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Bounded Real Lemma for Nonhomogeneous Markovian Jump Linear Systems

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Abstract— **In this note, a bounded real lemma is established for discrete-time Markov jump linear systems with nonhomogeneous finite state Markov chain. Different cases, depending on the time variation character of the transition probabilities (TPs), are considered. Namely, arbitrary variation and periodic variation.**

I. INTRODUCTION

The family of systems, referred to as Markovian jump linear systems (MJLS), is very appropriate to model plant whose structure is subject to random abrupt changes. The theory of stability, optimal and robust control, as well as important applications of such systems, can be found in several references in the current literature, for instance in [1], [3], [5], [6] and the reference therein.

Almost all the works in the field of MJLS assume that the Markov process (or chain) is homogeneous, that is the TPs are time-invariant. However, in some applications, this assumption is not verified. A typical example can be found when considering failure prone systems. Usually, in this class of systems, it is often assumed that the components failure rates (or probabilities) are time-independent and independent of system state. In other words, the underlying Markov chain used to model random failures is homogeneous. In reality, however, this assumption is often violated and the failure rate of a component usually depends on many factors, for example, its age, the degree of solicitation of this one, *etc*. Other examples can be found in networked control systems applications [**?**] or flight control applications [10], *etc*.

In this paper, we establish a bounded real lemma (BRL) for discrete-time MJLS with *nonhomogeneous* finite state Markov chain. Different cases, depending on the time variation character of the transition probabilities, are considered. Namely, arbitrary variation (*general case*) and periodic variation. Our main result can be viewed as an extension of the BRL for discrete-time MJLS with homogeneous finite state Markov chain (see [9]) to the nonhomogeneous case.

This paper is organized as follows: Section 2 describes the dynamical model of the system and gives some preliminaries. The main results are given in Section 3. Section 4 concludes this paper.

Notations. The notations used in this paper are quite standard. $\mathbb{R}^{m\times n}$ is the set of m-by-n real matrices. $\mathcal{A}^T$ is the transpose of the matrix $\mathcal{A}$. The notation $\mathcal{X} \geq \mathcal{Y}$ ($\mathcal{X} > \mathcal{Y}$, respectively), where $\mathcal{X}$ and $\mathcal{Y}$ are symmetric matrices, means that $\mathcal{X}-\mathcal{Y}$ is positive semi-definite (positive definite, respectively). $\mathbb{I}$ and $\mathbf{0}$ are identity and zero matrices of appropriate dimensions, respectively. In block matrices, $\star$ indicates symmetric terms: $\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{B}^T & \mathcal{C} \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \star \\ \mathcal{B}^T & \mathcal{C} \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \star & \mathcal{C} \end{bmatrix}$. The expression $\mathcal{M}\mathcal{N}\star$ is equivalent to $\mathcal{M}\mathcal{N}\mathcal{M}^T$. $\lambda_{\min}[\mathcal{P}]$ ($\lambda_{\max}[\mathcal{P}]$) denotes the minimal eigenvalue (maximal eigenvalue) of the matrix $\mathcal{P}$.

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II. DEFINITIONS AND PRELIMINARY RESULTS

A. Dynamical Model

Fix an underlying probability space $(\Omega, \mathcal{F}, P)$ and consider the following discrete-time MJLS

$$\Sigma : \begin{cases} x_{k+1} = \mathcal{A}(\theta_k)x_k + \mathcal{B}(\theta_k)w_k \\ z_k = \mathcal{C}(\theta_k)x_k + \mathcal{D}(\theta_k)w_k \end{cases} \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the system state, $w_k \in \mathbb{R}^m$ is the system external disturbance and z_k is the controlled output. The process $\{\theta_k, k \geq 0\}$ is described by a discrete-time Markov chain with finite state-space $\Xi = \{1, \dots, \sigma\}$ and mode transition probabilities

$$\pi_{ij}(k) = \Pr\{\theta_{k+1} = j \mid \theta_k = i\} \tag{2}$$

with the restrictions $\pi_{ij}(k) \geq 0$ and $\sum_{j=1}^{\sigma} \pi_{ij}(k) = 1$, $\forall k \geqslant 0$. The set Ξ comprises the operation modes of system (1) and for each possible value of $\theta_k = i$, $i \in \Xi$, we denote the matrices associated with the *i-th mode* by $\mathcal{M}_i = \mathcal{M}(\theta_k = i)$. For notation, we define $\Theta_k \triangleq \{\theta_0, \cdots, \theta_k\}$.

$\pi_{ij}(k)$ are the entries of the transition matrix Π_k. If $\Pi_k = \Pi$ for all $k \geq 0$, then the Markov chain is known as an *homogeneous Markov chain*. In the case of time dependent transition probabilities, the Markov chain is known as *nonhomogeneous*. For more details concerning nonhomogeneous Markov chains, one can refer to [8].

In the rest of the paper, we will refer to the case where there is no assumption on the time variation character of the transition matrix Π_k as the *general case*.

To end this section, let us define the space $\ell_2[(\Omega, \mathcal{F}, P)]$ of $\mathcal{F}$-measurable sequences $\{z_k\}_{k=0}^{\infty}$ for which

$$\| z \|_2 \triangleq \left[\sum_{k=0}^{\infty} \underset{\Theta_k}{E} \left[|z_k|^2 \right] \right]^{\frac{1}{2}} < \infty \tag{3}$$

where $\underset{\Theta_k}{E}[\cdot]$ denotes the expectation operator over Θ_k.

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B. Stochastic Stability

The concept of stochastic stability that we deal with in the rest of the paper is presented next.

