.$$

The calculated maximum allowable upper bound (MAUB) h_2 are tabulated in Table I under different values of lower bound h_1 and β_i.

Applying Theorem 1 with Remark 3, the constant time delay case $(h_1 = h_2)$ has been investigated through using Theorem 1. The last column of Table I shows that the proposed IO approach is less conservative than the previous results even in the large value β_i and non-fractioning ($m = 1$) setting. Moreover, when delay rate upper bounds β_i is small and $m > 1$, significant improvement can be observed.

When considering the fast varying delay case ($\tau = \infty$), the 2nd through 5th columns of Table I list the MAUBs derived from [4], [6], [14], [11] comparing with the method using Theorem 1 with Remark 2 under different values of h_1. It's obvious that our method is superior than the previous ones.

To further illustrate the advantages of the proposed method, Figure 1 draws the stability region in terms of h_1 and h_2 according the data in Table I. The dash line represents the border of $h_1 = h_2$. And the space between different decorated lines and the border are regions where the system is asymptotically stable. Note that the highest line which uses Theorem 1 with $\beta_i = 0.5$ and $m = 3$ apparently outperforms all of the other methods. And another merit of the proposed method lies in that the larger h_1 is, the less conservative the criterion is. This is ascribed to the fractioning of h_1 and is well illustrated by the slopes of the different lines. Note that the slope of the top line obtained using Theorem 1 increases as the lower bound h_1 becoming larger. In contrast, other lines almost keep the same slopes.

Table II compares the results of the slowly varying delay case where $h_2 = 0.4$ and $\tau = 0.1$. It's clear that Theorem 1 in this paper achieves better results than [4], [6], and far better than the fast varying case (2nd column of Table I).

Example 2: (Controller Design) Consider the T-S system in (2) with the following parameters, as in [18]:

$$A_1 = \begin{bmatrix} 0 & 0.6 \\ 0 & 1 \end{bmatrix},\ A_2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix},$$

$$A_{d1} = \begin{bmatrix} 0.5 & 0.9 \\ 0 & 2 \end{bmatrix},\ A_{d2} = \begin{bmatrix} 0.9 & 0 \\ 1 & 1.6 \end{bmatrix}.$$

and $B_1 = B_2 = [1, 1]^T$, where the delay is considered as *time invariant*.

This example is mainly used to compare the performances under different feedback controllers. The results are obtained using Theorem 2 with the delay partitioning number $m = 3$, the tuning parameter $\delta = 1$, and for different values of β_i. To allow this versatility in Theorem 2, simply set $\bar{K}_{\underline{d}}^i = \bar{K}_{\bar{d}}^i = 0$ for the memoryless control or $\bar{K}_i = 0$ for the purely delayed control.

The 2nd row shows the results of memory control where both $x(t-h_1)$ and $x(t-h_2)$ are employed to feedback the open-loop system in addition to the non-delay states $x(t)$. Therefore, the memory control strategy achieves the least conservatism compared to memoryless control and purely delayed control (4th row).

Note that the same example has been investigated in [14], [18] and the reference therein. However, the best result among these papers is 0.8420 of [18], which is conservative than the memoryless control and even more conservative than the memory control used here.

TABLE I

COMPARISONS OF MAUB h_2

THE FAST VARYING DELAY CASE

Method \ h_1		0.4	0.8	1.2	h_2
Lien *et al.* [6]		0.8829	1.0677	1.3181	1.5974
Tien *et al.* [14]		0.883	1.093	1.336	1.5974
Li *et al.* [4]		1.038	1.158	1.359	1.5974
Peng *et al.* [11]		1.1622	1.2808	1.4288	1.5974
Th. 1	$m=1$	1.3428	1.3607	1.4499	1.6183
$\beta_i = 0.5$	$m=3$	1.3802	1.4627	1.6066	2.1563
$\beta_i = 5 \times 10^k$	$m=1$	1.2713	1.2962	1.4215	1.6008
$k = 0, 1$	$m=3$	1.2988	1.3965	1.5603	2.0836

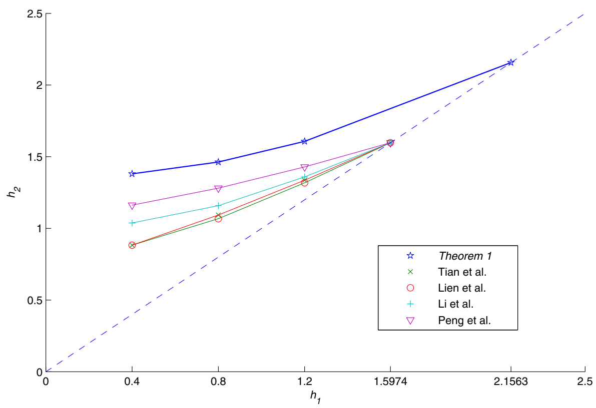

Fig. 1. Different Methods Comparing with Th. 1

V. CONCLUSION

This paper proposes an input-output framework for analysis and synthesis of T-S fuzzy systems with time varying delay. A novel approximation method has been employed to convert the original system into feedback interconnection form. Based on the scaled small gain theorem, new delay-range-dependent stability and stabilization conditions have been derived by studying the bounded real property and H_∞ control problem of the scaled forward subsystem. The given numerical examples demonstrated the advantages and less-conservatism over the existing results.

TABLE II

COMPARISONS OF MAUB h_2

THE SLOWLY VARYING DELAY CASE WITH $\tau = 0.1$

Method		$h_1 = 0.4$
Lien *et al.* [6]		1.4841
Li *et al.* [4]		1.4849
Th. 1	$m=1$	1.5211
$\beta_i = 0.5$	$m=3$	1.5465
$\beta_i = 5 \times 10^k$	$m=1$	1.4943
with $k = 0, 1 \dots 3$	$m=3$	1.5145

TABLE III

MAXIMUM ALLOWABLE CONSTANT TIME DELAY d

Controller	$\beta_i = 0.01$	$\beta_i = 0.1$	$\beta_i = 10^k$, $k = 0, 1$
K, $K_{\underline{d}}$, $K_{\bar{d}}$	1.432	1.400	1.397
K	1.414	1.382	1.366
$K_{\underline{d}}$, $K_{\bar{d}}$	0.371	0.368	0.365

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