2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Structural stability of equilibrium sets for a class of discontinuous vector fields

J. J. Benjamin Biemond, Nathan van de Wouw and Henk Nijmeijer

***Abstract*— A class of discontinuous vector fields is investigated, where equilibria are generically positioned in an interval in the phase space, and equilibria are not isolated points. The dynamics near such an equilibrium set is studied, and it is shown that the structural stability of trajectories near the equilibrium sets is determined by the local dynamics near the endpoints of this interval. Based on this result, sufficient conditions for structural stability of equilibrium sets in planar systems are given, and two new bifurcations are identified. The results are illustrated by application to a controlled mechanical system with dry friction.**

I. INTRODUCTION

In this paper we consider the structural stability of a class of discontinuous vector fields, which are discontinuous when one of the velocities is zero. The discontinuous term in the vector field compensates the other terms of the vector field when these are sufficiently small, such that equilibrium points are generically not isolated, but occur on an interval of a curve in the phase space. This feature, which is distinct from the behaviour of smooth differential equations, makes this class of discontinuous vector fields suitable to model nonlinear mechanical systems with dry friction. In these systems, equilibrium sets occur generically: a single mass experiencing dry friction and gravity is not expected to move towards a local minimum, but can stick to all positions where the slope of the surface is sufficiently small.

Dry friction appears at virtually all physical interfaces that are in contact, and the dynamics of systems with friction can be understood by studying the class of discontinuous vector fields presented in this paper. It is well known that dry friction may induce limit cycling, thereby deteriorating performance, cf. [1], [2]. Moreover, the presence of friction-induced equilibrium sets in engineering systems compromises position accuracy in motion control systems, such as robot positioning control, see e.g. [2]–[4]. Despite countermeasures such as friction compensation, steady-state errors induced by the existence of equilibrium sets can often not be avoided. Hence, the stability properties of the equilibrium sets are of interest and the effect of variations of control parameters on the equilibrium sets should be considered. The force exerted by dry friction can be modelled with a discontinuous friction law, such that the dynamics is described by a discontinuous vector field. Despite the fact that such a friction law is not accurately describing the microscopic interaction forces in the frictional contact, and can not show hysteretic behaviour, this friction law can yield an accurate description of the equilibrium set and the nearby dynamics, see e.g. [4]–[6].

The equilibrium sets of the discontinuous vector fields may be stable or unstable in the sense of Lyapunov. In addition, equilibrium sets may attract all nearby trajectories in finite time, cf. [7]. A natural question is to ask how changes in (control) system parameters influence these properties. To answer this question, structural stability of equilibrium sets is studied.

For this purpose, the local phase portrait near an equilibrium set is studied and possible bifurcations are identified. Under a non-degeneracy condition, the local dynamics is shown to be structurally stable near the equilibrium set, except for two specific points, namely the endpoints of the equilibrium set. With this result, the study of the structural stability of the dynamics near the equilibrium sets can be restricted to the analysis of the two endpoints of the equilibrium set; it is not necessary to consider the whole interval in the phase space. Local analysis near these endpoints yields a listing of the bifurcations that are possible.

Although quite some results exist on the asymptotic stability and attractivity of equilibrium sets of mechanical systems with dry friction, see [7]–[9], very few results exist that study structural stability and bifurcations of equilibrium sets, see [10]. In [7], sufficient conditions are presented for attractivity and asymptotic stability of equilibrium sets using Lyapunov theory and invariance results. In [8], conditions are presented under which trajectories converge to the equilibrium set in finite time. Using Lyapunov functions, the attractive properties of individual points in the equilibrium sets are analysed in [9]. Bifurcations of equilibrium sets are studied in [10] for systems with dry friction. In this reference, the appearance or disappearance of an equilibrium set is studied by solving an algebraic inclusion; however, nearby trajectories and stability properties are not considered.

Bifurcation results considering the larger family of differential inclusions, that contains the specific class of vector fields considered in this paper, are either focussing on bifurcations of limit cycles, or bifurcations of isolated equilibrium points. Bifurcations of limit cycles of discontinuous systems are studied using a return map, see [11]. However, this approach is only applicable when the friction interface is moving, such that the discontinuity surface does not contain equilibria. Bifurcations of equilibria in two or three dimensions are studied extensively, see [12] for review of existing results. Here, the dynamics is understood by following the

This work is supported by the Netherlands Organisation for Scientific Research (NWO).

Department of Mechanical Engineering, Eindhoven University of Technology, P.O. Box 513, 5600MB Eindhoven, the Netherlands. {j.j.b.biemond,n.v.d.wouw,h.nijmeijer}@tue.nl

trajectories that become tangent to a discontinuity boundary. In [12], a generic classification is presented of bifurcations with codimension one and two in planar differential inclusions. However, the special structure of differential inclusions describing mechanical systems with dry friction, which we analyse in the present paper, is considered to be non-generic in [12]. In (nonlinear) mechanical systems with dry friction, equilibrium sets occur generically, and persist when physically relevant perturbations are applied.