Definition 1 [6]. The system (1) with $w_k \equiv 0$, is said to be

- Exponentially stable in the mean square sense with conditioning of type I (ESMSCI) if there exist $\beta \geqslant 1,\ q \in (0,1)$ such that for any Markov chain $(\{\theta_k\}_{k\geqslant 0},\ \{\Pi_k\}_{k\geqslant 0},\ \Xi)$ we have

$$E\left[|\Phi(k,k_0)x_{k_0}|^2 \Big| \theta_{k_0}=i\right] \leqslant \beta |x_{k_0}|^2 q^{k-k_0}$$

$$\forall\ k \geqslant k_0 \geqslant 0,\ i \in \Xi_{k_0},\ x_{k_0} \in \mathbb{R}^n$$

where $\Phi(k,k_0)$ is the fundamental random matrix solution of system (1), and $\Xi_{k_0} = \{i \in \Xi | \Pr\{\theta_{k_0}=i\} > 0\}$.

- Exponentially stable in the mean square sense (ESMS), if there exist $\beta \geqslant 1,\ q \in (0,1)$ such that for any Markov chain $(\{\theta_k\}_{k\geqslant 0},\ \{\Pi_k\}_{k\geqslant 0},\ \Xi)$ we have

$$E\left[|\Phi(k,k_0)x_{k_0}|^2\right] \leqslant \beta |x_{k_0}|^2 q^{k-k_0}$$

$$\forall\ k \geqslant k_0 \geqslant 0,\ x_{k_0} \in \mathbb{R}^n$$

Proposition 1. We deduce the following results from [6]:

i) ESMSCI $\Rightarrow$ ESMS

ii) Under the following assumption:

H1) $\delta_k(i) = \Pr\{\theta_k = i\} > 0,\ \forall k \geqslant 0,\ \forall i \in \Xi.$

we have that system Σ is ESMSCI if and only if there exist matrices $\mathcal{P}_i(k) = \mathcal{P}_i^T(k)$ and a positive scalar ξ, such that

$$\mathcal{A}_i^T \tilde{\mathcal{P}}_i(k+1)\mathcal{A}_i - \mathcal{P}_i(k) \leqslant -\xi \mathbb{I}, \quad i \in \Xi,\ \forall k \geqslant 0 \quad (4)$$

where

$$\tilde{\mathcal{P}}_i(k+1) = \sum_{j=1}^{\sigma} \pi_{ij}(k)\mathcal{P}_j(k+1)$$

and

$$0 < \underline{\eta} \leqslant \lambda_{\min}[\mathcal{P}_i(k)] \leqslant \lambda_{\max}[\mathcal{P}_i(k)] \leqslant \overline{\eta} < +\infty$$

$$i \in \Xi,\ \forall k \geqslant 0$$

Remark 1. Note that in ii) above, the assumption $\mathbf{H}_1$ can be stressed to:

H2) The transition probability matrices Π_k are nondegenerate stochastic matrices $\forall k \geqslant 0$.

Note also that $\mathbf{H}_1$ is verified if and only if $\mathbf{H}_2$ is verified and $\delta_0(i) = \Pr\{\theta_0 = i\} > 0,\ \forall i \in \Xi$ (please reefer to [6] for more details).
We will need $\mathbf{H}_1$ to prove our main results (see Lamma A.2).

III. Main Results

Before giving the main results of this section, we first introduce the definition of the $\mathcal{H}_\infty$ norm for discrete-time MJLS.

Definition 2. Assume that system (Σ) is ESMSCI. let $x_0 = 0$ and define the $\mathcal{H}_\infty$ norm, denoted $\parallel \Sigma \parallel_\infty$, as

$$\parallel \Sigma \parallel_\infty := \sup_{\theta_0 \in \Xi} \sup_{w \in \ell_2[(\Omega,\mathcal{F},P)], \|w\|_2 \neq 0} \frac{\parallel z \parallel_2}{\parallel w \parallel_2} \quad (5)$$

To derive the bounded real lemma, we need the following definition of weak controllability of MJLS.

Definition 3. System (Σ) is weakly controllable if for every initial state/mode, (x_0,θ_0), and any final state/mode, (x_f,θ_f), there exists a finite time k_c and an input $w_c(k)$ such that

$$P\left[x(k_c) = x_f \text{ and } \theta(k_c) = \theta_f\right] > 0 \quad (6)$$

A. General case

Theorem 1. Assume that Σ is weakly controllable. Under assumption $\mathbf{H}_1$, Σ is ESMSCI and satisfies $\|\Sigma\|_\infty < \gamma$ if only if there exist matrices $\mathcal{P}_i(k) = \mathcal{P}_i^T(k)$, and a positive scalar ξ, such that

$$\Omega_i^k \leqslant -\xi \mathbb{I}, \quad i \in \Xi,\ \forall k \geqslant 0 \quad (7)$$

where

$$\Omega_i^k = \begin{bmatrix} \mathcal{A}_i & \mathcal{B}_i \\ \mathcal{C}_i & \mathcal{D}_i \end{bmatrix}^T \begin{bmatrix} \tilde{\mathcal{P}}_i(k+1) & \mathbf{0} \\ \mathbf{0} & \mathbb{I} \end{bmatrix} \star - \begin{bmatrix} \mathcal{P}_i(k) & 0 \\ 0 & \gamma^2 \mathbb{I} \end{bmatrix}$$

and: $0 < \underline{\eta} \leqslant \lambda_{\min}[\mathcal{P}_i(k)] \leqslant \lambda_{\max}[\mathcal{P}_i(k)] \leqslant \overline{\eta} < +\infty$, $i \in \Xi,\ \forall k \geqslant 0.$ □

