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New Stability Criteria for Switched Time-Varying Systems: Output-Persistently Exciting Conditions

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Abstract—This paper proposes three tools to facilitate the verification of the output-persistently exciting (OPE) condition and simultaneously, provides new asymptotic stability criteria for uniformly globally stable switched systems. By introducing some related reference systems, the OPE condition of the original system can be reduced or simplified. Both the ideas of classic LaSalle invariance principle and nested Matrosov theorem are used to generate such reference systems. The effectiveness and flexibility of the proposed methods are demonstrated by two applications. From these applications, it can be seen that the flexibility of the proposed method produces a novel set of tools for checking uniform asymptotic stability of switched time-varying systems.

I. Introduction

This paper presents some new tools to help the verification of the output-persistently exciting (OPE) condition and simultaneously, provides new stability criteria for uniformly globally stable (UGS) switched systems. The OPE condition is a generalization of weak zero-dectability [8] and δ-PE condition [14, 15], which are widely used to ensure uniform attractivity of nonlinear time-varying systems [5, 7-10, 14-15].

Stability analysis of switched systems has been a popular research area [1-2, 4, 7, 9, 11-12, 16]. Due to the complex behaviors of switched systems, the traditional Lyapunov functions based theory is often not very effective for switched systems [1, 11]. Several possible extensions have been proposed to help stability analysis, for instance, the use of multiple Lyapunov functions [1, 2], the extension of LaSalle invariance principle [4, 7, 12] and a type of Matrosov theorem [16]. It is worth to notice that the notion of output-persistent excitation (OPE) was proposed in [9] and used to derive a generalized Krasovskii-LaSalle theorem. This paper further contributes to stability of switched systems based on three novel tools for the verification of the OPE condition.

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As discussed in [9], OPE plays an important role when we want to verify uniform global asymptotic stability (UGAS) for a UGS switched system. In fact, it is possible to show that OPE is a necessary condition to ensure uniform global attractivity of the original system. Furthermore, it can be used as a sufficient condition to guarantee uniform global attractivity under a (necessary) assumption for UGS switched systems. Therefore, it is important to investigate the OPE condition.

In general, it is hard to verify the OPE condition directly by its definition. To overcome this difficulty, the key technique used in this paper is to provide some easily checked conditions so that we can guarantee the OPE condition of the original system by verifying *the OPE condition of a reduced and simpler reference system*. Moreover, by continuing this process, a sequence of reference systems can be generated such that the OPE condition of the next system is more easily verified than the former. As soon as the OPE condition for one of reference systems is verified, the OPE condition of the original system can then be guaranteed.

More precisely, a stability criterion proposed in [9] is revisited, see Theorem 1 below. Then, three different ways are proposed to generate reference systems. First, a reference system is generated by keeping the same dynamics while changing the output function. Second, we provide a reference system by simplifying the dynamics of the system while keeping the same output. These methods are proposed in the spirit of LaSalle Invariance Principle. Third, a reference system can be obtained by augmenting output functions and keeping the same dynamics. This is consistent with the insight provided by the well-known Matrosov-type Theorem [14-16]. Two illustrative examples show the effectiveness of the proposed methods. From these applications, it can be seen that the proposed framework provides alternative ways to guarantee the UGAS property for UGS switched systems.

This paper is organized as follows. Preliminary results are provided in Section II, including a stability criterion proposed in [9]. In Section III, three tools are proposed and they are followed by two examples in Section IV. Section V summarizes this work.

Notations

1. Throughout this paper, Λ denotes a finite set and $R_+ = [0, \infty)$.

2. For any function $g: R_+ \times R^p \times \Lambda \to R^q$ and any $\upsilon \in \Lambda$, denote $g_\upsilon(t,x) = g(t,x,\upsilon), \forall t \ge 0, x \in R^p$.

3. For any function $\rho: R_+ \times R^p \to R^q$, let $\nabla_t \rho(t, x_1, x_2, \cdots, x_p)$ and $\nabla_{x_i} \rho(t, x_1, x_2, \cdots, x_p), 1 \le i \le p$, denote the partial derivatives of ρ w. r. t. the parameters t and x_i, respectively. Moreover, let

$$\nabla_x \rho = (\nabla_{x_1}\rho, \nabla_{x_2}\rho, \cdots, \nabla_{x_p}\rho), \forall x = (x_1, x_2, \cdots, x_p)^T \in R^p.$$

4. A function $g: R_+ \times R^p \to R^q$ is said to be uniformly bounded if for any $r_1 > 0$, there exists a $r_2 > 0$ such that $\|g(t,x)\| \le r_2, \forall t \ge 0, \forall \|x\| \le r_1$.