In this paper both the existence of an equilibrium set and the structural stability of the local phase portrait are investigated. Sufficient conditions for structural stability of the phase portrait are given, where we restrict our attention to a neighbourhood of the equilibrium set. At system parameters where the conditions for structural stability are not satisfied, two bifurcations are identified that do not occur in smooth systems.

The outline of this paper is as follows. First, in Section II we introduce the class of discontinuous vector fields considered in this paper. In Section III the main result is presented, which states that structural stability of equilibrium sets is determined by the local dynamics of two specific points, which are the endpoints of the equilibrium set. Furthermore, classes of systems are identified that are structurally stable. In Section IV, two bifurcations of the equilibrium set of planar systems are presented. In Section V, the results of this paper are illustrated with an example of a controlled mechanical system with dry friction, where a bifurcation occurs when a control parameter is changed. Concluding remarks are given in Section VI.

II. MODELLING

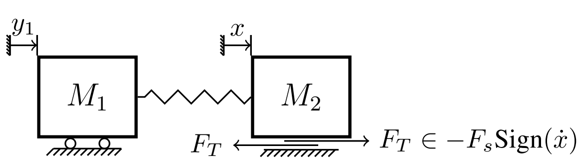


Fig. 1. Mechanical system subject to dry friction.

Consider a mechanical system that experiences friction on one interface between two surfaces that move relative to each other in a given direction. Let x denote the displacement in this direction and $\dot{x}$ denote the slip velocity, see Fig. 1 for an example. For an $n-$dimensional dynamical system this implies that $n-2$ other states y are required besides x and $\dot{x}$. These states contain the other positions and velocities of the mechanical system, and possibly controller and observer states, e.g. in the case of a feedback-controlled motion system. The system given in Fig. 1 can be modelled with the additional states $y = \begin{pmatrix} y_1 & \dot{y}_1 \end{pmatrix}^T$. Using the states $x, \dot{x}$ and y, the dynamics are described by the following differential inclusion, cf. [13]:

$$\begin{aligned} \ddot{x} - f(x,\dot{x},y) &\in -F_s\mathrm{Sign}(\dot{x}), \\ \dot{y} &= g(x,\dot{x},y), \end{aligned} \tag{1}$$

where f and g are sufficiently smooth, $F_s \neq 0$, and $\mathrm{Sign}(\cdot)$ denotes the set-valued sign function $\mathrm{Sign}(p) = p|p|^{-1}$, for $p \neq 0$ and $\mathrm{Sign}(0) = [-1,1]$.

Note that (1) also encompasses systems with other nonlinearities than dry friction, e.g. geometric nonlinearities in robotic systems. Introducing the state variables $q = \begin{pmatrix} x & \dot{x} & y^T \end{pmatrix}^T$, the dynamics of (1) can be reformulated as:

$$\dot{q} \in F(q), \tag{2}$$

$$F(q) = \begin{cases} F_1(q), & q \in S_1 := \{q \in \mathbb{R}^n : h(q) < 0\}, \\ F_2(q), & q \in S_2 := \{q \in \mathbb{R}^n : h(q) > 0\}, \\ \mathrm{co}(F_1(q), F_2(q)), & q \in \Sigma := \{q \in \mathbb{R}^n : h(q) = 0\}, \end{cases} \tag{3}$$

where $q \in \mathbb{R}^n$, $\mathrm{co}(a,b)$ denotes the smallest convex hull containing a and b, and F_1, F_2 and h are given by: $F_1(q) = \begin{pmatrix} \dot{x} & f(x,\dot{x},y) + F_s & g(x,\dot{x},y)^T \end{pmatrix}^T$, $F_2(q) = \begin{pmatrix} \dot{x} & f(x,\dot{x},y) - F_s & g(x,\dot{x},y)^T \end{pmatrix}^T$, and $h(q) = \dot{x}$.

In most existing bifurcation results for differential inclusions, see e.g. [11]–[13], parameter changes are considered that induce perturbations of the function F in (2). In these studies also the first component of F is perturbed, which implies that the case where the discontinuity surface coincides with the set where the first element of F is zero is considered non-generic by these authors. This implies that the existence of an equilibrium set in (2) is non-generic. However, parameter changes for the specific system (1) only yield perturbations of f and g in (2). We show that for the class of systems under study, i.e. mechanical systems with set-valued friction, equilibrium sets will occur, generically.

To study trajectories at the discontinuity surface Σ, the solution concept of Filippov is used, see [13]. Three domains are distinguished on the discontinuity surface. If trajectories on both sides arrive at the boundary, then we have a stable sliding region Σ^s. If one side of the boundary has trajectories towards the boundary, and trajectories on the other side leave the boundary, this domain is called the crossing region Σ^c (or transversal intersection). Otherwise, we have the unstable sliding motion on the domain Σ^u. The mentioned domains are identified as follows:

$$\begin{aligned} \Sigma &:= \{q \in \mathbb{R}^n : h(q) = 0\}, \\ \Sigma^s &:= \{q \in \Sigma : L_{F_1}h > 0 \text{ and } L_{F_2}h < 0\}, \\ \Sigma^u &:= \{q \in \Sigma : L_{F_1}h < 0 \text{ and } L_{F_2}h > 0\}, \\ \Sigma^c &:= \{q \in \Sigma : (L_{F_1}h)(L_{F_2}h) > 0\}, \end{aligned} \tag{4}$$

where $L_{F_i}h, i = 1,2$, denotes the directional derivative of h with respect to F_i, i.e. $L_{F_i}h = \nabla h F_i(q)$.

