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A scaling and squaring method for the discretisation of positive switched systems

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***Abstract*— In this paper we present a new method for approximating the matrix exponential of a Metzler matrix. This method is useful in discretising switched positive systems. In particular, the method preserves both linear and quadratic stability of the original continuous time system, as well as positivity of states starting for initial conditions in the positive orthant. The usefulness of the method is highlighted by illustrating some of the drawbacks of Padé approximations when applied to positive linear systems.**

I. Introduction

Switched and non-switched linear positive systems have been the subject of much recent attention in the control engineering and mathematics literature [1], [2], [3], [4], [5], [6], [7], [8]. An important problem in the study of such systems concerns how to obtain discrete time approximations to a given continuous time system. While a complete understanding of this problem exists for LTI systems [9], and while some results exist for switched linear systems [10], [11], the analogous problems for positive systems are more challenging since discretisation methods must preserve not only the stability properties of the original continuous time system, but also physical properties, such as state positivity. To the best of our knowledge, this is a relatively new problem in the literature, with only a few recent works on this topic [12].

Our objective in this paper is to summarise and the results of [13], and to present to the control community a new method of approximation to the matrix exponential. We show that this method is particularly suited to the discretisation of switched positive systems. In particular, both stability and positivity are preserved when using this method. The utility of the method is highlighted by outlining the unsuitability of Padé methods when dealing with positive systems.

The paper is organised as follows: In Section II the notation and preliminary definitions are introduced. In Section III we propose the squaring and scaling approximation, that proves efficient in terms of positivity and co-positive Lyapunov functions preservation. In Section IV we discuss some of the problems related to the use of diagonal Padé approximations.

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II. Mathematical preliminaries

A. Notation

Capital letters denote matrices, small letters denote vectors. For matrices or vectors, $(')$ indicates transpose and $(^*)$ the complex conjugate transpose. For matrices X or vectors x, the notation X or $x > 0$ (≥ 0) indicates that X, or x, has all positive (nonnegative) entries and it will be called a positive (non-negative) matrix or vector. The notation $X \succ 0$ $(X \prec 0)$ or $X \succeq 0$ $(X \preceq 0)$ indicates that the matrix X is positive (negative) definite or positive (negative) semi-definite. The sets of real and natural numbers are denoted by $\mathbb{R}$ and $\mathbb{N}$, respectively, while $\mathbb{R}_+$ denotes the set of nonnegative real numbers. A square matrix A_c is said to be Hurwitz stable if all its eigenvalues lie in the open left-half of the complex plane. A square matrix A_d is said to be Schur stable if all its eigenvalues lie inside the unit disc. A matrix A is said to be Metzler (or essentially nonnegative) if all its off-diagonal elements are nonnegative; moreover we ask that the diagonal entries are non-positive, with at least one negative diagonal entry. A matrix B is an M-Matrix if $B = -A$, where A is both Metzler and Hurwitz; if an M-matrix is invertible, then its inverse is nonnegative [14]. I denotes the identity matrix of appropriate dimensions.

B. Definitions

Generally speaking, we are interested in the evolution of the system

$$\dot{x}_c(t) = A_c(t)x_c(t),\, x_c(0) = x_0; \tag{1}$$

where $A_c(t) \in \{A_{c,1}, ..., A_{c,m}\}$, $x_c(t) \in R^{n\times 1}$, $m \geq 1$, and where the $A_{c,i}$ are Hurwitz stable Metzler matrices. Such a system is said to be a continuous-time positive system. Positive systems [1], [15] have the special property that any nonnegative input and nonnegative initial state generate a nonnegative state trajectory and output for all times. We are interested in obtaining from this system, a discrete-time representation of the dynamics:

$$x(k+1) = A(k)x(k),\ A(k) \in \{A_{d,1}, ..., A_{d,m}\},\ x(0) = x_0. \tag{2}$$

Positivity in discrete time is ensured if each $A_{d,i}$ is a nonnegative matrix. One standard method to obtain $A_{d,i}$ from $A_{c,i}$ is via the Padé approximation to the exponential function $e^{A_{c,i}h}$, where h is the sampling time. Notice that, since (1) is a system switching according to an arbitrarily switching signal $\sigma(t) \in \{1, 2, \ldots, m\}$, it is not true, even in the ideal case $A_{d,i} = e^{A_{c,i}h}$, that

$x_c(kh) = x(k)$. This property is of course recovered when $t_k = kh$, where t_k is the generic switching instant of $\sigma(t)$.