5. A function $g: R_+ \times R^p \to R^q$ is said to satisfy the local Caratheodory condition if the following hold:

a) For any x in R^p, $g(\cdot, x)$ is measurable.

b) For almost all t in R_+, $g(t,\cdot)$ is continuous.

c) For any closed interval $I = [a,b] \subseteq R_+$ and any compact subset $K \subset R^p$, there is a Lebesgue integrable function $\rho_{I,K}: I \to R_+$ such that

$$\|g(t,x)\| \le \rho_{I,K}(t), \forall t \in I, \forall x \in K.$$

II. Preliminaries

A. The OPE condition and uniform global asymptotic stability

In this subsection, we revisit some necessary notations and a key stability result presented in [9]. We refer readers to that paper for more details.

Throughout this paper, we study a switched time-varying system described as follows:

$$\dot{x} = f(t,x,\lambda) \quad (1)$$

$$y = h(t,x,\lambda) \quad (2)$$

where $t \in R_+$, $x \in R^p$ is the state vector and λ is a Λ-valued switching signal; $h: R_+ \times R^p \times \Lambda \to R^q$ is an output function and $f: R_+ \times R^p \times \Lambda \to R^p$ is a function with the property that for each $\upsilon \in \Lambda, f_\upsilon(\cdot,\cdot) = f(\cdot,\cdot,\upsilon)$ satisfies the local Caratheodory condition.

Let Φ denote a set of pairs (x,λ) with $\lambda: [t_0, \infty) \to \Lambda$ being a switching signal and x, starting at $t = t_0 \ge 0$, being a complete solution of (1) w. r. t. λ. For convenience, we denote the initial time instant t_0 as $t_0(x)$. Moreover, a time instant $t > t_0(x)$ is said to be a jump time if $\lim_{\tau \to t^-} \lambda(\tau) \ne \lambda(t)$. Following [9], we denote

$$\Phi^{st} = \{x \mid (x,\lambda) \in \Phi \ \ for\ some\ \ \lambda\}$$

and $$\Phi^{sw} = \{\lambda \mid (x,\lambda) \in \Phi\ for\ some\ x\}.$$

Several definitions of stability are recalled as follows.

Definition 1. a) The origin is said to be *uniformly Lyapunov stable* (ULS) w. r. t. Φ if, for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that for any $x \in \Phi^{st}$ with $\|x(s)\| < \delta$, we have $\|x(t)\| < \varepsilon$ for any $t_0(x) \le s \le t$.

b) All solutions are *uniformly globally bounded* (UGB) w. r. t. Φ if, for any $M > 0$ there exists a $\widetilde{M}(M) > 0$ such that for any $x \in \Phi^{st}$ with $\|x(s)\| < M$, we have $\|x(t)\| < \widetilde{M}$ for any $t_0(x) \le s \le t$.

c) The origin is said to be *uniformly globally stable* (UGS) w. r. t. Φ if, it is ULS w. r. t. Φ, and all solutions are UGB w. r. t. Φ.

d) The origin is said to be *uniformly globally attractive* (UGA) w. r. t. Φ if for any $\varepsilon > 0$ and any $r > 0$, there exists a $T(\varepsilon, r) > 0$ such that for any $x \in \Phi^{st}$ and any $s \ge t_0(x)$ with $\|x(s)\| \le r$, we have $\|x(t)\| < \varepsilon$ for any $t \ge s + T$.

e) The origin is said to be *uniformly globally asymptotically stable* (UGAS) w. r. t. Φ if, it is UGS and UGA w. r. t. Φ.

In this paper, the following definition related to persistent excitation (PE) condition plays a central role

Definition 2. The pair (h,f) is *output-persistently exciting* (OPE) w. r. t. Φ if, for any $a > 0$ and any $b > a$, there exist two positive constants $T(a,b)$ and $r(a,b)$ such that for any $(x,\lambda) \in \Phi$ and any $t \ge t_0(x)$, the following implication holds:

$$a \le \|x(\tau)\| \le b, \forall t \le \tau \le t+T, \Rightarrow \int_t^{t+T} \|h(\tau, x(\tau), \lambda(\tau))\|^2 d\tau \ge r. \quad (3)$$

To state the main result presented in [9], the following assumption is needed.