The vector field $\dot{q} = F^s(q)$ during sliding motion at $q \in \Sigma^u \cup \Sigma^s$ is defined using Filippov, [13], as follows. For each q, the vector $F^s(q)$ is the vector on the segment between $F_1(q)$ and $F_2(q)$ that is tangent to Σ at q:

$$\dot{q} = F^s(q) := \frac{L_{F_1}h(q)\, F_2(q) - L_{F_2}h(q)\, F_1(q)}{L_{F_1}h(q) - L_{F_2}h(q)}, \tag{5}$$

$$= \begin{pmatrix} 0 & 0 & g(x,0,y)^T \end{pmatrix}^T. \tag{6}$$

Since $L_{F_1}h = L_{F_2}h + 2F_s$, it follows from (4) that $F_s > 0$ implies that no unstable sliding occurs, and $F_s < 0$ implies

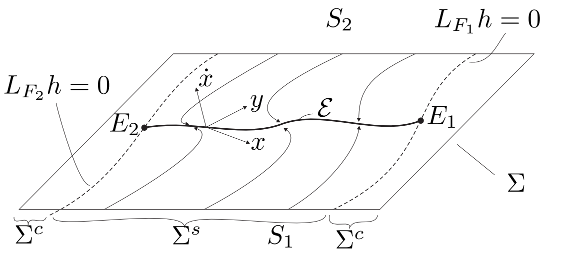


Fig. 2. Sketch of discontinuity surface Σ of (2) with $n = 3$ and $F_s > 0$, containing an equilibrium set $\mathcal{E}$.

that no stable sliding occurs. The resulting phase space is shown schematically in Fig. 2 for the case $n = 3$.

The equilibrium set is a segment of a curve on the discontinuity surface Σ when we adopt the following assumption.

Assumption 1: The functions f and g are such that $f(0,0,0) = 0$, $g(0,0,0) = 0$ and $\begin{pmatrix} \frac{\partial f(x,0,y)}{\partial x} & \frac{\partial f(x,0,y)}{\partial y} \\ \frac{\partial g(x,0,y)}{\partial x} & \frac{\partial g(x,0,y)}{\partial y} \end{pmatrix}$ is invertible. Furthermore, the map $\begin{pmatrix} f \\ g \end{pmatrix}$ is proper[1].

For systems satisfying this assumption, the equilibrium set $\mathcal{E}$ of (1) is a one-dimensional curve as shown in Fig. 2. The equilibrium set of a differential inclusion is given by $0 \in F(q)$, which is equivalent with $(q \in \Sigma^s \cup \Sigma^u$ and $L_{F_2}h\, F_1(q) - L_{F_1}h\, F_2(q) = 0)$, since $(q_2 = 0$ and $0 \in \mathrm{co}\{f(q) - F_s, f(q) + F_s\})$ is equivalent with $q \in \Sigma^s \cup \Sigma^u$ and $g(q) = 0$ is equivalent with $L_{F_2}h\, F_1(q) - L_{F_1}h\, F_2(q) = 0$ for $q \in \Sigma^s \cup \Sigma^u$, see (5).

The equilibrium set $\mathcal{E}$ is divided in interior points $p \in I$ and the two endpoints E_1, E_2 as follows:

$$\begin{aligned} \mathcal{E} &:= \{q \in \Sigma^u \cup \Sigma^s : L_{F_2}h\, F_1 - L_{F_1}h\, F_2 = 0\}, \\ I &:= \{q \in \mathcal{E} : F_1 \neq 0 \text{ and } F_2 \neq 0\}, \\ E_i &:= \{q \in \mathcal{E} : F_i = 0\},\ i = 1, 2. \end{aligned} \tag{7}$$

Note, that interior points are called pseudo-equilibria in [11]. The endpoints E_1 and E_2 satisfy $L_{F_1}h = 0$ or $L_{F_2}h = 0$, respectively, hence they are positioned on the boundary of the stable or unstable sliding mode as given by (4).

III. Structural stability of the system near the equilibrium set

In this section, trajectories near the equilibrium set are studied, and the influence of perturbations of (1) on this phase portrait is considered. For this purpose, we define the topological equivalence of phase portraits of (1) in Definition 1. We note that this definition is equal to the definition for smooth systems, see, e.g., [14].

Definition 1 ([13]): Two dynamical systems in open domains G_1 and G_2, respectively, are *topologically equivalent* if there exist a homeomorphism from G_1 to G_2 which carries, as does its inverse, trajectories into trajectories.

