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# Asymptotic Stability of Piecewise Affine Systems with Sampled-data Piecewise Linear Controllers

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*Abstract*— **This paper addresses stability analysis of closed-loop sampled-data piecewise affine (PWA) systems. In particular, we study the case in which a PWA plant is in feedback with a sampled-data piecewise linear (PWL) controller. We consider the sampled-data system as a continuous-time system with a variable time delay. The contributions of this work are threefold. First, we present a modified Lyapunov-Krasovskii functional (LKF) for studying PWA systems with time delay. Second, based on the new LKF, sufficient conditions are provided for asymptotic stability of PWA systems in feedback with sampled-data PWL controllers. Finally, following the time-delay approach, we formulate the problem of finding a lower bound on the maximum delay that preserves asymptotic stability to the origin as an optimization program in terms of LMIs. The new results are successfully applied to a unicycle example.**

## I. Introduction

PWA systems are considered as a framework for modeling and approximating nonlinearities that arise in physical systems [1]. Stability analysis of continuous-time PWA systems is addressed in [2], [3], [4], [5] using Lyapunov-based methods. Designing continuous-time controllers for PWA systems has received increasing interest since the late nineties [3], [4], [6], [7]. However, the resulting continuous-time controllers must be emulated as a discrete-time controller to be implementable in microprocessors. We refer the reader to [8] for more discussion on designing nonlinear controllers via the emulation method. While sampled-data control of linear systems is a well-studied subject [9], its extension to PWA systems has not received many research contributions. The term "sampled-data PWA system" was probably used for the first time in [10], [11], although the system described there does not possess the typical structure of a continuous-time plant being controlled by a discrete-time controller. In the problem addressed by [10], [11] the controller is implemented in continuous-time and the switching events are the ones controlled by the system logic inside a computer. In other words, it is assumed that the designer has command over the switching times of the system.

By contrast, [12] addresses the classical structure of a sampled-data system in which a continuous-time system is controlled in discrete-time inside a computer. Assuming constant sampling rate, the author studies the stability of sampled-data PWA systems using a quadratic Lyapunov function. The paper provides a set of linear matrix inequalities (LMIs) as sufficient conditions for exponential convergence of the sampled-data system to an invariant set containing the origin.

In sampled-data systems, the discrete-time controller can also be modeled as a continuous-time controller with time varying delay. The time-delay representation has been implemented in nonlinear and linear sampled-data systems using Razumikhin-type theorems [13], and Lyapunov-Krasovskii functionals (LKFs) [14]. Following the time-delay approach, reference [15] studies the stability of sampled-data PWA systems with variable sampling rate. The paper uses a LKF to prove that if a set of LMIs are satisfied, the trajectories of the sampled-data system converge to an invariant set containing the origin.

In contrast to previous work, we address asymptotic stability to the origin rather than stability to an invariant set for sampled-data PWA systems when the feedback controller is PWL. To the best of our knowledge, asymptotic stability of sampled-data PWA systems was not proved before. We study a continuous-time PWA plant in feedback with a PWL controller that appears between a sampler, with variable sampling rate, and a zero-order-hold. The contributions of this work are threefold. First, we present a modified LKF for studying PWA systems with time delay. Second, based on the new LKF, sufficient conditions are provided for asymptotic stability of PWA systems in feedback with sampled-data PWL controllers. Finally, following the time-delay approach, we formulate the problem of finding a lower bound on the maximum delay that preserves asymptotic stability to the origin as an optimization program in terms of LMIs.

The paper is organized as follows. Section II presents basic information about sampled-data PWA systems. In Section III, first a modified LKF is introduced. Next, we present a theorem that provides sufficient conditions for asymptotic stability of PWA systems with sampled-data PWL controllers. Furthermore, we formulate the problem of finding a lower bound on the maximum delay that preserves asymptotic stability to the origin as an optimization program in terms of LMIs. Finally, the new results are applied to a unicycle example in Section IV.

## II. Preliminaries

Consider the PWA system

$$\dot{x}(t) = A_i x(t) + a_i + Bu(t), \text{ for } x(t) \in \mathcal{R}_i \text{ and } i \in I, \quad (1)$$

where $x \in \mathbb{R}^{n_x}$ denotes the state vector, $A_i \in \mathbb{R}^{n_x \times n_x}$, $a_i \in \mathbb{R}^{n_x}$, $B \in \mathbb{R}^{n_x \times n_u}$, $u \in \mathbb{R}^{n_u}$ is the control input, and $I = \{1, ..., M\}$ is the set of indices of the regions $\mathcal{R}_i$ that partition the state space. Each region $\mathcal{R}_i$ is defined as the intersection of $p_i$ open half spaces in $\mathbb{R}^{n_x}$, i.e.