A method of choice in control engineering for system discretisation is to use so-called Padé approximations. Such approximations are used in calculating the matrix exponential in Matlab, and when designing and simulating dynamic systems.

Definition 1: [16] The $[L/M]$ order Padé approximation to the exponential function $e^s$is the rational function C_{LM} defined by

$$C_{LM}(s) = Q_L(s)Q_M^{-1}(-s)$$

where

$$Q_L(s) = \sum_{k=0}^{L} l_k s^k,\ Q_M(s) = \sum_{k=0}^{M} m_k s^k, \qquad (3)$$

$$l_k = \tfrac{L!(L+M-k)!}{(L+M)!k!(L-k)!} \text{ and } m_k = \tfrac{M!(L+M-k)!}{(L+M)!k!(M-k)!}. \qquad (4)$$

Thus, the p-th order diagonal Padé approximation to $e^{A_c h}$ (the matrix exponential with sampling time h) is obtained by setting $L = M = p$

$$C_p(A_c h) = Q_p(A_c h)Q_p^{-1}(-A_c h), \qquad (5)$$

where $Q_p(A_c h) = \sum_{k=0}^{p} c_k (A_c h)^k$ and $c_k = \frac{p!(2p-k)!}{(2p)!k!(p-k)!}$.

Much is known in general about the Padé maps in the context of LTI systems. In particular, it is known that diagonal Padé approximations are A-stable [17]; namely, they map the open left-half of the complex plane to the interior of the unit disc, preserving in this way the stability from the continuous-time to the discrete-time system. Notice that for $p = 1$ the diagonal Padé approximation is

$$C_1(A_c h) = \left(I + \frac{A_c h}{2}\right)\left(I - \frac{A_c h}{2}\right)^{-1}. \qquad (6)$$

This is the celebrated *bilinear transformation* or *Tustin operator*.

We shall see in the sequel that Padé are not very useful for discretising positive systems [18]. Motivated by this fact we now propose another approximation to the matrix exponential. This method, which is a variation on the squaring and scaling method for calculating the matrix exponential [19], is of great use when dealing with positive systems.

Definition 2: Given $h \geq 0$, the SS_P approximation to the exponential matrix is the map $SS_p:\ A_c h \to A_d$ given by

$$SS_p(A_c h) = \left[\left(I + \frac{A_c h}{2p}\right)\left(I - \frac{A_c h}{2p}\right)^{-1}\right]^p,\ p \in \mathbb{N} \qquad (7)$$

Writing $A_{ad} = \left(I + \frac{A_c h}{2p}\right)^p \left(I - \frac{A_c h}{2p}\right)^{-p}$, and applying the binomial expansion to each of the two factors in that expression, we find readily that A_{ad} converges to $e^{A_c h}$ as $p \to \infty$. The scaling and squaring method (sse [19]) exploits the fact that for a square matrix M and $j \in \mathbb{N}, e^M = (e^{M/2^j})^{2^j}$. Accordingly, the scaling and squaring method proceeds by scaling the original matrix by a power of two, computing a Padé approximant of the resulting matrix, and then successively squaring that approximant to produce an approximation to the exponential of the original matrix.

Comment : Observe that for $p = 1$ the SS_1 transformation (7) is the bilinear transformation (6). Even though the bilinear transformation is the lowest order Padé approximant, it has very special properties. These properties make it very useful in the context of positive systems.

III. Lyapunov stability and positivity preservation of the SS_p approximation

Recently, it was shown in [11] that quadratic Lyapunov functions are preserved under discretization for sets of matrices that arise in the study of systems of the form of Equation (1). We now ask whether co-positive Lyapunov functions are preserved when discretising an LTI positive system using the SS_p approximations. In particular, our attention focuses on co-positive Lyapunov functions, linear and quadratic. Since trajectories of positive systems are constrained to lie in the positive orthant, the stability of such a system is completely captured by Lyapunov functions whose derivative is decreasing for all such positive trajectories and one can always associate a linear, or a quadratic co-positive Lyapunov function, with any given stable linear time-invariant positive system [1].

A. Bilinear transformation for positive time-invariant systems

Here we consider the bilinear transformation, or, equivalently the SS_1 transformation. The results will be instrumental for the main result concerning the effect of the SS_p discretisation on switched positive linear systems. We begin with the following elementary result that establishes the preservation of linear co-positive and linear quadratic Lyapunov functions under a bilinear transformation.