Assumption 1. For any $0 < a < b$ and any $c > 0$, there exists a positive constant $M(a,b,c)$ such that for any $(x,\lambda) \in \Phi$ and any $t_0(x) \le s \le t$, with $a \le \|x(\tau)\| \le b$, $\forall s \le \tau \le t$, the following integral inequality holds:

$$\int_s^t \|h(\tau, x(\tau), \lambda(\tau))\|^2 d\tau \le M + c(t-s). \quad (4)$$

The following result presented in Theorem 1 of [9] is recalled. It can be viewed as a generalized Krasovskii-LaSalle theorem, see more detailed discussions in that paper.

Theorem 1. Consider the switched time-varying system (1)-(2). Let Φ denote a set of pairs (x,λ) with λ being a switching signal and x being a complete solution of (1) w. r. t. λ. Suppose the origin is UGS w. r. t. Φ and Assumption 1 holds. If the pair (h,f) is OPE w. r. t. Φ, then the origin is UGAS w. r. t. Φ.

B. Zeroing and nonpositive pairs

In this subsection, we give some definitions and results that are needed in next section. First, the following definition characterizes some relationships between two functions.

Definition 3. Let $h: R_+ \times R^p \to R^q$ and $\hat{h}: R_+ \times R^p \to R^{\hat{q}}$ be given.

a) The pair $(h,\hat{h})$ is said to be a nonpositive pair if, for any time sequence $t_n \to \infty$, any constants $0<a<b$ and any sequence $\{x_n\} \subset R^p$, with $a \le \|x_n\| \le b$, $\forall n \in N$, the following implication holds:

$$\lim_{n\to\infty} h(t_n,x_n)=0 \Rightarrow \limsup_{n\to\infty} \hat{h}_i(t_n,x_n) \le 0, \forall 1 \le i \le \hat{q}, \quad (5)$$

where $\hat{h}_i$ is the i^th component of $\hat{h}$.

b) The pair $(h,\hat{h})$ is said to be a zeroing pair if, for any time sequence $t_n \to \infty$, any constants $0<a<b$ and any sequence $\{x_n\} \subset R^p$, with $a \le \|x_n\| \le b$, $\forall n \in N$, the following implication holds:

$$\lim_{n\to\infty} h(t_n,x_n)=0 \Rightarrow \lim_{n\to\infty} \hat{h}(t_n,x_n)=0. \quad (6)$$

As will be shown later, both nonpositive pair and zeroing pair will play an important role in verifying the OPE condition. The following result presents some related properties. The proof is omitted to save space.

Proposition 1. Let $h_1: R_+ \times R^p \to R^{q_1}$, $h_2: R_+ \times R^p \to R^{q_2}$ and $h_3: R_+ \times R^p \to R^{q_3}$ be three functions. Then, the following conditions hold.

a) **(Transitivity)** Assume that (h_1,h_2) is a zeroing pair. If (h_2,h_3) is a zeroing (nonpositive) pair, then (h_1,h_3) is also a zeroing (nonpositive) pair.

b) Suppose $q_2 = q_3$ and for any $0<a<b$, there is a function $\alpha_{a,b}: R^{q_1} \to R_+$, zero at zero and continuous at zero, such that $\|h_2(t,x)-h_3(t,x)\| \le \alpha_{a,b}(h_1(t,x))$, $\forall t \ge 0$, $\forall a \le \|x\| \le b$. Then, (h_1,h_2) is a zeroing (nonpositive) pair if and only if (h_1,h_3) is a zeroing (nonpositive) pair.

c) Assume that $q_1 = 1$, h_1 is uniformly bounded, and (h_1,h_3) and (h_2,h_3) are both zeroing (nonpositive) pairs. Then $(h_1 h_2, h_3)$ is also a zeroing (nonpositive) pair. □

III. Verifying the OPE Condition: Three Tools

In view of Theorem 1, we need to check three things to verify UGAS of the origin for switched time-varying system: (1) uniform global stability, (2) Assumption 1 and (3) the OPE condition. Generally speaking, it is possible to verify the first two conditions through some Lyapunov-like functions as shown in [9]. Therefore, checking the OPE condition becomes crucial. In that paper, several criteria have been proposed to check the OPE condition. In this section, three different tools will be proposed to verify the OPE condition. Then, two interesting examples will be presented in Section IV to illustrate the merits of the proposed new tools.

A. The first tool: Changing output functions

In this subsection, the first tool is presented. It shows that by changing the output function in (2), the OPE condition of the original system can be verified by checking the OPE condition of a "new" system with the new output.

To this end, we need the following definition.