$$\mathcal{R}_i = \{x | E_i x + e_i \succ 0\}, \tag{2}$$

with $E_i \in \mathbb{R}^{p_i \times n_x}$, $e_i \in \mathbb{R}^{p_i}$, and $\succ$ represents an elementwise inequality. Each polytopic region $\mathcal{R}_i$ can be outer approximated by a (possibly degenerate) quadratic curve as

$$\mathcal{R}_i \subseteq \epsilon_i = \{x | \overline{x}^T \overline{E}_i^T \Lambda_i \overline{E}_i \overline{x} > 0\}, \tag{3}$$

where $\Lambda_i \in \mathbb{R}^{p_i \times p_i}$ is a matrix with non-negative entries and

$$\overline{x} = \begin{bmatrix} x \\ 1 \end{bmatrix}, \; \overline{E}_i = \begin{bmatrix} E_i & e_i \end{bmatrix}. \tag{4}$$

Let a PWL controller for (1) be defined by

$$u(t) = K_i x(t), \text{ for } x(t) \in \mathcal{R}_i,$$

where $K_i \in \mathbb{R}^{n_u \times n_x}$. Furthermore, let $I^* = \{i | 0 \in \overline{\mathcal{R}}_i\}$, where $\overline{\mathcal{R}}_i$ denotes the closure of $\mathcal{R}_i$.

*Assumption 1:* The vector field of the open-loop system (1) for $u(t) = 0$ is continuous across the boundaries of any neighboring regions.

*Assumption 2:* The open-loop system is linear in the regions that contain the origin in their closure, i.e. $a_i = 0$ for $i \in I^*$.

*Assumption 3:* The measurements for computing the control input are taken in a sample-and-hold fashion. Therefore, the control input can be rewritten as

$$u(t) = K_j x_{t_n}, \text{ for } t \in [t_n, t_{n+1}), \; x_{t_n} \in \mathcal{R}_j, \text{ and } j \in I,$$

where $t_n$ and $t_{n+1}$, $n \in \mathbb{N}$, are two consecutive sampling times, and $x_{t_n} = x(t_n)$.

We denote the time elapsed since the last sampling instant by

$$\rho(t) = t - t_n, \text{ for } t \in [t_n, t_{n+1}), \tag{5}$$

and the shortest and longest intervals between two consecutive sampling times by $\epsilon > 0$ and $\tau > 0$, respectively, i.e,

$$\epsilon = \inf_{n \in \mathbb{N}} (t_{n+1} - t_n), \; \tau = \sup_{n \in \mathbb{N}} (t_{n+1} - t_n). \tag{6}$$

Assuming $x(t) \in \mathcal{R}_i$ and $x_{t_n} \in \mathcal{R}_j$, we can rewrite (1) as

$$\dot{x}(t) = A_i x(t) + a_i + B K_j x_{t_n} \tag{7a}$$
$$= A_i x(t) + a_i + B K_i x_{t_n} + B w(t), \tag{7b}$$

where $w \in \mathbb{R}^{n_u}$ is a piecewise constant vector defined by

$$w(t) = (K_j - K_i) x_{t_n}, \text{ for } x(t) \in \mathcal{R}_i \text{ and } x_{t_n} \in \mathcal{R}_j. \tag{8}$$

The vector $w$ represents the input for the closed-loop system due to the fact that $x(t)$ and $x_{t_n}$ might be in different regions.

We denote the $m \times m$ identity matrix by $I_m$ and define a non-negative scalar $\Delta K$ as

$$\Delta K = \max_{i,j \in I} ||K_j - K_i||. \tag{9}$$

## III. Main Results

Let $V : \mathbb{R}^{n_x} \times \mathcal{C}([t-\tau, t], \mathbb{R}^{n_x}) \times [0, \tau] \to \mathbb{R}^+$ be an LKF where $\mathcal{C}([t-\tau, t], \mathbb{R}^{n_x})$ is the Banach space of absolutely continuous functions mapping the interval $[t-\tau, t]$ to $\mathbb{R}^{n_x}$ and let $x_t \in \mathcal{C}([t-\tau, t], \mathbb{R}^{n_x})$ be defined as $x(t+r)$ for $-\tau \le r \le 0$ (see [16], Section 2.1). We define

$$V(x, x_t, \rho) = V_1(x) + V_2(x_t) + V_3(x_t, \rho), \tag{10}$$

with

$$\begin{aligned} V_1(x) =& x^T(t) P x(t), \\ V_2(x_t) =& \int_{-\tau}^{0} \int_{t+r}^{t} (\dot{x}(s) - B K_j x_{t_n})^T \\ & \qquad R (\dot{x}(s) - B K_j x_{t_n}) \, \mathrm{d}s \, \mathrm{d}r, \\ V_3(x_t, \rho) =& (\tau - \rho)(x(t) - x_{t_n})^T X (x(t) - x_{t_n}), \end{aligned}$$

where $P$, $R$, and $X$ are symmetric positive definite matrices in $\mathbb{R}^{n_x \times n_x}$, $t_n \le t$ is the most recent sampling instant, and $x_{t_n} \in \mathcal{R}_j$.