This equivalence relation allows for homeomorphisms that do not preserve the parameterisation of the trajectory with time, as required for *topological conjugacy* defined in [14]. Throughout this paper, we assume that f and g smoothly depend on system parameters. When a parameter variation of a dynamical system A yields a system $\tilde{A}$ which is not topologically equivalent to A, then the dynamical system undergoes a bifurcation.

With the definitions given above, we can formulate our main result in the following theorem.

Theorem 1: Assume (1) satisfies Assumption 1. If $\left.\frac{\partial g}{\partial y}\right|_p$ has no eigenvalue λ with $\mathrm{real}(\lambda) = 0$ for any $p \in \mathcal{E}$, then the dynamical system (1), in a neighbourhood of the equilibrium set, can only experience bifurcations near the endpoints E_1 or E_2.

Proof: The proof of Theorem 1 is given in [15]. ■

To prove Theorem 1, the influence of perturbations on systems (1) are studied. If perturbations of f and g of (1) can not yield a dynamical system which is not topologically equivalent to the original system, then the occurrence of bifurcations is excluded. Hence, structural stability of (1) is investigated, which is defined as follows.

Definition 2: A system A given by (1) is *structurally stable for perturbations in f and g* if there exists an $\epsilon > 0$ such that any perturbed system $\tilde{A}$, given by (1) with $\tilde{f}$ and $\tilde{g}$, is topologically equivalent to system A when

$$|f - \tilde{f}| < \epsilon,\ \left\| \frac{\partial (f - \tilde{f})}{\partial q} \right\| < \epsilon,\ \|g - \tilde{g}\| < \epsilon,\ \left\| \frac{\partial (g - \tilde{g})}{\partial q} \right\| < \epsilon, \tag{8}$$

holds for all $q \in \mathbb{R}^n$.

If these properties hold locally in a neighbourhood of a set $J \subset \mathbb{R}^n$, then the local phase portrait near J is structurally stable. Note, that this definition corresponds to C^1-structural stability as defined by [16], and is tailored to dynamical systems described using second-order time derivatives of the state x.

Note that perturbations of (1) in f and g do not cause perturbations of the first component of $F(\cdot)$ in (2), as observed e.g. in [16] or [13, page 226]. One consequence of this fact is that equilibrium sets occur generically in systems (1), although they are non-generic in systems (2). In experiments on mechanical systems with dry friction, such equilibrium sets are found to occur generically, see e.g. [17]. For this reason, perturbations satisfying (8) are used throughout this paper. System A can be structurally stable for perturbations in f and g, whereas the corresponding system (2) is not structurally stable for general perturbations of F. Small changes of system parameters cause small perturbations of f and g and their derivatives. However, the first equation of (2) will not change under parameter changes. Namely, this equation represents the kinematic relationship between position and velocity of a mechanical system, such that perturbation of this equation does not make sense for the class of physical systems under study. Hence, structural stability for perturbations in f and g excludes the occurrence of bifurcations for small variations of the system parameters.

Small perturbations of system (1) cause the equilibrium set $\mathcal{E}$ to deform, but the equilibrium set of the perturbed system

[1] A continuous map is proper if the inverse image of any compact set is compact.

remains a smooth curve. Hence, there exist a smooth coordinate transformation that transforms the original equilibrium set $\mathcal{E}$ to the equilibrium set of the perturbed system.

A. Structural stability of planar systems

In this section, sufficient conditions are presented for the structural stability of planar systems, restricted to a neighbourhood of equilibrium sets. In the planar case, (1) and (2) reduce to, respectively:

$$\ddot{x} - f(x, \dot{x}) \in -F_s \mathrm{Sign}(\dot{x}), \tag{9}$$

$$\dot{q} \in F(q) = \begin{cases} F_1(q) = \begin{pmatrix} q_2 \\ -f(q_1, q_2) + F_s \end{pmatrix}, & h(q) < 0 \\ F_2(q) = \begin{pmatrix} q_2 \\ -f(q_1, q_2) - F_s \end{pmatrix}, & h(q) > 0 \\ \mathrm{co}(F_1(q), F_2(q)), & h(q) = 0, \end{cases} \tag{10}$$

where $q = \begin{pmatrix} x & \dot{x} \end{pmatrix}^T$ and $h(q) = q_2$. In this case, the Filippov solution $\dot{q} = F^s(q) = 0, \forall q \in \Sigma^s \cup \Sigma^u$, see (5), such that the set of interior points of the equilibrium set satisfies $I = \Sigma^s \cup \Sigma^u$.

Throughout this section it is assumed that $F_s > 0$, which corresponds to the practically relevant case where dry friction dissipates energy. According to Theorem 1, structural stability of (9), restricted to a neighbourhood of the equilibrium set, is determined by the trajectories of (9) near the endpoints. Analogous to the Hartman-Grobman theorem, which presents sufficient conditions for structural stability of the phase portrait near an equilibrium point in smooth systems based on the linearised dynamics near this point, sufficient conditions for structural stability of (9) are formulated based on the linearisation of F_1 and F_2 near the endpoints of the equilibrium set.