Lemma 1: [13] Let A_c be a Metzler and Hurwitz stable matrix and let α be a positive real number. Define $A_d(h) = (\alpha(h)I + A_c)(\alpha(h)I - A_c)^{-1}$, where $\alpha(h) = \frac{\alpha}{h}$ and $h > 0$, and assume that $A_d(h)$ is a nonnegative matrix. Then the following statements are true.

1) If $v(x) = x'Px$, with $P = P' \succ 0$, is a quadratic Lyapunov function for A_c, that is

$$x'(A_c'P + PA_c)x < 0,\ \forall\, x \geq 0,\ x \neq 0; \qquad (8)$$

then $v(x)$ is a quadratic Lyapunov function for $A_d(h)$; that is

$$x'(A_d'PA_d - P)x < 0,\ \forall\, x \geq 0,\ x \neq 0. \qquad (9)$$

2) If $v(x) = w'x$, $w > 0$ is a linear co-positive Lyapunov function for A_c; that is $w'A_c < 0$, then $v(x)$ is a linear co-positive Lyapunov function for $A_d(h)$; namely, $w'A_d < w'$.

■

We now turn our attention to providing sufficient conditions under which, for a given Metzler and Hurwitz matrix A, the bilinear transformation results in a nonnegative matrix.

Lemma 2: [13] Let $A_c = \{a_{ij}\}$ be the Metzler and Hurwitz stable matrix. Suppose that $\alpha_0 > 0$, set $\alpha(h) = \frac{\alpha_0}{h}$, and define A_d by

$$A_d = (\alpha(h)I + A_c)\,(\alpha(h)I - A_c)^{-1}. \tag{10}$$

If

$$h \leq \min_{i\,:\,a_{ii}\neq 0} \frac{\alpha_0}{|a_{ii}|} \tag{11}$$

then A_d is nonnegative and Schur stable.

Corollary 1: Let A_c be a Metzler and Hurwitz matrix. If $h \leq \min_{i\,:\,a_{ii}\neq 0} \frac{2}{|a_{ii}|}$, then $C_1(hA_c)$ is a nonnegative and Schur stable matrix.

B. SS_p approximation for positive switched systems

We now show that the SS_p approximation that has the following important properties: one can always find a sampling time such that positivity is preserved, and in addition, for any h, both linear and quadratic co-positive Lyapunov functions are preserved. Here is a key result.

Theorem 1: Let $\{A_{c,1}, \ldots, A_{c,m}\}$ be a set of Metzler and Hurwitz stable matrices. For each $i = 1, \ldots, m$, let $A_{ad,i}(h) = SS_p(A_{c,i}h)$ be the $p-th$ order of the approximation to exponential matrix $e^{A_{c,i}h}$ defined in Equation (7). Then the following properties hold:

1. Fix an i between 1 and m, and suppose that

$$0 < h \leq h_i = \min_j \frac{2p}{|a_{jj,i}|}, \tag{12}$$

where $a_{jj,i}$ are the elements on the main diagonal of the matrix $A_{c,i}$. Then $A_{ad,i}$ is both nonnegative and stable.

2. Consider the following continuous-time switching positive system

$$\dot{x}(t) = A_c(t)x(t), \quad x(0) = x_0, \tag{13}$$

where $x(t) \in \mathbb{R}^n_+$, $x_0 \in \mathbb{R}^n_+$ is the initial condition and $A_c(t)$ belongs to $\{A_{c,1}, \ldots, A_{c,m}\}$. Suppose that (12) holds. Then the discretised system

$$x(k+1) = A(k)x(k) \tag{14}$$

is positive, where $A(k) \in \{SS_p(A_{c,1}h), ..., SS_p(A_{c,m}h)\}$. Moreover, if there exists a common quadratic or linear co-positive Lyapunov function for system (13), then the origin $x = 0$ is globally uniformly exponentially stable for system (14).