Note that the second component of the LKF introduced in (10) is different from its corresponding term in previously studied LKFs such as [14], [15]. By subtracting $BK_j x_{t_n}$ from $\dot{x}$ in the definition of $V_2$, we omit an unfavorable positive definite term involving $w^T w$ from $\dot{V}$. This modification considerably improves the stability results as shown in Section IV. The following theorem provides a set of sufficient conditions for which the trajectories of a PWA system in feedback with an emulated PWL controller asymptotically converge to the origin.

*Theorem 1:* Consider the closed-loop sampled-data PWA system defined in (7) and (8) with sampling intervals smaller than $\tau$. The system is asymptotically stable to the origin if there exist symmetric positive definite matrices $P$, $R$, and $X$, symmetric matrices $\Lambda_i$, $i \notin I^*$, with non-negative entries, matrices $N_i$, $i \notin I^*$, and $\overline{N}_i$, $i \in I^*$, with appropriate dimensions, and positive scalars $\gamma$, $0 < \theta < 1$, and $\eta$, satisfying

$$\Delta K^2 \gamma < \theta \tag{11}$$

- for all $i \notin I^*$

$$\Omega_i + \tau(M_{1i} + M_{2i}) + S_i < 0 \tag{12}$$

$$\begin{bmatrix} \Omega_i + \tau(M_{2i} + M_{3i}) + S_i & \tau N_i \\ \tau N_i^T & -\tau R \end{bmatrix} < 0 \tag{13}$$

- for all $i \in I^*$

$$\overline{\Omega}_i + \tau(\overline{M}_{1i} + \overline{M}_{2i}) < 0 \tag{14}$$

$$\begin{bmatrix} \overline{\Omega}_i + \tau(\overline{M}_{2i} + \overline{M}_{3i}) & \tau \overline{N}_i \\ \tau \overline{N}_i^T & -\tau R \end{bmatrix} < 0 \tag{15}$$

where

$$\Omega_i = \begin{bmatrix} \Psi_i & \begin{bmatrix} P \\ 0 \end{bmatrix} \begin{bmatrix} B & a_i \end{bmatrix} \\ \begin{bmatrix} B^T \\ a_i^T \end{bmatrix} \begin{bmatrix} P & 0 \end{bmatrix} & \begin{bmatrix} -\gamma I_{n_u} & 0 \\ 0 & \eta \end{bmatrix} \end{bmatrix}$$

$$
-\begin{bmatrix} I_{n_x} \\ -I_{n_x} \\ 0 \\ 0 \end{bmatrix} N_i^T - N_i \begin{bmatrix} I_{n_x} & -I_{n_x} & 0 & 0 \end{bmatrix},
$$

$$
\Psi_i = \begin{bmatrix} A_i^T \\ K_i^T B^T \end{bmatrix} \begin{bmatrix} P & 0 \end{bmatrix} + \begin{bmatrix} P \\ 0 \end{bmatrix} \begin{bmatrix} A_i & BK_i \end{bmatrix}
- \begin{bmatrix} I_{n_x} \\ -I_{n_x} \end{bmatrix} X \begin{bmatrix} I_{n_x} & -I_{n_x} \end{bmatrix} + \begin{bmatrix} \eta I_{n_x} & 0 \\ 0 & I_{n_x} \end{bmatrix},
$$

$$
M_{1i} = \begin{bmatrix} A_i^T \\ K_i^T B^T \\ B^T \\ a_i^T \end{bmatrix} X \begin{bmatrix} I_{n_x} & -I_{n_x} & 0 & 0 \end{bmatrix}
+ \begin{bmatrix} I_{n_x} \\ -I_{n_x} \\ 0 \\ 0 \end{bmatrix} X \begin{bmatrix} A_i & BK_i & B & a_i \end{bmatrix},
$$

$$
M_{2i} = \begin{bmatrix} A_i^T \\ 0 \\ 0 \\ a_i^T \end{bmatrix} R \begin{bmatrix} A_i & 0 & 0 & a_i \end{bmatrix},
$$

$$
M_{3i} = \begin{bmatrix} 0 \\ K_i^T B^T \\ B^T \\ 0 \end{bmatrix} N_i^T + N_i \begin{bmatrix} 0 & BK_i & B & 0 \end{bmatrix},
$$

$$
S_i = \begin{bmatrix} E_i^T \\ 0 \\ 0 \\ e_i^T \end{bmatrix} \Lambda_i \begin{bmatrix} E_i & 0 & 0 & e_i \end{bmatrix},
$$