Proof.
Sufficiency. Assume that there exist matrices $\mathcal{P}_i(k) = \mathcal{P}_i^T(k)$, and a positive scalar ξ, such that (7) is verified. Then, it follows from Proposition 1 that Σ is ESMSCI under $\mathbf{H}_1$. Let us now define the following cost functional

$$\mathcal{J}_\tau^\gamma = \sum_{k=0}^{\tau} \underset{\Theta_k}{E}\left[z_k^T z_k - \gamma^2 w_k^T w_k\right] \quad (8)$$

Note that $\parallel \Sigma \parallel_\infty < \gamma$ if and only if $\mathcal{J}_\infty^\gamma < 0$, where $\mathcal{J}_\infty^\gamma$ corresponds to the cost functional (8) for $\tau \longrightarrow \infty$.
Define $\mathcal{V}_{\theta_k}(k,x_k) \triangleq x_k^T \mathcal{P}(k,\theta_k)x_k$, then one has

$$\begin{aligned}\mathcal{J}_\tau^\gamma = &\sum_{k=0}^{\tau} \underset{\Theta_{k+1}}{E}\left[z_k^T z_k - \gamma^2 w_k^T w_k - \mathcal{V}_{\theta_k}(k,x_k)\right.\\ &\left.+\mathcal{V}_{\theta_{k+1}}(k+1,x_{k+1})\right]\\ &+\sum_{k=0}^{\tau} \underset{\Theta_{k+1}}{E}\left[\mathcal{V}_{\theta_k}(k,x_k) - \mathcal{V}_{\theta_{k+1}}(k+1,x_{k+1})\right]\\ =&\sum_{k=0}^{\tau} \underset{\Theta_k}{E}\left[\begin{pmatrix} x_k^T & w_k^T \end{pmatrix} \Omega_{\theta_k}^k \star\right]\\ &- \underset{\Theta_{\tau+1}}{E}\left[\mathcal{V}_{\theta_{\tau+1}}(\tau+1,x_{\tau+1})\right] + \underset{\Theta_0}{E}\left[\mathcal{V}_{\theta_0}(0,x_0)\right] \quad (9)\end{aligned}$$

Given $x_0 = 0$, $\mathcal{V}_{\theta_0}(0, x_0) = 0$ for any initial mode $\theta_0 \in \Xi$, hence

$$\begin{aligned}\mathcal{J}_\tau^\gamma &= \sum_{k=0}^{\tau} \underset{\Theta_k}{E}\left[\begin{pmatrix} x_k^T & w_k^T \end{pmatrix} \Omega_{\theta_k}^k \star\right] \\ &\quad - \underset{\Theta_{\tau+1}}{E}\left[\mathcal{V}_{\theta_{\tau+1}}(\tau+1, x_{\tau+1})\right] \\ &\leqslant \sum_{k=0}^{\tau} \underset{\Theta_k}{E}\left[\begin{pmatrix} x_k^T & w_k^T \end{pmatrix} \Omega_{\theta_k}^k \star\right] \\ &\leqslant -\xi \sum_{k=0}^{\tau} \underset{\Theta_k}{E}\left[|w_k|^2\right] \end{aligned} \tag{10}$$

One knows that $\|w\|_2 \neq 0$. This yield, as $\tau \longrightarrow \infty$, $\mathcal{J}_\infty^\gamma < 0$.

Necessity. First, we will show that if $\|\Sigma\|_\infty < \gamma$ then there exist matrices $\{\mathcal{P}_i(k)\} \geqslant 0$, with $\lambda_{\max}[\mathcal{P}_i(k)] \leqslant \overline{\eta} < +\infty$, that satisfy the following backward generalized Riccati equations (for $i \in \Xi$, $\forall k \geqslant 0$)

$$\begin{aligned}\mathcal{P}_i(k) &= \mathcal{A}_i^T \tilde{\mathcal{P}}_i(k+1)\mathcal{A}_i + \mathcal{C}_i^T\mathcal{C}_i \\ &\quad + \left[\mathcal{B}_i^T \tilde{\mathcal{P}}_i(k+1)\mathcal{A}_i + \mathcal{D}_i^T C_i\right]^T V_i(k+1)^{-1}\star \end{aligned} \tag{11}$$

where $V_i(k+1) = \gamma^2\mathbb{I} - \mathcal{B}_i^T\tilde{\mathcal{P}}_i(k+1)\mathcal{B}_i - \mathcal{D}_i^T\mathcal{D}_i$. We will first prove that (11) is well defined. For every $\tau \geqslant 0$, Let $\{\mathcal{P}_i^\tau(k)\}$ be the solution of (11) with the given final value condition $\{\mathcal{P}_i^\tau(\tau+1)\} = 0$. We will show in a first step that if $\|\Sigma\|_\infty < \gamma$, then there exists $\alpha > 0$ such that, $\forall \tau \geqslant 0$, $V_i^\tau(k+1) \geqslant \alpha\mathbb{I}$, $i \in \Xi$, $0 \leqslant k \leqslant \tau$, where $V_i^\tau(k+1) = \gamma^2\mathbb{I} - \mathcal{B}_i^T\tilde{\mathcal{P}}_i^\tau(k+1)\mathcal{B}_i - \mathcal{D}_i^T\mathcal{D}_i$. This will be shown using contradiction.