Definition 4. Let $h: R_+ \times R^p \to R^q$ be given. It is said to be almost uniformly bounded if for any $b>a>0$, there exist a positive constant $\eta(a,b)$ and a measure zero subset $S_{a,b}$ of R_+ such that $\|h(t,x)\| \le M(a,b)$, $\forall t \in R_+ - S_{a,b}$, $\forall a \le \|x\| \le b$.

The following result provides the first tool for the verification of OPE conditions. To save space, the proof is omitted.

Theorem 2. (*Changing output functions*) Consider the switched time-varying system (1)-(2) and a set Φ of pairs (x,λ) where x is a complete solution of (1) w. r. t. λ. Let $\hat{h}: R_+ \times R^p \times \Lambda \to R^{\hat{q}}$ be any function. Suppose for each $\upsilon \in \Lambda$, $(h_\upsilon,\hat{h}_\upsilon)$ is a zeroing pair, $\hat{h}_\upsilon$ satisfies the local Caratheodory condition and is almost uniformly bounded. Then, (h,f) is OPE w. r. t. Φ provided that $(\hat{h},f)$ is OPE w. r. t. Φ. □

B. The second tool: Simplifying the dynamics of systems

Theorem 2 shows how to generate a reference system by selecting different output functions. In this subsection, another tool is proposed to simplify the dynamics of a system. Let

$$\overline{\Phi} = \left\{(c,\lambda) \middle| \, c \in R^p, \lambda \in \Phi^{sw} \right\}, \quad (7)$$

where for any $(c,\lambda) \in \overline{\Phi}$, c is viewed as a solution of $\dot{x}=0$ w. r. t. λ. Now, the following criterion is stated. Its proof is omitted to save space.

Theorem 3. (*Simplifying the dynamics*) Consider the switched time-varying system (1)-(2) and a set Φ of pairs (x,λ) where x is a complete solution of (1) w. r. t. λ. Let $\overline{\Phi}$ be the set defined in (7). Suppose that for each $\upsilon \in \Lambda$, (h_υ, f_υ) is a zeroing pair, $(h_\upsilon^T, f_\upsilon^T)^T$ is almost uniformly bounded, h_υ satisfies the local Caratheodory condition and for all $x \in R^p, t \in R_+$, $h_\upsilon(t,x)$ is continuous in x, uniformly in t. Then, (h,f) is OPE w. r. t. Φ provided that $(h,0)$ is OPE w.r.t. $\overline{\Phi}$ where $0: R_+ \times R^p \times \Lambda \to R^p$ represents the zero function. □

C. The third tool: Extending output functions

Besides changing output functions, it is possible to generate a reference system by augmenting outputs functions. This follows the same idea as the Matrosov-type theorems [14-16].

Consider the switched time-varying system (1)-(2) and the following condition where a function $g: R_+ \times R^p \times \Lambda \to R^{\hat{q}}$ is said to be almost uniformly bounded and continuously differentiable if for each $\upsilon \in \Lambda$, g_υ is almost uniformly bounded and continuously differentiable.

(C1) Suppose there exist a uniformly bounded and continuously differentiable function $V: R_+ \times R^p \times \Lambda \to R$, and an almost uniformly bounded function $\hat{h}: R_+ \times R^p \times \Lambda \to R$ such that the following conditions hold.

a) For any $(x,\lambda) \in \Phi$ and any jump time *t*, we have

$$V(t, x(t), \lambda(t)) \le \lim_{\tau \to t^-} V(\tau, x(\tau), \lambda(\tau)). \tag{8}$$

b) The following inequality holds:

$$\dot{V}(t,x,\upsilon) = \nabla_t V_\upsilon(t,x) + \nabla_x V_\upsilon(t,x) \le \hat{h}(t,x,\upsilon), \quad \forall t \ge 0, \forall x \in R^n, \forall \upsilon \in \Lambda. \tag{9}$$

c) For any $\upsilon \in \Lambda$, the pair $(h_\upsilon, \hat{h}_\upsilon)$ is a nonpositive pair.

Now, the following theorem can be proposed. Its proof is omitted

Theorem 4. (*Extending output functions*) Consider the switched time-varying system (1)-(2) and a set Φ of the pairs (x,λ) where x is a complete solution of (1) w. r. t. λ. Let $\widetilde{h} = (h^T, \sqrt{\|\dot{V}\|})^T$. Suppose (C1) holds. Then, (h, f) is OPE w. r. t. Φ provided that $(\widetilde{h}, f)$ is OPE w. r. t. Φ. □

Remark 1. Suppose $f \equiv 0$. In this case, several assumptions could be relaxed. For example, the assumption that V is continuously differentiable can be relaxed as V is uniformly bounded and $\nabla_t V$ exists and is bounded. □

Remark 2. Theorem 4 can be recursively used to derive a "nested Matrosov theorem." Due to limited space, we omit the detailed discussion and refer readers to [10] for some related discussion, also see [14-16]. □

In addition to some regularity assumption, the following result is obtained.