$$
\overline{\Omega}_i = \begin{bmatrix} \Psi_i & \begin{bmatrix} P \\ 0 \end{bmatrix} B \\ B^T \begin{bmatrix} P & 0 \end{bmatrix} & -\gamma I_{n_u} \end{bmatrix} - \begin{bmatrix} I_{n_x} \\ -I_{n_x} \\ 0 \end{bmatrix} \overline{N}_i^T
- \overline{N}_i \begin{bmatrix} I_{n_x} & -I_{n_x} & 0 \end{bmatrix},
$$

$$
\overline{M}_{1i} = \begin{bmatrix} A_i^T \\ K_i^T B^T \\ B^T \end{bmatrix} X \begin{bmatrix} I_{n_x} & -I_{n_x} & 0 \end{bmatrix}
+ \begin{bmatrix} I_{n_x} \\ -I_{n_x} \\ 0 \end{bmatrix} X \begin{bmatrix} A_i & BK_i & B \end{bmatrix},
$$

$$
\overline{M}_{2i} = \begin{bmatrix} A_i^T \\ 0 \\ 0 \end{bmatrix} R \begin{bmatrix} A_i & 0 & 0 \end{bmatrix},
$$

$$
\overline{M}_{3i} = \begin{bmatrix} 0 \\ K_i^T B^T \\ B^T \end{bmatrix} \overline{N}_i^T + \overline{N}_i \begin{bmatrix} 0 & BK_i & B \end{bmatrix}.
$$

*Proof:* Similar to the approach in [15], it can be shown that the LKF (10) is positive definite and decrescent. The first two components, $V_1$ and $V_2$, are continuous functions. The last component, $V_3$, is equal to zero at the sampling instants ($x(t)|_{t=t_n} = x_{t_n}$) and greater than zero at other times. Therefore, the LKF is non-increasing at the sampling times. To prove asymptotic stability of the trajectories to the origin, it suffices to show that inequalities (11)-(15) are sufficient conditions for $V$ to be strictly decreasing between any two consecutive sampling times.

The time derivative of $V$ for $t \in [t_n, t_{n+1})$ is composed of three terms computed as follows. First, the time derivative of $V_1$ is

$$
\dot{V}_1 = \dot{x}^T P x + x^T P \dot{x} \tag{16}
$$

Second, applying the Leibniz integral rule to $V_2$ yields

$$
\begin{aligned}
\dot{V}_2 =& \int_{-\tau}^{0} \left(\dot{x} - BK_j x_{t_n}\right)^T R \left(\dot{x} - BK_j x_{t_n}\right) \mathrm{d}r \\
&- \int_{-\tau}^{0} \left(\dot{x}(t+r) - BK_j x_{t_n}\right)^T \\
&\qquad R \left(\dot{x}(t+r) - BK_j x_{t_n}\right) \mathrm{d}r \\
=& \tau \left(\dot{x} - BK_j x_{t_n}\right)^T R \left(\dot{x} - BK_j x_{t_n}\right) \\
&- \int_{-\tau}^{0} \left(\dot{x}(t+r) - BK_j x_{t_n}\right)^T \\
&\qquad R \left(\dot{x}(t+r) - BK_j x_{t_n}\right) \mathrm{d}r.
\end{aligned}
$$

According to (5) and (6), we have $\rho < \tau$. Therefore,

$$
\begin{aligned}
\dot{V}_2 \leq& \tau \left(\dot{x} - BK_j x_{t_n}\right)^T R \left(\dot{x} - BK_j x_{t_n}\right) \\
&- \int_{-\rho}^{0} \left(\dot{x}(t+r) - BK_j x_{t_n}\right)^T \\
&\qquad R \left(\dot{x}(t+r) - BK_j x_{t_n}\right) \mathrm{d}r \\
=& \tau \left(\dot{x} - BK_j x_{t_n}\right)^T R \left(\dot{x} - BK_j x_{t_n}\right) \\
&- \int_{t-\rho}^{t} \left(\dot{x}(v) - BK_j x_{t_n}\right)^T R \left(\dot{x}(v) - BK_j x_{t_n}\right) \mathrm{d}v.
\end{aligned} \tag{17}
$$

Since $R$ is positive definite, for any arbitrary time varying vector $h_i(t) \in \mathbb{R}^{n_x}$ we can write

$$
\begin{bmatrix} \left(\dot{x}(v) - BK_j x_{t_n}\right)^T & h_i^T \end{bmatrix}
\begin{bmatrix} R & -I_{n_x} \\ -I_{n_x} & R^{-1} \end{bmatrix} \begin{bmatrix} \dot{x}(v) - BK_j x_{t_n} \\ h_i \end{bmatrix} \geq 0.
$$