Suppose that for each $\epsilon > 0$, there exists $\tau_\epsilon \geqslant 0$ such that $\lambda_{\min}\left[V_{i_\epsilon}^{\tau_\epsilon}(k_\epsilon+1)\right] < \epsilon$, for some $k_\epsilon \in [0, \tau_\epsilon]$ and some $i_\epsilon \in \Xi$. This can hold in three cases, which are resumed in Lemma A.2, Lemma A.3 and Lemma A.4, respectively (see Appendix A). Considering these lemmas, it results by contradiction that if $\|\Sigma\|_\infty < \gamma$ then there exists $\alpha > 0$ such that $V_i^\tau(k+1) \geqslant \alpha\mathbb{I}$, $\forall\tau \geqslant 0$, $i \in \Xi$, $0 \leqslant k \leqslant \tau$. Also, by contradiction, one can show that $\|\Sigma\|_\infty < \gamma$ implies that there exists $\overline{\eta} > 0$ such that $\forall \tau \geqslant 0$, one has $\lambda_{\max}[\mathcal{P}_i^\tau(k)] \leqslant \overline{\eta} < +\infty$, $i \in \Xi$, $0 \leqslant k \leqslant \tau+1$ (Lemma A.5 in Appendix A).

Using induction, one can show that, $\forall \tau \geqslant 0$, $\{\mathcal{P}_i^\tau(k)\} \geqslant 0$, $i \in \Xi$, $0 \leqslant k \leqslant \tau+1$. Let us now define $\mathcal{Y}_i^\tau(k) = \mathcal{P}_i^{\tau+1}(k) - \mathcal{P}_i^\tau(k)$, $i \in \Xi$, $0 \leqslant k \leqslant \tau+1$. Using induction and Lemma A.1, one can also show that $\mathcal{Y}_i^\tau(k) \geqslant 0$, $\forall\tau \geqslant 0$, $i \in \Xi$, $0 \leqslant k \leqslant \tau+1$. That is, $\mathcal{P}_i^\tau(k) \leqslant \mathcal{P}_i^{\tau+1}(k)$, $i \in \Xi$, $0 \leqslant k \leqslant \tau+1$. Hence, the matrix sequences $\{\mathcal{P}_i^\tau(k)\}$ are increasing sequences in τ, thus, uniform boundedness of these sequences (Lemma A.5) implies convergence. Let $\mathcal{P}_i(k) = \lim_{\tau\longrightarrow\infty}\mathcal{P}_i^\tau(k)$. It is clear from the analysis above that the matrices $\{\mathcal{P}_i(k)\} \geqslant 0$ satisfy the generalized Riccati equation (11) with $\lambda_{\max}[\mathcal{P}_i(k)] \leqslant \overline{\eta} < +\infty$, $i \in \Xi$, $\forall k \geqslant 0$.

Now, let $0 < \rho^2 < \gamma^2 - \|\Sigma\|_\infty^2$ and set $\hat{\gamma} = (\gamma^2 - \rho^2)^{\frac{1}{2}}$. We have $\|\Sigma\|_\infty < \hat{\gamma}$. To conclude the proof of necessity, define the perturbed plant Σ_ϵ

$$\Sigma_\epsilon : \begin{cases} x_{k+1} = \mathcal{A}(\theta_k)x_k + \mathcal{B}(\theta_k)w_k \\ z_k^\epsilon = \mathcal{C}^\epsilon(\theta_k)x_k + \mathcal{D}^\epsilon(\theta_k)w_k \end{cases} \tag{12}$$

where $\epsilon = (\rho^2 + \epsilon_0^2)^{\frac{1}{2}}$ and

$$\mathcal{C}^\epsilon(\theta_k) = \begin{bmatrix} \mathcal{C}(\theta_k) \\ \epsilon\mathbb{I} \end{bmatrix}, \quad \mathcal{D}^\epsilon(\theta_k) = \begin{bmatrix} \mathcal{D}(\theta_k) \\ \mathbf{0} \end{bmatrix}.$$

For sufficiently small $\epsilon_0 > 0$ and $\rho > 0$, Σ_ϵ is ESMS and $\|\Sigma_\epsilon\|_\infty < \hat{\gamma}$. Using the same arguments as above, one conclude that there exist matrices $\{\mathcal{P}_i^\epsilon(k)\} \geqslant 0$, with $\lambda_{\max}[\mathcal{P}_i^\epsilon(k)] \leqslant \overline{\eta} < +\infty$, that satisfy the following generalized Riccati equations (for $i \in \Xi$, $\forall k \geqslant 0$)

$$\begin{aligned}\mathcal{P}_i^\epsilon(k) &= \mathcal{A}_i^T\tilde{\mathcal{P}}_i^\epsilon(k+1)\mathcal{A}_i + (\mathcal{C}_i^\epsilon)^T\mathcal{C}_i^\epsilon \\ &\quad + \left[\mathcal{B}_i^T\tilde{\mathcal{P}}_i^\epsilon(k+1)\mathcal{A}_i + (\mathcal{D}_i^\epsilon)^T C_i^\epsilon\right]^T (V_i^\epsilon(k+1))^{-1}\star \end{aligned} \tag{13}$$

where $V_i^\epsilon(k+1) = \hat{\gamma}^2\mathbb{I} - \mathcal{B}_i^T\tilde{\mathcal{P}}_i^\epsilon(k+1)\mathcal{B}_i - (\mathcal{D}_i^\epsilon)^T\mathcal{D}_i^\epsilon$. After multiplying out all the matrices, one obtaines

$$\begin{aligned}&\mathcal{P}_i^\epsilon(k) - \mathcal{A}_i^T\tilde{\mathcal{P}}_i^\epsilon(k+1)\mathcal{A}_i - \mathcal{C}_i^T\mathcal{C}_i \\ &- \rho^2\mathbb{I} - \left[\mathcal{B}_i^T\tilde{\mathcal{P}}_i^\epsilon(k+1)\mathcal{A}_i + \mathcal{D}_i^T C_i\right]^T \left(\tilde{V}_i^\epsilon(k+1)\right)^{-1}\star \\ &= \epsilon_0^2\mathbb{I} > 0 \end{aligned} \tag{14}$$