Note that if p is chosen as a power of 2, then (7) coincides exactly with the scaling and squaring method, where the Padé approximant computed is the first order diagonal Padé approximant. Following the analysis given in section 11.3.1 of [19], we find that if $p = 2^j$ is chosen so that $||hA_c||_\infty \leq 2^{j-1}$, then taking A_{ad} equal to the matrix $SS_p(A_ch)$ of (7) has the property that

$$\frac{||e^{A_c} - A_{ad}||_\infty}{||e^{A_c}||_\infty} \leq \frac{h}{6}||A_c||_\infty e^{\frac{h}{6}||A_c||_\infty}.$$

In particular, for small values of h, A_{ad} approximates e^{hA_c} with high relative accuracy, in addition to the above mentioned features that A_{ad} preserves both positivity and linear/quadratic co-positive Lyapunov functions.

C. A computational algorithm

We can now propose a computational scheme for defining a sampling time h such that the discretised switched system is stable under arbitrary switching. Consider a switched system in continuous-time characterized by Metzler and Hurwitz matrices $A_{c,i}, i = 1, 2, ..., m$. It follows that there are positive vectors $c_i', i = 1, \ldots, m$ such that $c_i'A_{c,i} < 0', i = 1, \ldots, m$. Since each $A_{c,i}$ is Hurwitz, we find that $e^{A_{c,i}h} \to 0$ as $h \to \infty$. Consequently, we can find an $h_0 > 0$ such that for any $h > h_0$ and any pair of indices i and j between 1 and m, we have that

$$c_j' e^{A_{c,i}h} < c_i'. \tag{15}$$

Now fix an $h > h_0$. As noted after Definition 2, for each $i = 1, \ldots, m, SS_p(A_{c,i}h) \to e^{A_{c,i}h}$ as $p \to \infty$. Consequently, it follows from (15) that there is a p_0 (depending on h) such that for any pair of indices i and j between 1 and m, we have $c_j'SS_{p_0}(A_{c,i}h) < c_i'$. Set $A_{d,i}$ equal to $SS_{p_0}(A_{c,i}h)$, and consider the Lyapunov function

$$V(x) = c_\sigma' x,\ \sigma = 1, 2, \cdots, m.$$

Since $c_j'A_{d,i} - c_i' < 0'$ for any i and j between 1 and m, we see that that

$$V\big(x(k+1)\big) - V\big(x(k)\big) = \Big(c'_{\sigma(k+1)}A_d, \sigma(k) - c'_{\sigma(k)}\Big)x(k) < 0,$$

for each possible switching sequence $\sigma(k)$. Thus the discrete time switched system obtained for sampling time h is stable under arbitrary switching. This system is also positive if it is possible to choose an h guaranteeing that the matrices $A_{d,i}$ are nonnegative. Notice that $c_\sigma' x$ defines a piecewise linear co-positive Lyapunov function that is preserved by sampling.

Comment : Note also from the discussion in the previous subsection that a sufficient condition for positivity preservation by the SS_p transformation is that

$$h < 2p \min_j \frac{1}{|a_{jj,i}|} := h_{SS_p}$$

Then, it is always possible to find p such that $h_0 < h_{SS_p}$ and choose $h \in (h_0, h_{SS_p})$. These values are such that the resulting positive discrete-time switched system is positive and stable under arbitrary switching.

IV. Padé Approximations

Lemma 1 says that the first order diagonal Padé approximation is a robust approximation to the original system. That is, for every h, linear and quadratic stability is preserved. This result seems like good news since it says that the most basic Padé approximation to the matrix exponential, preserves stability and (under certain conditions) positivity, and consequently one might hope, as is the case for general matrices, that higher orders of diagonal Padé approximations will also preserve co-positive linear and quadratic stability and positivity. Unfortunately, rather surprisingly, this is not true as we shall now see. We begin with a surprising example that illustrates that not even positivity is a robust property of Padé approximations.

Example 1: Consider a chain of first order linear systems. Such systems are are of interest in the context of biological systems in the systems community [20], [4], and appear in the design of cascade filters [21]. Specifically: we consider a chain of n linear first order systems described by

$$\dot{x} = Ax$$

where

$$A = \begin{bmatrix} -\alpha_1 & k_1 & 0 & \cdots & \cdots & 0 \\ 0 & -\alpha_2 & k_2 & 0 & \cdots & 0 \\ 0 & 0 & -\alpha_3 & k_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & k_{n-1} \\ k_n & 0 & 0 & 0 & & -\alpha_n \end{bmatrix}. \quad (16)$$