Therefore,

$$
\begin{aligned}
&- \left(\dot{x}(v) - BK_j x_{t_n}\right)^T R \left(\dot{x}(v) - BK_j x_{t_n}\right) \\
&\quad \leq h_i^T R^{-1} h_i - \left(\dot{x}(v) - BK_j x_{t_n}\right)^T h_i \\
&\quad - h_i^T \left(\dot{x}(v) - BK_j x_{t_n}\right).
\end{aligned}
$$

Integrating both sides from $t - \rho$ to $t$, we have

$$
\begin{aligned}
&- \int_{t-\rho}^{t} \left(\dot{x}(v) - BK_j x_{t_n}\right)^T R \left(\dot{x}(v) - BK_j x_{t_n}\right) \mathrm{d}v \\
&\quad \leq \rho h_i^T R^{-1} h_i - \left(x - x_{t_n} - \rho BK_j x_{t_n}\right)^T h_i \\
&\quad - h_i^T \left(x - x_{t_n} - \rho BK_j x_{t_n}\right).
\end{aligned} \tag{18}
$$

Here, we used the facts that for $v \in [t-\rho, t]$, $K_j x_{t_n}$ is constant and $\dot{x}(v)$ is continuous, and $t - \rho = t_n$. Replacing (18) in (17), we have

$$
\begin{aligned}
\dot{V}_2 \leq& \tau \left(\dot{x} - BK_j x_{t_n}\right)^T R \left(\dot{x} - BK_j x_{t_n}\right) \\
&+ \rho h_i^T R^{-1} h_i - \left(x - x_{t_n} - \rho BK_j x_{t_n}\right)^T h_i \\
&- h_i^T \left(x - x_{t_n} - \rho BK_j x_{t_n}\right).
\end{aligned} \tag{19}
$$

Now, we use (8) to replace $K_j x_{t_n}$ by $K_i x_{t_n} + w$ in the last two components of (19) and reach

$$\begin{aligned}\dot{V}_2 \leq& \tau\left(\dot{x} - BK_j x_{t_n}\right)^T R\left(\dot{x} - BK_j x_{t_n}\right) + \rho h_i^T R^{-1} h_i \\ &- \left(x - x_{t_n} - \rho B(K_i x_{t_n} + w)\right)^T h_i \\ &- h_i^T\left(x - x_{t_n} - \rho B(K_i x_{t_n} + w)\right). \end{aligned} \tag{20}$$

From (5) we have $\dot{\rho} = 1$. Hence, the time derivative of $V_3$ is computed as

$$\begin{aligned}\dot{V}_3 =& -\left(x - x_{t_n}\right)^T X(x - x_{t_n}) + (\tau - \rho)\left(\dot{x}^T X(x - x_{t_n})\right) \\ &+ (\tau - \rho)\left((x - x_{t_n})^T X\dot{x}\right). \end{aligned} \tag{21}$$

Since $\dot{V} = \dot{V}_1 + \dot{V}_2 + \dot{V}_3$, adding (16), (20), and (21) yields

$$\begin{aligned}\dot{V} \leq& \dot{x}^T P x + x^T P\dot{x} + \tau\left(\dot{x} - BK_j x_{t_n}\right)^T R\left(\dot{x} - BK_j x_{t_n}\right) \\ &+ \rho h_i^T R^{-1} h_i - \left(x - x_{t_n} - \rho B(K_i x_{t_n} + w)\right)^T h_i \\ &- h_i^T\left(x - x_{t_n} - \rho B(K_i x_{t_n} + w)\right) \\ &- \left(x - x_{t_n}\right)^T X(x - x_{t_n}) + (\tau - \rho)\left(\dot{x}^T X(x - x_{t_n})\right) \\ &+ (\tau - \rho)\left((x - x_{t_n})^T X\dot{x}\right). \end{aligned} \tag{22}$$

We divide the state space into two parts:

1) $x(t) \in \mathcal{R}_i$ and $i \notin I^*$,
2) $x(t) \in \mathcal{R}_i$ and $i \in I^*$.

In the rest of the proof, we study $\dot{V}$ for $t \in [t_n, t_{n+1})$ in each part of the state space.

• *Part 1:* For $x(t) \in \mathcal{R}_i$ and $i \notin I^*$, based on (7), we have

$$\dot{x}(t) = \begin{bmatrix} A_i & BK_i & B & a_i \end{bmatrix} \zeta(t), \tag{23}$$

and

$$\dot{x}(t) - BK_j x_{t_n} = \begin{bmatrix} A_i & 0 & 0 & a_i \end{bmatrix} \zeta(t), \tag{24}$$

with $\zeta(t) = \begin{bmatrix} x^T(t) & x_{t_n}^T & w^T(t) & 1 \end{bmatrix}^T \in \mathbb{R}^{2n_x + n_u + 1}$.