where $\tilde{V}_i^\epsilon(k+1) = \hat{\gamma}^2\mathbb{I} - \mathcal{B}_i^T\tilde{\mathcal{P}}_i^\epsilon(k+1)\mathcal{B}_i - \mathcal{D}_i^T\mathcal{D}_i$. Clearly, $\mathcal{P}_i^\epsilon(k) \geqslant \epsilon^2\mathbb{I} > 0$, $i \in \Xi$, $\forall k \geqslant 0$. Finally, apply the Schur complement property to show that $\{\mathcal{P}_i^\epsilon(k)\}$ satisfy

$$\begin{bmatrix} \mathcal{A}_i & \mathcal{B}_i \\ \mathcal{C}_i & \mathcal{D}_i \end{bmatrix}^T \begin{bmatrix} \tilde{\mathcal{P}}_i^\epsilon(k+1) & \mathbf{0} \\ \mathbf{0} & \mathbb{I} \end{bmatrix}\star - \begin{bmatrix} \mathcal{P}_i^\epsilon(k) & 0 \\ 0 & \gamma^2\mathbb{I} \end{bmatrix} < -\rho^2\mathbb{I}, \quad i \in \Xi,\ \forall k \geqslant 0$$

Hence, the proof is complete. ■

B. Periodic case

In [7], the authors considered a class of nonhomogeneous MJLS with p-periodic probability transition matrix satisfying $\Pi_{k+p} = \Pi_k$, and proposed testable stability conditions for this class of systems. They pointed out that when p tends to infinity, the periodic representation may be used as an approximation of the general case.
In what follows, we present a bounded real lemma for this class of nonhomogeneous MJLS with p-periodic probability transition matrix.

Before doing this, let us recall the following stability result for this class of systems.

Proposition 2 [6].

i) ESMSCI $\Leftrightarrow$ ESMS

ii) System Σ is ESMS if and only if there exist p-periodic symmetric positive definite matrices $\mathcal{P}_i(k) = \mathcal{P}_i^T(k) > 0,\ \mathcal{P}_i(k+p) = \mathcal{P}_i(k)$, such that

$$\mathcal{A}_i^T \tilde{\mathcal{P}}_i(k+1)\mathcal{A}_i - \mathcal{P}_i(k) < 0, \quad i \in \Xi,\ \forall k \geqslant 0 \quad (15)$$

□

We are now in position to introduce the main result of this section.

Proposition 3. Assume that Σ is weakly controllable. Under assumption $\mathbf{H}_1$, Σ is ESMS and satisfies $\|\Sigma\|_\infty < \gamma$ if only if there exist p-periodic symmetric positive definite matrices $\mathcal{P}_i(k) = \mathcal{P}_i^T(k) > 0,\ \mathcal{P}_i(k+p) = \mathcal{P}_i(k)$, such that

$$\begin{bmatrix} \mathcal{A}_i & \mathcal{B}_i \\ \mathcal{C}_i & \mathcal{D}_i \end{bmatrix}^T \begin{bmatrix} \tilde{\mathcal{P}}_i(k+1) & \mathbf{0} \\ \mathbf{0} & \mathbb{I} \end{bmatrix} \star - \begin{bmatrix} \mathcal{P}_i(k) & 0 \\ 0 & \gamma^2\mathbb{I} \end{bmatrix} < 0 \quad (16)$$

$$i \in \Xi,\ \forall k \geqslant 0$$

□

Proof. The proof follows from Theorem 1 and the periodicity property of the probability transition matrix. ■

An equivalent formulation of condition (16) is given as follows.

Corollary 1. Assume that Σ is weakly controllable. Under assumption $\mathbf{H}_1$, Σ is ESMS and satisfies $\|\Sigma\|_\infty < \gamma$ if only if there exist positive-definite matrices $\mathcal{P}_i^l = (\mathcal{P}_i^l)^T$ such that for $l \in \{0 \cdots p-1\}$

$$\begin{bmatrix} \mathcal{A}_i & \mathcal{B}_i \\ \mathcal{C}_i & \mathcal{D}_i \end{bmatrix}^T \begin{bmatrix} \tilde{\mathcal{P}}_i^{l+1} & \mathbf{0} \\ \mathbf{0} & \mathbb{I} \end{bmatrix} \star - \begin{bmatrix} \mathcal{P}_i^l & 0 \\ 0 & \gamma^2\mathbb{I} \end{bmatrix} < 0, \quad i \in \Xi \quad (17)$$

and

$$\mathcal{P}_i^p = \mathcal{P}_i^0, \quad i \in \Xi \quad (18)$$

where

$$\tilde{\mathcal{P}}_i^{l+1} = \sum_{j=1}^{\sigma} \pi_{ij}(l)\mathcal{P}_j^{l+1}$$

□

Remark 2. Corollary 1 requires the solution of $\sigma \times p$ linear matrix inequalities (LMIs) and σ equality constraints. In practice, however, only the following $\sigma \times p$ LMIs are necessary

$$\begin{bmatrix} \mathcal{A}_i & \mathcal{B}_i \\ \mathcal{C}_i & \mathcal{D}_i \end{bmatrix}^T \begin{bmatrix} \tilde{\mathcal{P}}_i^{l+1} & \mathbf{0} \\ \mathbf{0} & \mathbb{I} \end{bmatrix} \star - \begin{bmatrix} \mathcal{P}_i^l & 0 \\ 0 & \gamma^2\mathbb{I} \end{bmatrix} < 0$$

$$i \in \Xi,\ l \in \{0 \cdots p-2\}$$

and

$$\begin{bmatrix} \mathcal{A}_i & \mathcal{B}_i \\ \mathcal{C}_i & \mathcal{D}_i \end{bmatrix}^T \begin{bmatrix} \tilde{\mathcal{P}}_i^{0} & \mathbf{0} \\ \mathbf{0} & \mathbb{I} \end{bmatrix} \star - \begin{bmatrix} \mathcal{P}_i^{p-1} & 0 \\ 0 & \gamma^2\mathbb{I} \end{bmatrix} < 0$$

$$i \in \Xi$$

where

$$\tilde{\mathcal{P}}_i^{0} = \sum_{j=1}^{\sigma} \pi_{ij}(p-1)\mathcal{P}_j^{0}$$