By choosing $\alpha_i \geq 0$ and $k_i \geq 0$ one obtains that A is Metzler. For convenience we assume $n = 8$, $k_1 = k_2 = ... = k_7 = 1$, $k_8 = \alpha_1 = ... = \alpha_8 = 0$. In this case our system becomes a chain of homogeneous integrators connected in open loop. In spite of the fact that this is a very elementary system, it turns out that preserving positivity of this elementary system is far from trivial. Specifically, we shall now consider the second order diagonal Padé approximation $C_2(hA) = (I+\frac{1}{2}hA+\frac{1}{12}h^2A^2)(I-\frac{1}{2}hA+\frac{1}{12}h^2A^2)^{-1}$. Notice that the function $C_2(x) = (1+\frac{1}{2}x+\frac{1}{12}x^2)(1-\frac{1}{2}x+\frac{1}{12}x^2)^{-1}$, can be written as a power series in x as $C_2(x) = \sum_{j=0}^{\infty} \beta_j x^j$. Computing the first few coefficients in that power series, we find that $\beta_0 = 1, \beta_1 = 1, \beta_2 = \frac{1}{2}, \beta_3 = \frac{1}{6}, \beta_4 = \frac{1}{24}, \beta_5 = \frac{1}{144}, \beta_6 = 0$, and $\beta_7 = -\frac{1}{1728}$. Since $A^8 = 0$, it now follows that for every $h > 0$, $C_2(hA)$ has a negative entry and that **positivity is lost**.

The previous example illustrates a fact that Padé approximations do not take special care of positivity. One might ask whether it is true that stability is preserved (assuming that positivity has been ensured). As we shall now see, concrete statements concerning stability can only be made under stringent assumptions (a full discussion of the following results can be found in [13]). Indeed, as one increases the order of approximation to the matrix exponential, one can in fact lose preservation of a given Lyapunov function of the original system, even when positivity is preserved. To analyse this phenomenon, we decompose a generic Padé map C_p into a suitable product of bilinear functions. We summarise some evident results in this direction with the next lemma.

Lemma 3: [13] Let A_c be a Metzler and Hurwitz matrix, and suppose that $\hat{\lambda}$ is a complex number with positive real part. For each $h > 0$, let $\lambda(h) = \frac{\hat{\lambda}}{h}$, and consider the following matrices:

$$\begin{aligned} \Theta_1 &= (\lambda(h)I + A_c)(\lambda^*(h)I + A_c); \quad (17) \\ \Theta_2 &= (\lambda(h)I - A_c)(\lambda^*(h)I - A_c); \\ A_d(h) &= (\lambda(h)I + A_c)(\lambda^*(h)I + A_c) \\ &\quad \times (\lambda^*(h)I - A_c)^{-1}(\lambda(h)I - A_c)^{-1} = \Theta_1\Theta_2^{-1}. \end{aligned}$$

Suppose that there is an $h_0 > 0$ such that for all $0 < h \leq h_0, \Theta_2$ is an M-matrix and $A_d(h)$ is a nonnegative matrix. Then, the following statements are true.

1) If $v(x) = x'Px$, with $P = P' \succ 0$, is a co-positive quadratic Lyapunov function for A_c, i.e.,

$$x'(A_c'P + PA_c)x < 0, \ \forall\, x \geq 0, \ x \neq 0 \quad (18)$$

then there is an $h_1 > 0$ such that for all $0 < h \leq h_1$, $v(x)$ is a quadratic Lyapunov function for $A_d(h)$, i.e.,

$$x'(A_d'PA_d - P)x < 0, \ \forall\, x \geq 0, \ x \neq 0. \quad (19)$$

2) If $v(x) = w'x$, $w > 0$, is a linear co-positive Lyapunov function for A_c, that is $w'A_c < 0$ then for $0 < h \leq h_0$, $v(x)$ is a linear co-positive Lyapunov function for $A_d(h)$; namely, $w'A_d < w'$.

We can now state the following result, which formalises the intuition that stability, for a switched linear system, is indeed preserved provided h is chosen to be small enough (fast enough sampling), for diagonal Padé approximations. To state this result, recall the continuous-time switched linear positive system

$$\dot{x}_c(t) = A_c(t)x_c(t), \quad x_c(0) = x_0, \quad (20)$$

where $x_c(t) \in \mathbb{R}^n_+$, $x_0 \in \mathbb{R}^n_+$ is the initial condition, and $A_c(t)$ belongs to the set $\{A_{c,1}, \ldots, A_{c,m}\}$. We then have the following result.