Replacing (23) and (24) in (22) and setting $h_i(t) = N_i^T \zeta(t)$ with $N_i \in \mathbb{R}^{(2n_x + n_u + 1) \times n_x}$, we can write

$$\begin{aligned}\dot{V} \leq \zeta^T \Bigg(& \begin{bmatrix} A_i^T \\ K_i^T B^T \\ B^T \\ a_i^T \end{bmatrix} P \begin{bmatrix} I_{n_x} & 0 & 0 & 0 \end{bmatrix} \\ &+ \begin{bmatrix} I_{n_x} \\ 0 \\ 0 \\ 0 \end{bmatrix} P \begin{bmatrix} A_i & BK_i & B & a_i \end{bmatrix} \\ &+ \tau \begin{bmatrix} A_i^T \\ 0 \\ 0 \\ a_i^T \end{bmatrix} R \begin{bmatrix} A_i & 0 & 0 & a_i \end{bmatrix} + \rho N_i R^{-1} N_i^T \\ &- \begin{bmatrix} I_{n_x} \\ -I_{n_x} - \rho K_i^T B^T \\ -\rho B^T \\ 0 \end{bmatrix} N_i^T \\ &- N_i \begin{bmatrix} I_{n_x} & -I_{n_x} - \rho BK_i & -\rho B & 0 \end{bmatrix} \\ &- \begin{bmatrix} I_{n_x} \\ -I_{n_x} \\ 0 \\ 0 \end{bmatrix} X \begin{bmatrix} I_{n_x} & -I_{n_x} & 0 & 0 \end{bmatrix} \\ &+ (\tau - \rho) \begin{bmatrix} A_i^T \\ K_i^T B^T \\ B^T \\ a_i^T \end{bmatrix} X \begin{bmatrix} I_{n_x} & -I_{n_x} & 0 & 0 \end{bmatrix} \\ &+ (\tau - \rho) \begin{bmatrix} I_{n_x} \\ -I_{n_x} \\ 0 \\ 0 \end{bmatrix} X \begin{bmatrix} A_i & BK_i & B & a_i \end{bmatrix} \Bigg) \zeta. \end{aligned} \tag{25}$$

Hence, for $\rho = 0$, LMI (12) implies

$$\dot{V} < -\eta x^T x - x_{t_n}^T x_{t_n} + \gamma w^T w - \eta - \zeta^T S_i \zeta. \tag{26}$$

Using Schur complement, LMI (13) implies that (26) holds for $\rho = \tau$. Since (25) is affine in $\rho$, LMIs (12) and (13) are sufficient conditions for (26) to hold for any $\rho \in [0, \tau)$.

Recalling (8) and (9), we can write

$$||w|| \leq \Delta K ||x_{t_n}||. \tag{27}$$

Considering (11), we have

$$||w|| < \sqrt{\theta/\gamma}\,||x_{t_n}||. \tag{28}$$

Adding and subtracting $\theta x_{t_n}^T x_{t_n}$, $0 < \theta < 1$ to inequality (26) and using (28), we get

$$\dot{V} < -\eta x^T x - (1 - \theta) x_{t_n}^T x_{t_n} - \eta - \zeta^T S_i \zeta. \tag{29}$$

It follows from (3) that $\zeta^T S_i \zeta > 0$ if $x(t) \in \mathcal{R}_i$. Hence, LMIs (11)-(13) are sufficient conditions for $V$ to be strictly decreasing for any $t \in [t_n, t_{n+1})$, $x(t) \in \mathcal{R}_i$, and $i \notin I^*$.

• *Part 2:* For $x(t) \in \mathcal{R}_i$ and $i \in I^*$, based on Assumption 2, we have $a_i = 0$. Replacing $N_i$ by $\begin{bmatrix} \overline{N}_i^T & 0_{n_x \times 1} \end{bmatrix}^T$, $\overline{N}_i \in \mathbb{R}^{(2n_x + n_u) \times n_x}$, and setting $a_i = 0$ in (25), LMI (14) implies

$$\dot{V} < -\eta x^T x - x_{t_n}^T x_{t_n} + \gamma w^T w \tag{30}$$

for $\rho = 0$. Using Schur complement, LMI (15) implies that (30) holds for $\rho = \tau$. Since (25) is affine in $\rho$, LMIs (14) and (15) are sufficient conditions for (30) to hold for any $\rho \in [0, \tau)$. Adding and subtracting $\theta x_{t_n}^T x_{t_n}$, $0 < \theta < 1$ to inequality (30) and using (28), we get

$$\dot{V} < -\eta x^T x - (1 - \theta) x_{t_n}^T x_{t_n}. \tag{31}$$

Hence, LMIs (11), (14), and (15) are sufficient conditions for $V$ to be strictly decreasing for any $t \in [t_n, t_{n+1})$, $x(t) \in \mathcal{R}_i$, and $i \in I^*$.