IV. Conclusion

This note presented a bounded real lemma for some classes of discrete-time Markov jump linear systems with time varying transition probabilities. Different cases, depending on the time variation character of the transition probabilities have been considered. The developed results can be viewed as an extension of the bounded real lemma of MJLS with homogeneous Markov chains to the nonhomogeneous case.

Appendix A

Lemma A.1. Let $\{\mathcal{P}_i^\tau(k)\}$ and $\{\hat{\mathcal{P}}_i^\tau(k)\}$ be solutions of (11), corresponding to γ and $\hat{\gamma}$, with the given final value conditions $\{\mathcal{P}_i^\tau(\tau+1)\} = \{\hat{\mathcal{P}}_i^\tau(\tau+1)\} = 0$, respectively. Define $\Delta\mathcal{P}_i^\tau(k) = \hat{\mathcal{P}}_i^\tau(k) - \mathcal{P}_i^\tau(k)$. Then, $\forall i \in \Xi$ and $0 \leqslant k \leqslant \tau$, we have

$$\begin{aligned} \Delta\mathcal{P}_i^\tau(k) &= \tilde{\mathcal{A}}_i^T(k+1)\Delta\tilde{\mathcal{P}}_i^\tau(k+1)\tilde{\mathcal{A}}_i(k+1) \\ &+ (\gamma^2 - \hat{\gamma}^2)(\mathbb{K}_i^\tau)^T(k+1)\mathbb{K}_i^\tau(k+1) \\ &+ \left[\mathcal{B}_i^T\Delta\tilde{\mathcal{P}}_i^\tau(k+1)\tilde{\mathcal{A}}_i(k+1) + (\gamma^2-\hat{\gamma}^2)\mathbb{K}_i^\tau(k+1)\right]^T \\ &\times (\hat{V}_i^\tau(k+1))^{-1}\star \end{aligned} \quad (19)$$

where

$$\begin{cases} \Delta\tilde{\mathcal{P}}_i^\tau(k+1) = \check{\mathcal{P}}_i^\tau(k+1) - \tilde{\mathcal{P}}_i^\tau(k+1) \\ \check{\mathcal{P}}_i^\tau(k+1) = \sum_{j=1}^{\sigma}\pi_{ij}(k)\hat{\mathcal{P}}_j^\tau(k+1) \\ \mathbb{K}_i^\tau(k+1) = (V_i^\tau(k+1))^{-1}\left[\mathcal{B}_i^T\tilde{\mathcal{P}}_i^\tau(k+1)\mathcal{A}_i + \mathcal{D}_i^T\mathcal{C}_i\right] \\ \hat{V}_i^\tau(k+1) = \hat{\gamma}^2\mathbb{I} - \mathcal{B}_i^T\check{\mathcal{P}}_i^\tau(k+1)\mathcal{B}_i - \mathcal{D}_i^T\mathcal{D}_i \\ \tilde{\mathcal{A}}_i(k+1) = \mathcal{A}_i + \mathcal{B}_i\mathbb{K}_i^\tau(k+1) \end{cases}$$

□

Proof. The proof follows the same algebraic manipulation as in Lemma 3.1 in [4]. ■

Lemma A.2. Let $\{\mathcal{P}_i^\tau(k)\}$ be the solution of (11) with the given final value condition $\{\mathcal{P}_i^\tau(\tau+1)\} = 0$. Assume that there exists $\tau \geqslant 0$ and $0 \leqslant k_0 \leqslant \tau$ such that $\{V_i^\tau(k+1)\} > 0$ for $k_0+1 \leqslant k \leqslant \tau$ and $V_{i_0}^\tau(k_0+1)$ has a negative eigenvalue

$$w_k = \begin{cases} 0, & 0 \leqslant k \leqslant k_0 - 1 \\ \chi\{\theta_{k_0} = i_0\}\nu_{k_0}, & k = k_0 \\ \left(V_{\theta_k}^\tau(k+1)\right)^{-1}\left(\mathcal{B}_{\theta_k}^T\tilde{\mathcal{P}}_{\theta_k}^\tau(k+1)\mathcal{A}_{\theta_k} + \mathcal{D}_{\theta_k}^T\mathcal{C}_{\theta_k}\right)x_k, & k_0+1 \leqslant k \leqslant \tau \\ 0 & \text{else} \end{cases}$$

for some $i_0 \in \Xi$. Then, $\|\Sigma\|_\infty > \gamma$. □

Proof. Let $\lambda_{k_0} < 0$ be the negative eigenvalue of $V_{i_0}^{\tau}(k_0+1)$ and ν_{k_0} the corresponding eigenvector, *i.e.* $V_{i_0}^{\tau}(k_0+1)\nu_{k_0} = \lambda_{k_0}\nu_{k_0}$. Define the disturbance input shown at the bottom of the previous page, where $\chi\{\theta_{k_0} = i_0\} = \begin{cases} 1 & \text{if } \theta_{k_0} = i_0 \\ 0 & \text{else} \end{cases}$.
Note that, applying this disturbance to the system with $x_0 = 0$, one has $x_k = 0$ for $0 \leqslant k \leqslant k_0$. This yields the following