Theorem 2: [13] Consider the system (20). Suppose that $A_{c,i}$ is a Metzler and Hurwitz stable matrix for each $i = 1, \ldots, m$ and let $A_{di}(h) = C_p(A_{c,i}h)$ be the $p-th$ order diagonal Padé approximation of $e^{A_{c,i}h}$. Suppose also that there is an $h_0 > 0$ such that for all $0 < h \leq h_0$, and each complex pole λ of $C_p(x)$, and each $i = 1, \ldots, m$, we have that $(\frac{\lambda}{h}I - A_{c,i})(\frac{\lambda^*}{h}I - A_{c,i})$ is an M-matrix and $A_{di}(h)$ is a nonnegative matrix, as is all Padé factors of the form given in 1 and Lemma 3 associated with the real and complex poles of $C_p(A_{c,i}h)$ respectively. Finally, suppose there exists a common linear co-positive Lyapunov function for system (20). Then, for all $0 < h \leq h_0$, the system

$$x(k+1) = A(k)x(k),\ \mu = \in \{1, 2, \ldots, m\}, \qquad (21)$$

with $A(k) \in \{C_p(A_{c,1}h), ..., C_p(A_{c,m}h)\}$, shares the same common linear co-positive Lyapunov function.

Comment : An analogous statement may be made for co-positive quadratic stability.

The hypotheses of Lemma 3 include the condition that Θ_2 is an M-matrix for all sufficiently small $h > 0$. It is natural to wonder when that condition holds. To do this suppose that λ_0 is a complex number with $Re(\lambda_0) > 0$. Set $\lambda(h) = \frac{\lambda_0}{h}$, and define A_d via

$$\begin{aligned} A_d &= (\lambda(h)I + A_c)(\lambda^*(h)I + A_c) & (22)\\ &\quad \times (\lambda(h)I - A_c)^{-1}(\lambda^*(h)I - A_c)^{-1}. & (23)\end{aligned}$$

Set

$$\begin{aligned} \Theta_1 &= \left(|\lambda(h)|^2 I + 2Re(\lambda(h))A_c + A_c^2\right) & (24)\\ \Theta_2 &= \left(|\lambda(h)|^2 I - 2Re(\lambda(h))A_c + A_c^2\right), & (25)\end{aligned}$$

so that $A_d = \Theta_1\Theta_2^{-1}$. Define $A_c = \{a_{ij}\}$ and $A_c^2 = \{b_{ij}\}$ then let $\mathcal{P}$ be the set of indices $i, j,\ i \neq j$, such that $b_{ij} \neq 0$. Then we have the following result.

Lemma 4: [13] Let $A_c = \{a_{ij}\}$ be a Metzler and Hurwitz stable matrix and A_d the matrix achieved through the transformation (22). If

$$h \leq 2Re(\lambda_0) \min_{i,j \in \mathcal{P}} \frac{a_{ij}}{|b_{ij}|}, \qquad (26)$$

then Θ_1 of (24) is a nonnegative matrix, Θ_2 of (24) is an M-matrix, and A_d is nonnegative and Schur stable.

Comment : Note that if A_c has a zero entry in an off-diagonal position where B has a positive entry, then the right-hand side of (26) is 0. Clearly in that situation, Lemma 4 does not yield a useful conclusion.

Lemmas 2 and 4 will now yield the following result regarding the nonnegativity of a p-th order diagonal Padé approximation.

Theorem 3: Let A_c be a Metzler and Hurwitz stable matrix and $A_d(h) = C_p(A_c h)$ be the $p-th$ order diagonal Padé approximation to $e^{A_c h}$. Let $\alpha_l, l = 1, \ldots, m$ denote the real poles of $C_p(x)$, and let $\lambda_k, \lambda_k^*, k = 1, \ldots, \frac{n}{2}$ denote the complex conjugate pairs of poles $C_p(x)$. If $m \geq 1$, we define $\hat{\alpha} = \min_{l=1,\ldots,m} \alpha_l$, and if $n \geq 2$, we define $\hat{\lambda} = \min_{k=1,\ldots,\frac{n}{2}} Re(\lambda_k)$. Then $A_d(h)$ is nonnegative and Schur stable for every $h \leq h^*$, where