For ease of notation, we define $A_k := \left.\frac{\partial F_k}{\partial q}\right|_{q=E_k}$, $k = 1, 2$, which determines the linearised dynamics in S_k near the endpoints of the equilibrium set, with $S_k,\ k = 1, 2$, defined in (3). In the other domain, i.e. S_{3-k}, it follows from (10) that the vector field is pointing towards the discontinuity surface. To study the structural stability of planar systems (9), we adopt the following assumption.

Assumption 2:

(i) The dry friction force satisfies $F_s > 0$;

(ii) The eigenvalues of $A_k,\ k = 1, 2$, are distinct and nonzero.

Observe that *(i)* implies that the equilibrium point $\mathcal{E}$ persists under perturbation of f, whereas *(ii)* concerns the linearised vector field near the endpoints of $\mathcal{E}$. Furthermore, *(ii)* implies that A_k is invertible. The following theorem presents sufficient conditions for structural stability of (9), restricted to a neighbourhood of the equilibrium set.

Theorem 2: Consider a system A given by (9) that satisfies Assumptions 1 and 2. Restricted to a neighbourhood of the equilibrium set, system A is structurally stable for perturbations in f.

Proof: The proof of Theorem 2 is given in [15]. ■

The theorem implies that one can identify a number of different types of systems (9) which, restricted to a neighbourhood of the equilibrium set, are structurally stable.

IV. Bifurcations

Since the class of systems (1) is a special case of systems (2), many bifurcations of (2), e.g. those observed in [18], can not occur in (1). In this section, bifurcations of (1) are studied, restricting ourselves to planar systems, given by (9).

Theorem 2 states that Assumptions 1 and 2 together imply structural stability. Hence, it seems a reasonable step to consider parameter-dependent systems, and study the parameters where the conditions on the differential inclusion (9), as given in Assumption 2, no longer hold. In the following sections, two bifurcations of planar systems (1) are identified when Assumption 2*(ii)* is violated.

A. Real or complex eigenvalues

Consider system (9) where the eigenvalues of A_1 change from real to complex eigenvalues under a parameter variation. From (9) it follows that $A_1 = \begin{pmatrix} 0 & 1 \\ a_{21} & a_{22} \end{pmatrix}$, hence the eigenvalues of A_1 are distinct when $a_{21} \neq -\frac{1}{4}a_{22}^2$ and the eigenvalues are both nonzero given $a_{21} \neq 0$. Now, let the first part of Assumption 2.(ii) be violated, such that the eigenvalues are equal. In that case, we obtain $a_{21} = -\frac{1}{4}a_{22}^2$.

Arbitrarily close to systems with $a_{21} = -\frac{1}{4}a_{22}^2$ there exist topologically distinct systems, since a system where A_1 has complex eigenvalues is topologically distinct from a system where A_1 has real eigenvalues. This follows from the observation that there exists a stable or unstable manifold containing E_1 if and only if A_1 has real eigenvalues. When no stable or unstable manifold exist, there exists only one trajectory converging to E_1, which emanates from S_2. This case is topologically distinct from the situation where a stable or unstable manifold exist, since in that case there exist more trajectories converging to, or emanating from this point.

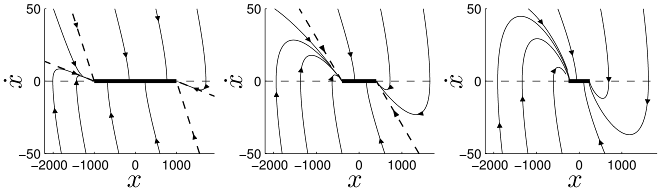
This bifurcation is illustrated with the exemplary system:

$$\ddot{x} - a_{21}x - a_{22}\dot{x} \in -F_s \mathrm{Sign}(\dot{x}), \tag{11}$$

with $F_s = 1$, $a_{22} = -0.1$ and varying a_{21}. In this example, the matrices A_1 and A_2 are equal, such that both endpoints undergo a bifurcation at the same value for a_{21}. This system shows a bifurcation when $a_{21} = -0.0025$, as shown in Fig. 3. We refer to this bifurcation as a focus-node bifurcation. According to [8], all trajectories of system (11) will arrive in the equilibrium set $\mathcal{E}$ in finite time if and only if $a_{21} < -\frac{1}{4}a_{22}^2 = -0.0025$. For $a_{21} \geq 0.0025$, the matrices A_1 and A_2 have a real eigenvector corresponding to an eigenvalue λ. The span of this eigenvector contains trajectories that converge exponentially according to $x(t) - E_i = e^{\lambda t}(x(0) - E_i),\ i = 1, 2$, which consequently does not converge in finite time. Hence, this change of the attractivity properties of the equilibrium set coincides with a bifurcation, defined using topological equivalence as used in this paper.

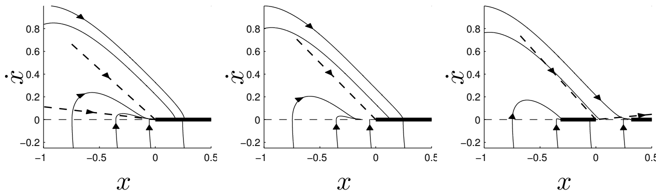
B. Zero eigenvalue

Consider system (9) where an eigenvalue of A_1 becomes zero under parameter variation, where $A_1 = \begin{pmatrix} 0 & 1 \\ a_{21} & a_{22} \end{pmatrix}$. This matrix has an eigenvalue equal to zero when $a_{21} = 0$.