$$
\begin{aligned}
\mathcal{J}_\infty^\gamma &= \sum_{k=0}^{\infty} \underset{\Theta_k}{E}\left[z_k^T z_k - \gamma^2 w_k^T w_k\right] \\
&\geqslant \sum_{k=0}^{\tau} \underset{\Theta_k}{E}\left[z_k^T z_k - \gamma^2 w_k^T w_k\right] \\
&= \underset{\Theta_{k_0}}{E}\left[z_{k_0}^T z_{k_0} - \gamma^2 w_{k_0}^T w_{k_0}\right] + \sum_{k=k_0+1}^{\tau} \underset{\Theta_{k+1}}{E}\Big[\mathcal{V}_{\theta_k}^\tau(k, x_k) \\
&\quad -\mathcal{V}_{\theta_{k+1}}^\tau(k+1, x_{k+1})\Big] \\
&\quad + \sum_{k=k_0+1}^{\tau} \underset{\Theta_{k+1}}{E}\Big[z_k^T z_k - \gamma^2 w_k^T w_k - \mathcal{V}_{\theta_k}^\tau(k, x_k) \\
&\quad +\mathcal{V}_{\theta_{k+1}}^\tau(k+1, x_{k+1})\Big] \\
&= \underset{\Theta_{k_0}}{E}\left[z_{k_0}^T z_{k_0} - \gamma^2 w_{k_0}^T w_{k_0}\right] \\
&\quad + \underset{\Theta_{k_0+1}}{E}\left[\mathcal{V}_{\theta_{k_0+1}}^\tau(k_0+1, x_{k_0+1})\right] \\
&\quad - \underset{\Theta_{\tau+1}}{E}\left[\mathcal{V}_{\theta_{\tau+1}}^\tau(\tau+1, x_{\tau+1})\right] \\
&= \underset{\Theta_{k_0}}{E}\Big[w_{k_0}^T \Big(\mathcal{D}_{\theta_{k_0}}^T \mathcal{D}_{\theta_{k_0}} - \gamma^2 \mathbb{I} \\
&\quad + \mathcal{B}_{\theta_{k_0}}^T \Big(\sum_{j=1}^{\sigma} \pi_{\theta_{k_0} j}(k_0) \mathcal{P}_j^\tau(k_0+1)\Big) \mathcal{B}_{\theta_{k_0}}\Big) w_{k_0}\Big] \\
&= \underset{\Theta_{k_0}}{E}\left[-w_{k_0}^T V_{\theta_{k_0}}^\tau(k_0+1) w_{k_0}\right] \\
&= -\delta_{k_0}(i_0)\lambda_{k_0}|\nu_{k_0}|^2 > 0. \qquad (20)
\end{aligned}
$$

where the last inequality follows from $\mathbf{H}_1$. Hence $\|\Sigma\|_\infty > \gamma$. ■

Lemma A.3. Let $\{\mathcal{P}_i^\tau(k)\}$ be the solution of (11) with the given final value condition $\{\mathcal{P}_i^\tau(\tau+1)\} = 0$. Assume that there exists $\tau \geqslant 0$ and $0 \leqslant k_0 \leqslant \tau$ such that $\{V_i^\tau(k+1)\} > 0$ for $k_0+1 \leqslant k \leqslant \tau$ and $V_{i_0}^\tau(k_0+1)$ has a zero eigenvalue for some $i_0 \in \Xi$. Then, $\|\Sigma\|_\infty \geqslant \gamma$. □

Proof. Let $\{\mathcal{P}_i^\tau(k)\}$ be the solution of (11) corresponding to γ. By assumption $\{V_i^\tau(k+1)\} > 0$ for $k_0+1 \leqslant k \leqslant \tau$ and $V_{i_0}^\tau(k_0+1)$ has a zero eigenvalue for some $i_0 \in \Xi$. Given $\epsilon > 0$, let $\{\hat{\mathcal{P}}_i^\tau(k)\}$ be a second solution of (11), corresponding to $(\gamma - \epsilon)$, with given final condition $\{\hat{\mathcal{P}}_i^\tau(\tau+1)\} = 0$. If $\epsilon > 0$ is sufficiently small, then $\{\hat{V}_i^\tau(k+1)\} > 0$ for $k_0+1 \leqslant k \leqslant \tau$. Now, using induction and Lemma A.1, one can show that $\Delta\mathcal{P}_i^\tau(k) \geqslant 0$ for $k_0+1 \leqslant k \leqslant \tau+1$. It follows that $\hat{V}_{i_0}^\tau(k_0+1) < V_{i_0}^\tau(k_0+1)$ and hence $\hat{V}_{i_0}^\tau(k_0+1)$ has a negative eigenvalue. By Lemma A.2, we have $\|\Sigma\|_\infty > (\gamma - \epsilon)$. Since $\epsilon > 0$ can be sufficiently small, then $\|\Sigma\|_\infty \geqslant \gamma$. ■

Lemma A.4. Let $\{\mathcal{P}_i^\tau(k)\}$ be the solution of (11) with the given final value condition $\{\mathcal{P}_i^\tau(\tau+1)\} = 0$. Assume that $\forall\tau \geqslant 0$, $\{V_i^\tau(k+1)\} > 0$ for $0 \leqslant k \leqslant \tau$. Suppose that for some $i_0 \in \Xi$, $\forall\epsilon > 0$, there exists $\tau_\epsilon \geqslant 0$ such that $0 < \lambda_{\min}\left[V_{i_0}^{\tau_\epsilon}(k_\epsilon+1)\right] < \epsilon$, for some $k_\epsilon \in [0,\ \tau_\epsilon]$. Then $\|\Sigma\|_\infty \geqslant \gamma$. □