Note that Zeno phenomenon does not occur since there exists $\epsilon > 0$ such that $t_{n+1} - t_n > \epsilon$ (see (6)). Therefore, inequalities (11)-(15) are sufficient conditions for the LKF to be strictly decreasing between any two consecutive sampling times in the whole state space. Since the LKF is non-increasing at sampling times, the closed-loop sampled data PWA system is asymptotically stable to the origin. ■

Based on Theorem 1, the problem of finding a lower bound on the longest interval between two consecutive sampling times that preserves asymptotic stability is formulated as

*Problem 1:*

$$\begin{aligned} \text{maximize} \quad & \tau \\ \text{subject to} \quad & P > 0,\ R > 0,\ X > 0,\ \Lambda_i \succeq 0,\ \text{for}\ \ i \notin I^*, \\ & \gamma > 0,\ 0 < \theta < 1,\ \eta > 0, \\ & (11) - (15). \end{aligned}$$

We denote the solution of Problem 1 by $\tau_{\max}$.

## IV. Numerical Example

Consider the path following example of [6], whose objective is to control a unicycle to follow the line $y = 0$ in the $x - y$ plane (see Fig. 1). The dynamics of the system are represented by

$$\begin{bmatrix} \dot{\psi} \\ \dot{r} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -k/I & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \psi \\ r \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ v\sin(\psi) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/I \\ 0 \end{bmatrix} u, \tag{32}$$

where $\psi$ and $r$ are the heading angle and its time derivative, respectively, $y$ is the distance from the line $y = 0$, $v$ represents the unicycle's velocity, $u$ is the torque input about the $z$ axis, $I = 1$ (kgm$^2$) is the unicycle's moment of inertia with respect to its center of mass, and $k$ =0.01 (Nms) is the damping coefficient. The state vector of the system is represented by $z = \begin{bmatrix} \psi & r & y \end{bmatrix}^T$. We assume that the unicycle has a constant velocity $v$ =1 (m/s) and the heading angle $\psi$ is restricted to the interval $[-3\pi/5, 3\pi/5]$.

The system's nonlinearity, $\sin(\psi)$, is approximated by a PWA function. The PWA approximation is defined over the following five regions:

$$\begin{aligned} \mathcal{R}_1 &= \left\{ z \in \mathbb{R}^3 | \psi \in (-3\pi/5, -\pi/5) \right\}, \\ \mathcal{R}_2 &= \left\{ z \in \mathbb{R}^3 | \psi \in (-\pi/5, -\pi/15) \right\}, \\ \mathcal{R}_3 &= \left\{ z \in \mathbb{R}^3 | \psi \in (-\pi/15, \pi/15) \right\}, \\ \mathcal{R}_4 &= \left\{ z \in \mathbb{R}^3 | \psi \in (\pi/15, \pi/5) \right\}, \\ \mathcal{R}_5 &= \left\{ z \in \mathbb{R}^3 | \psi \in (\pi/5, 3\pi/5) \right\}. \end{aligned}$$

It is shown in [6] that the continuous-time PWL controller

$$u = K_i z,\ \text{for}\ z \in \mathcal{R}_i, \tag{33}$$

with

$$\begin{aligned} K_1 &= [-49.907\ \ -9.468\ \ -13.925], \\ K_2 &= [-48.315\ \ -9.330\ \ -13.812], \\ K_3 &= [-50.147\ \ -9.468\ \ -13.742], \\ K_4 &= [-48.316\ \ -9.330\ \ -13.812], \\ K_5 &= [-49.907\ \ -9.468\ \ -13.925], \end{aligned}$$

stabilizes the system to the origin. Assuming that the controller is implemented via sample-and-hold in a microprocessor, our goal is to find a lower bound on the longest interval between two consecutive sampling times such that asymptotic stability is preserved.

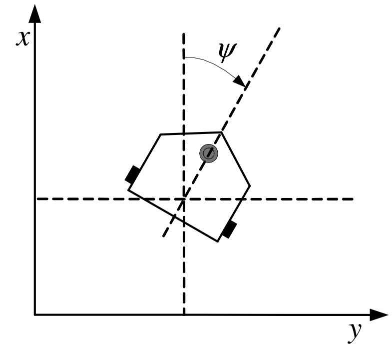

Fig. 1. Unicycle path following example

Solving Problem 1, we get

$$\tau_{\max} = 0.126\,(\text{sec})$$

and

$$P = \begin{bmatrix} 28.598 & 1.233 & 8.247 \\ 1.233 & 0.289 & 0.353 \\ 8.247 & 0.353 & 13.538 \end{bmatrix},$$

$$R = \begin{bmatrix} 10.491 & 4.987 & 1.689 \\ 4.987 & 163.543 & -18.386 \\ 1.689 & -18.386 & 32.819 \end{bmatrix},$$

$$X = \begin{bmatrix} 226.028 & 18.681 & 79.734 \\ 18.681 & 2.047 & 7.003 \\ 79.734 & 7.003 & 57.143 \end{bmatrix}.$$

Theorem 1 guarantees that if controller (33) is implemented in the unicycle via sample-and-hold, with variable sampling rates greater than $1/\tau_{\max} = 7.9$ (Hz), the system asymptotically converges to the origin.