(a) $a_{21}=-0.001$ (b) $a_{21}=-0.0025$ (c) $a_{21}=-0.004$

Fig. 3. System (11) with $F_s=1$ and $a_{22}=-0.1$ showing a focus-node bifurcation. The equilibrium set $\mathcal{E}$ is given by a bold line, the real eigenvectors of the stable eigenvalues of A are represented with dashed lines. The real eigenvectors are distinct for $a_{21}>-0.0025$, collide at $a_{21}=-0.0025$ and subsequently become imaginary.



(a) $a_{21}=-0.1$ (b) $a_{21}=0$ (c) $a_{21}=0.1$

Fig. 4. System (12) with $F_s=1$, $a_{22}=-1$ and varying a_{21}, showing a bifurcation where an eigenvalue becomes zero. A neighbourhood of the origin is depicted, that does not contain the complete equilibrium set. The equilibrium set $\mathcal{E}$ is given by a bold line, the eigenvectors of stable or unstable eigenvalues of $\left.\frac{\partial F_2}{\partial q}\right|_{q=0}$ are represented with dashed lines.

At the point E_1 of the equilibrium set the vector field satisfies $F_1(E_1)=0$. By definition, the point E_1 is an endpoint of $\mathcal{E}$, such that trajectories on Σ on one side of E_1 can cross Σ without sliding. This implies that the second component of F_1, denoted $F_{1,2}$, evaluated on the curve Σ changes sign at E_1. Assuming that $a_{21}=0$, since F_1 is smooth and $F_{1,2}$ changes sign at E_1, we obtain $\left.\frac{\partial^k F_{1,2}}{\partial x^k}\right|_{E_1}\neq 0$ for an odd integer $k\geq 3$, and $\left.\frac{\partial^i F_{1,2}}{\partial x^i}\right|_{E_1}=0$, for $i=1,\ldots,k-1$. The equilibrium set $\mathcal{E}$ is the subset of Σ where $F_{2,2}<0$ and $F_{1,2}>0$. The mentioned characteristics of $F_{1,2}$ imply that a small parameter change can create two distinct domains where $F_{1,2}>0$ near E_1, such that two equilibrium sets are created. Analogue observations apply to E_2.

This bifurcation is illustrated with the exemplary system:

$$\ddot{x}-a_{21}x-a_{22}\dot{x}+F_s+x^3\in -F_s\mathrm{Sign}(\dot{x}), \tag{12}$$

with $F_s=1$, $a_{22}=-1$ and varying a_{21}. The system is designed such that the origin is always the endpoint of an equilibrium set. The resulting phase portrait is given in Fig. 4, and shows the mentioned bifurcation. At the equilibrium points of this systems, the dry friction force (i.e., $F_s(1+\mathrm{Sign}(\dot{x}))$) can compensate the remaining forces (i.e., $-a_{21}x+x^3$) when $\dot{x}=0$. Hence, at the equilibrium set $-a_{21}x+x^3\in -F_s(1+\mathrm{Sign}(0))=[-2,0]$ holds, such that a bifurcation occurs when a_{21} crosses zero. For $a_{21}=-0.1$, one compact equilibrium set exists. For $a_{21}=0$, an eigenvalue of the system becomes zero and the corresponding eigenvector is parallel to the equilibrium set. Note that, in this case, both Assumptions 1 and 2(ii) are violated. For $a_{21}>0$, the equilibrium set splits in two separated equilibrium sets where $-a_{21}x+x^3\in -F_s(1+\mathrm{Sign}(0))$, cf. Fig. 4(c). Another bifurcation occurs when F_1 and F_2 are linear systems. In that case, the equilibrium set grows unbounded when $a_{21}\to 0$, and becomes the complete line Σ.

V. Illustrative Example

Theorem 1 is an important tool to study structural stability and bifurcations of higher-dimensional systems as well. This is illustrated with an observer-based control system, where a single mass is controlled using a velocity observer. The system is given by:

$$\dot{q}=Aq+B(u+f(q)), \tag{13}$$

with $q=\begin{pmatrix}x & \dot{x}\end{pmatrix}^T\in\mathbb{R}^2$, measurement $z=x$, control input u and friction force $f(q)\in -F_s\mathrm{Sign}(q_2)$, as shown schematically in Fig. 5. We assume $M=1$. The matrix A is given by $A=\begin{pmatrix}0 & 1\\ 0 & -c\end{pmatrix}$, with $c>0$, and $B=\begin{pmatrix}0 & 1\end{pmatrix}^T$. For

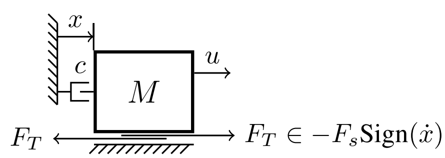


Fig. 5. Example of mechanical system subject to dry friction.