Proof. It follows the same arguments as for the proof of Lemma A.3. ■

Lemma A.5. Assume that Σ is weakly controllable. Let $\{\mathcal{P}_i^\tau(k)\}$ be the solution of (11) with given final value condition $\{\mathcal{P}_i^\tau(\tau+1)\} = 0$, and assume that, $\forall\tau \geqslant 0$, $\{V_i^\tau(k+1)\} > 0$, for $0 \leqslant k \leqslant \tau$. Suppose that for some $\theta^* \in \Xi$, $\forall\epsilon > 0$, there exists $\tau_\epsilon \geqslant 0$ such that $\lambda_{\max}\left[\mathcal{P}_{\theta^*}^{\tau_\epsilon}(k_\epsilon)\right] > \epsilon$ for some $k_\epsilon \in [0,\ \tau_\epsilon]$. Then $\|\Sigma\|_\infty > \gamma$. □

$$
w_k^{j_l} = \begin{cases} w_k^c, & 0 \leqslant k \leqslant k_c - 1 \\ w_k^l, & \text{if } (x_{k_c}, \theta_{k_c}) = (\nu^*, \theta^*),\ k_c \leqslant k \leqslant k_c + \tau_{j_l} - k_{j_l} \\ 0 & \text{else} \end{cases}
$$

where $w_k^l = \left(V_{\theta_k}^{\tau_{j_l}}(k + k_{j_l} + 1 - k_c)\right)^{-1} \left(\mathcal{B}_{\theta_k}^T \tilde{\mathcal{P}}_{\theta_k}^{\tau_{j_l}}(k + k_{j_l} + 1 - k_c)\mathcal{A}_{\theta_k} + \mathcal{D}_{\theta_k}^T \mathcal{C}_{\theta_k}\right) x_k$

Proof. By assumption, one knows that $\exists\theta^* \in \Xi$ and sequences $\{\tau_j\}_{j=0}^{\infty}$, $\{k_j\}_{j=0}^{\infty}$ ($k_j \in [0, \tau_j]$) such that $\lambda_{\max}\left[\mathcal{P}_{\theta^*}^{\tau_j}(k_j)\right] \longrightarrow \infty$ as $j \longrightarrow \infty$. Let ν_j be the eigenvector associated to $\lambda_{\max}\left[\mathcal{P}_{\theta^*}^{\tau_j}(k_j)\right]$ normalized to $|\nu_j| = 1$. Then, there exists ν^* and a subsequence j_l such that $\lim\limits_{j_l \longrightarrow \infty} \nu_{j_l} = \nu^*$.

Applying the weak controllability assumption, one knows that given x_0 and any initial mode θ_0, there exists a finite time k_c and an input w_k^c such that

$$P\left[x(k_c) = \nu^* \text{ and } \theta(k_c) = \theta^*\right] = \mu > 0 \tag{21}$$

Define the disturbance input given at the bottom of the previous page. Applying this disturbance, one gets

$$\begin{aligned}
&\mathcal{J}_\infty^\gamma \geqslant \sum_{k=0}^{k_c+\tau_{j_l}-k_{j_l}} \underset{\Theta_k}{E}\left[z_k^T z_k - \gamma^2 \left(w_k^{j_l}\right)^T w_k^{j_l}\right] \\
&= \sum_{k=0}^{k_c-1} \underset{\Theta_k}{E}\left[z_k^T z_k - \gamma^2 (w_k^c)^T w_k^c\right] \\
&+ \sum_{k=k_c}^{k_c+\tau_{j_l}-k_{j_l}} \underset{\Theta_{k+1}}{E}\Big[\mathcal{V}_{\theta_k}^{\tau_{j_l}}(k + k_{j_l} - k_c, x_k) \\
&-\mathcal{V}_{\theta_{k+1}}^{\tau_{j_l}}(k + k_{j_l} + 1 - k_c, x_{k+1})\Big] \\
&+ \sum_{k=k_c}^{k_c+\tau_{j_l}-k_{j_l}} \underset{\Theta_{k+1}}{E}\Big[z_k^T z_k - \gamma^2 \left(w_k^{j_l}\right)^T w_k^{j_l} \\
&-\mathcal{V}_{\theta_k}^{\tau_{j_l}}(k + k_{j_l} - k_c, x_k) + \mathcal{V}_{\theta_{k+1}}^{\tau_{j_l}}(k + k_{j_l} + 1 - k_c, x_{k+1})\Big] \\
&= \sum_{k=0}^{k_c-1} \underset{\Theta_k}{E}\left[z_k^T z_k - \gamma^2 (w_k^c)^T w_k^c\right] + \underset{\Theta_{k_c}}{E}\left[\mathcal{V}_{\theta_{k_c}}^{\tau_{j_l}}(k_{j_l}, x_{k_c})\right] \\
&= \sum_{k=0}^{k_c-1} \underset{\Theta_k}{E}\left[z_k^T z_k - \gamma^2 (w_k^c)^T w_k^c\right] + \mu(\nu^*)^T \mathcal{P}_{\theta^*}^{\tau_{j_l}}(k_{j_l})\nu^*
\end{aligned} \tag{22}$$

The summation term above is a fixed cost $\forall j_l$ while, by assumption, the second term can be made arbitrary large (as $j_l \longrightarrow \infty$). Then, there exists j_l such that $\mathcal{J}_\infty^\gamma > 0$. Hence $\|\Sigma\|_\infty > \gamma$. ■

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