$$\begin{aligned} h^* &= \min_{i\,,:\, a_{ii} \neq 0} \frac{\hat{\alpha}}{|a_{ii}|}, \text{if } n = 0,\ m \geq 1 & (27)\\ h^* &= 2\hat{\lambda} \min_{i,j \in \mathcal{P}} \frac{a_{ij}}{|b_{ij}|}, \text{if } m = 0,\ n \geq 2 & (28)\\ h^* &= \min_{i\,:\, a_{ii} \neq 0} \frac{\hat{\alpha}}{|a_{ii}|}, 2\hat{\lambda} \min_{i,j \in \mathcal{P}} \frac{a_{ij}}{|b_{ij}|}, \text{if m} \geq 1,\ n \geq 2 & (29)\end{aligned}$$

where a_{ij} and b_{ij} denote the (i, j) element of A_c and A_c^2 respectively.

Proof: We begin by noting that $\alpha_l > 0, l = 1, \ldots, m$ and $Re(\lambda_k) > 0, k = 1 \ldots, \frac{n}{2}$, and that $m + n = p$. Decomposing the $p-th$ order diagonal Padé approximation into real and complex conjugate pairs of poles [11], we have:

$$\begin{aligned} A_d(h) = &\prod_{l=1}^{m} (\alpha_l(h)I + A_c)\\ &\times \prod_{k=1}^{n/2} \left(|\lambda_k(h)|^2 I + 2Re(\lambda_k(h))A_c + A_c^2\right)\\ &\times \prod_{l=1}^{m} (\alpha_l(h)I - A_c)^{-1}\\ &\times \prod_{k=1}^{n/2} \left(|\lambda_k(h)|^2 I - 2Re(\lambda_k(h))A_c + A_c^2\right)^{-1}, \end{aligned} \qquad (30)$$

where $\alpha_l(h) = \frac{\alpha_l}{h}, l = 1, \ldots, m$ and $\lambda_k(h) = \frac{\lambda_k}{h}$, $\lambda_k^*(h) = \frac{\lambda_k^*}{h}, k = 1, \ldots, \frac{n}{2}$. For each l, we may apply Lemma 2 to the factor $(\alpha_l(h)I + A_c)(\alpha_l(h)I - A_c)^{-1}$ to deduce that it is nonnegative. Similarly, for each k we apply Lemma 4 to the factor $(|\lambda_k(h)|^2 I + 2Re(\lambda_k(h))A_c + A_c^2)(|\lambda_k(h)|^2 I - 2Re(\lambda_k(h))A_c + A_c^2)^{-1}$ to find that it is also nonnegative. We find immediately that $A_d(h)$ is nonnegative. Finally, since $A_d(h)$ is a diagonal Padé approximation, it is necessarily Schur stable [17]. ■

Comment : We note that in the case that $n \geq 2$, the quantity h^* in Theorem 3 is positive if and only if, for each nonzero offdiagonal entry in A_c^2, the corresponding entry in A_c is also nonzero. A discussion in terms of directed graphs is given in [13]. We note that the condition is always met for 2×2 matrices, see [18].

Example 2: We now give a somewhat contrived example to illustrate some of the pitfalls that can arise when using the Padé method of approximating the matrix exponential in the context of simulating a switched system. We are interested in simulating the following switched system:

$$\dot{x} = A(t)x \; ; A(t) \in \{A_1, A_2\}, \tag{31}$$

where

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{32}$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{33}$$

System (31) is a positive system. We will now show how this need not be the case when using Padé approximations to construct a discrete time equivalent. To this end, suppose we wish to simulate the periodic system where the matrix A_1 is active for 0.5s followed by the matrix A_2 for 4.5s. This system can be approximated after one period $T = 5$s via

$$x(T) = A_{d2}^9 A_{d1} x_0, \tag{34}$$

We then simulate the system starting from an initial condition $x(0)' = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 100 \end{bmatrix}$. Using the second order Padé approximations, it is easy to check that the evolution of the first component of the state vector x is takes negative values, and so the discrete time approximation is not a positive system.

We now repeat the above simulation using the SS_2 method to approximate the matrix exponential. It follows that the discrete-time system is a positive system for all initial conditions in the nonnegative orthant.

V. Conclusions

In this paper we examine the suitability of diagonal Padé transformations for discretising positive systems. We show that positivity and stability preservation are only guaranteed under very restrictive conditions. However, it is shown that the newly developed SS_p transformation exhibits performance that avoids these pitfalls.

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