this system, a linear state feedback controller of PD-type is designed, yielding $u=k_pz+k_dy$, with proportional gain k_p, differential gain k_d, and y an estimate of the velocity $\dot{x}$. This estimate is obtained with a reduced-order observer, cf. [19], that is designed for the system without friction and given by:

$$\dot{y}=-cy+u, \tag{14}$$

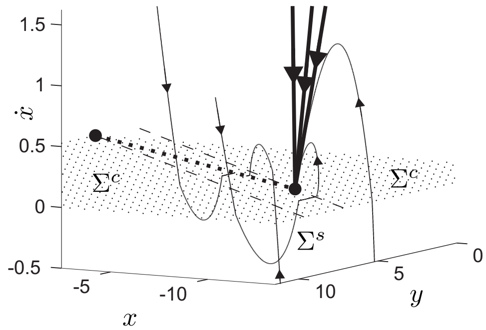
The resulting closed-loop system is given by

$$\begin{pmatrix}\dot{x}\\ \ddot{x}\\ \dot{y}\end{pmatrix}\in A_c\begin{pmatrix}x\\ \dot{x}\\ y\end{pmatrix}+\begin{pmatrix}0\\ -F_s\mathrm{Sign}(\dot{x})\\ 0\end{pmatrix}, \tag{15}$$

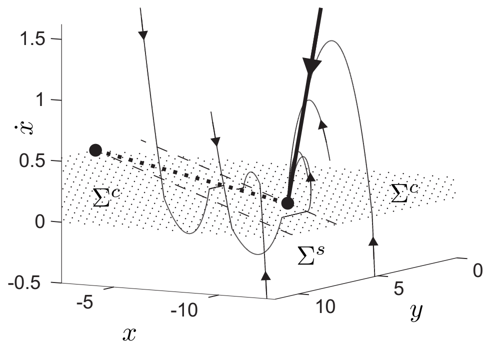
with $A_c=\begin{pmatrix}0 & 1 & 0\\ -k_p & -c & -k_d\\ -k_p & 0 & -c-k_d\end{pmatrix}$, which is equivalent with (2), where $f(x,\dot{x},y)=-k_px-c\dot{x}-k_dy$ and $g=-k_px-(c+k_d)y$. Assumption 1 implies $ck_p\neq 0$. If this is satisfied, system (15) has the equilibrium set $\{(x\ \dot{x}\ y)\in\mathbb{R}^3:(x\ \dot{x}\ y)=\alpha(-(\frac{1}{k_p}+\frac{k_d}{ck_p})\ 0\ \frac{1}{c}),\ \alpha\in[-F_s,F_s]\}$.

Since $\frac{\partial g}{\partial y}=-c-k_d$, Theorem 1 shows that when $-c-k_d\neq 0$, bifurcations can only occur at the endpoints E_1 and E_2 , given by $(x\ \dot{x}\ y)=\pm(-(\frac{1}{k_p}+\frac{k_d}{ck_p})F_s\ 0\ \frac{F_s}{c})$.

The structural stability of trajectories near the endpoints of an equilibrium set is studied in the present paper only for planar systems, while the current example is 3-dimensional. However, we will still present a bifurcation of trajectories near the endpoints. Similar to the approach used in Section IV, the linearisation of the vector field near the endpoints is used. Here, matrices A_1 and A_2 coincide with A_c, which has eigenvalues $\lambda_1=-c$ and $\lambda_{2,3}=-\frac{c+k_d}{2}\pm$



(a) $k_d = 1.6$



(b) $k_d = 1.4$

Fig. 6. System (15) with $c = 0.5$, $k_p = 1$ and $F_s = 2$, showing a bifurcation near the endpoints at $k_d = 1.5$. The equilibrium set $\mathcal{E}$ is given by a dotted line, and the real eigenvectors of A_2 are represented with thick lines.

$\frac{1}{2}\sqrt{(c+k_d)^2 - 4k_p}$. The eigenvalues $\lambda_{2,3}$ change from real to complex when $k_d = -c + 2\sqrt{k_p}$. At this point a bifurcation occurs similar to the focus-node bifurcation observed in Section IV-A. When two eigenvalues are complex, for both endpoints $E_i,\ i = 1, 2$, there exist only one trajectory that converges to the endpoints E_i from domain S_i for $t \to \infty$ or $t \to -\infty$. When eigenvalues $\lambda_{2,3}$ are real, more trajectories exist with this property. Hence, a bifurcation occurs when k_d crosses the value $-c + 2\sqrt{k_p}$. This bifurcation is illustrated in Fig. 6 where the parameters $c = 0.5,\ k_p = 1,\ F_s = 2$ are used. At these parameters, the mentioned bifurcation occurs at $k_d = 1.5$. For the used system parameters, the eigenvalues of A_c have negative real part. In Fig. 6, only trajectories near the endpoint E_2 are shown. Since the system is symmetric, the same bifurcation occurs near endpoint E_1.

These results suggest that using the linearisation of the dynamics near the endpoints, sufficient conditions for structural stability of trajectories can be constructed for higher-dimensional systems, analogously to the results in Sections III-A and IV for planar systems.