Figures 2- 4, illustrate the simulation results for the unicycle system (32) with PWL feedback (33). The initial condition is $z(0) = [\pi/2, 0, 0.5]^T$. Simulations are performed for sampling times of $T_s = \tau_{\max} = 0.126$ (sec) and $T_s = 0$. Fig. 2 and Fig. 3 show that the state vector asymptotically converges to the origin. As expected, based on Fig. 4, more control energy is required for stabilizing the system with the sample-and-hold controller.

Simulating the system with the same initial condition $z(0)$ for $T_s = 0.213$ (sec), we can see that the closed-loop sampled-data trajectories do not converge to the origin. Therefore, in this example, the error in the computed lower bound on the longest sampling interval that preserves asymptotic stability is at most 41%.

Table I compares the result of Theorem 1 with the result of [15]. Our proposed theorem provides a stronger stability result (asymptotic stability) and increases $\tau_{\max}$ with respect to the one obtained from Theorem 1 in [15].

## V. Conclusion

In this paper we presented a theorem that provides sufficient conditions for asymptotic stability of a closed-loop

TABLE I

COMPARISON OF TWO STABILITY THEOREMS APPLIED TO THE UNICYCLE PROBLEM

| Method | Stability Result | $\tau_{\max}$ (sec) |
|---|---|---|
| Theorem 1 in [15] | Convergence to the invariant set $\{z \mid V(z, z_t, \rho) \leq 0.0645\}$ | 0.098 |
| Theorem 1 in this paper | Asymptotic stability to the origin | 0.126 |

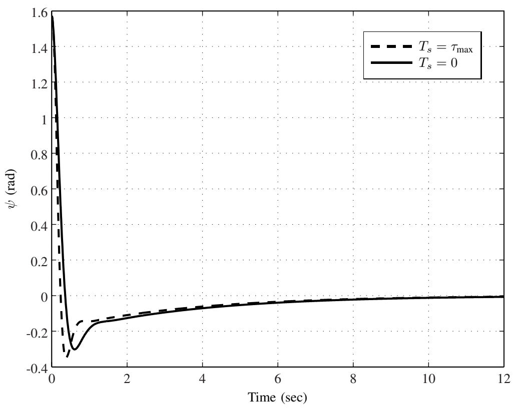


Fig. 2. Unicycle's heading angle for $T_s = \tau_{\max}$ and $T_s = 0$.

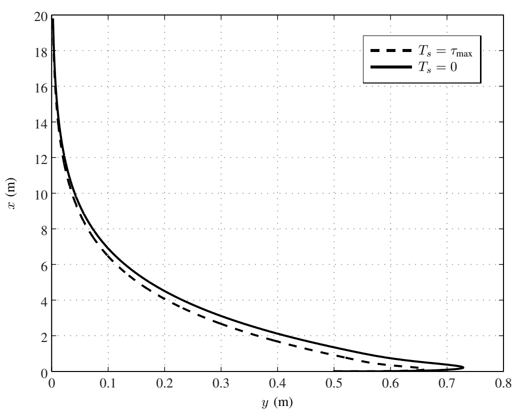


Fig. 3. Trajectory of the unicycle for $T_s = \tau_{\max}$ and $T_s = 0$.

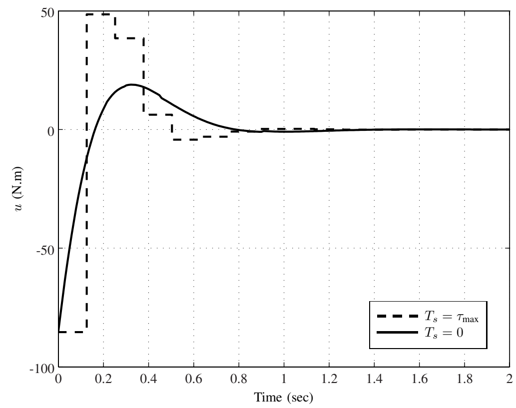


Fig. 4. Control input for $T_s = \tau_{\max}$ and $T_s = 0$.

sampled-data PWA system to the origin. The problem of finding a lower bound on the longest sampling interval that preserves asymptotic stability was formulated as an optimization problem subject to linear matrix inequalities. It was shown that our results compare favorably with previous research.

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