VI. Discussion

In this paper, structural stability is studied of a class of discontinuous vector fields, which show the presence of equilibrium sets. It has been shown in Theorem 1 that the structural stability of equilibrium sets of a class of nonlinear mechanical systems with a single frictional interface is determined by the trajectories near the endpoints of the equilibrium set. Hence, local techniques can be applied in a neighbourhood of these points. For differential inclusions given by (1), the linearisation of vector fields is only applicable to the part of the state space where the vector field is described by a smooth function. A careful study of this linearisation has given insight in the topological nature of solutions of the differential inclusion near the equilibrium set. Hence, in the neighbourhood of equilibrium sets the result of Theorem 1 significantly simplifies the further study of structural stability and bifurcations for this class of mechanical systems with friction. Using this approach, sufficient conditions are derived for structural stability of planar systems given by (1), restricted to a neighbourhood of the equilibrium set. The results are illustrated by application to a controlled mechanical system with friction.

References

[1] R. H. A. Hensen, M. J. G. van de Molengraft, and M. Steinbuch, “Friction induced hunting limit cycles: A comparison between the LuGre and switch friction model,” *Automatica*, vol. 39, no. 12, pp. 2131–2137, 2003.

[2] D. Putra, H. Nijmeijer, and N. van de Wouw, “Analysis of undercompensation and overcompensation of friction in 1DOF mechanical systems,” *Automatica*, vol. 43, no. 8, pp. 1387–1394, 2007.

[3] G. Morel, K. Iagnemma, and S. Dubowsky, “The precise control of manipulators with high joint-friction using base force/torque sensing,” *Automatica*, vol. 36, no. 7, pp. 931 – 941, 2000.

[4] N. Mallon, N. van de Wouw, D. Putra, and H. Nijmeijer, “Friction compensation in a controlled one-link robot using a reduced-order observer,” *IEEE Trans. Control Syst. Technol.*, vol. 14, no. 2, pp. 374–383, 2006.

[5] C. Glocker, *Set-valued force laws*. Springer, Berlin, 2001.

[6] V. Yakubovich, G. Leonov, and A. Kh. Gelig, *Stability of stationary sets in control systems with discontinuous nonlinearities*. World Scientific, Singapore, 2004.

[7] R. I. Leine and N. van de Wouw, “Stability properties of equilibrium sets of non-linear mechanical systems with dry friction and impact,” *Nonlinear Dyn.*, vol. 51, no. 4, pp. 551–583, 2008.

[8] A. Cabot, “Stabilization of oscillators subject to dry friction: finite time convergence versus exponential decay results,” *Trans. Amer. Math. Soc.*, vol. 360, no. 1, pp. 103–121, 2008.

[9] Q. Hui, W. M. Haddad, and S. P. Bhat, “Semistability, finite-time stability, differential inclusions, and discontinuous dynamical systems having a continuum of equilibria,” *IEEE Trans. Autom. Control*, vol. 54, no. 10, pp. 2465–2470, 2009.

[10] A. P. Ivanov, “Bifurcations in systems with friction: Basic models and methods,” *Regul. Chaotic Dyn.*, vol. 14, no. 6, pp. 656–672, 2009.

[11] M. di Bernardo, C. J. Budd, A. R. Champneys, P. Kowalczyk, A. B. Nordmark, G. O. Tost, and P. T. Piiroinen, “Bifurcations in nonsmooth dynamical systems,” *SIAM Rev.*, vol. 50, no. 4, pp. 629–701, 2008.

[12] M. Guardia, T. Seara, and M. Teixeira, “Generic bifurcations of low codimension of planar Filippov systems,” *J. Differential Equations*, vol. 250, no. 4, pp. 1967–2023, 2011.

[13] A. F. Filippov, *Differential equations with discontinuous righthand sides*. Kluwer Academic, Dordrecht, 1988.

[14] J. Guckenheimer and P. Holmes, *Nonlinear oscillations, dynamical systems and bifurcations of vector fields*. Springer, New York, 1983.

[15] J. J. B. Biemond, N. van de Wouw, and H. Nijmeijer, “Bifurcations of equilibrium sets in mechanical systems with dry friction,” *Physica D, in Press*, 2011. [Online]. Available: http://dx.doi.org/10.1016/j.physd.2011.05.006

[16] J. Sotomayor, “Structurally stable second order differential equations,” in *Differential equations*. Springer, Berlin, 1982, pp. 284–301.

[17] H. Olsson, K. Åström, C. Canudas De Wit, M. Gäfvert, and P. Lischinsky, “Friction models and friction compensation,” *Eur. J. Control*, vol. 4, no. 3, pp. 176–195, 1998.

[18] Yu. A. Kuznetsov and S. Rinaldi and A. Gragnani, “One-parameter bifurcations in planar Filippov systems,” *Internat. J. Bifur. Chaos*, vol. 13, no. 8, pp. 2157–2188, 2003.

[19] A. Gelb, Ed., *Applied optimal estimation*. MIT Press, Cambridge, 1974.