Corollary 1. Consider the switched time-varying system (1)-(2) and a set Φ of pairs (x,λ) where x is a complete solution of (1) w. r. t. λ. Suppose (C1) holds. Let $\overline{h} = (h^T, \hat{h})^T$. If for each $\upsilon \in \Lambda$, $\overline{h}_\upsilon$ satisfies the local Caratheodory condition and is almost uniformly bounded, then (h, f) is OPE w. r. t. Φ provided that $(\overline{h}, f)$ is OPE w. r. t. Φ. □

IV. Examples

Two examples are presented to demonstrate the use of the developed new tools. The first example considers a time-varying system. It can also be treated as a special case of a switched system with a single point index set. This example demonstrates how to generate appropriate reference systems using well selected Lyapunov-like functions to guarantee UGAS. The second example is used to illustrate how to obtain stability results to systems with arbitrary switching case. For this case, it is in general difficult to use LaSalle-type invariance principle. By constructing a sequence of reference systems, it is not necessary to search for common Lyapunov functions that are not easily found for the considered systems.

A. The first example

A third order system is considered as follows:

$$\begin{aligned} \dot{z}_1 &= \kappa(t, z_1, z_2, z_3) \\ \dot{z}_2 &= z_3 \kappa(t, z_1, z_2, z_3) \\ \dot{z}_3 &= -z_2 \kappa(t, z_1, z_2, z_3) - z_3 \end{aligned} \tag{10}$$

where $\kappa = -z_1 + \alpha(t, z_2, z_3)$ with $\alpha: R_+ \times R^2 \to R$ being continuously differentiable and satisfying the following regularity assumption:

(C2) Suppose $\alpha(t,0,0) = 0$, $\forall t \ge 0$, and $(\alpha, \nabla_t \alpha, \nabla_t \nabla_t \alpha, \nabla_{z_2} \alpha, \nabla_{z_3} \alpha)$ are uniformly bounded.

System (10) represents a closed-loop system of a 3rd order chained-form system and was studied in [14] to illustrate the use of the nested Matrosov theorem. Here, each of our main results is applied to show uniform global attractivity of the system. In the remainder of this subsection, Φ denotes the set of all complete solutions of (10). The following steps are needed.

- *Step1: Let* $V_1 = (z_2^2 + z_2^2)/2$. Differentiating V_1 along the trajectories of (10), we get $\dot{V}_1 = -z_3^2 \le 0$. Using Condition (C2), it is not difficult to check that uniform global stability holds [5, 14]. Particularly, Φ consists of all solutions of (10). Integrating the two sides of $\dot{V}_1 = -z_3^2$, it can be seen that Assumption 1 holds with the virtual output function being chosen as $h = z_3$. To apply Theorem 1, the OPE condition is needed to guarantee UGAS of the origin.

The proposed new tools will be used to check the OPE conditions by generating a sequence of reference systems. In this example, all involved output functions will satisfy the local Caratheodory condition and be uniformly bounded under condition (C2). Thus, the required regularity assumptions hold. Let

$$f = (\kappa(t, z_1, z_2, z_3), z_3 \kappa(t, z_1, z_2, z_3), -z_2 \kappa(t, z_1, z_2, z_3) - z_3)^T$$

- *Step 2: Extending the output function* z_3 *to* $(z_3, z_2 \kappa)^T$. Take $V_2 = z_3 z_2 \kappa(t, z_1, z_2, z_3)$. Differentiating V_2 along the trajectories of (10), we get

$$\dot{V}_2 = -[z_2 \kappa(t, z_1, z_2, z_3)]^2 + z_3 \eta(t, z_1, z_2, z_3) \tag{11}$$

for some uniformly bounded function η. It is trivial that $(z_3, -z_2^2 \kappa^2)$ is a nonpositive pair. Since η is uniformly bounded, we have that for any $b > 0$, there is a positive constant M_b satisfying $\|\dot{V}_2(t,z) - (-z_2^2 \kappa^2)\| \le M_b \|z_3\|$ for any $\|z\| = \|(z_1, z_2, z_3)^T\| \le b$. By (b) of Proposition 1, $(z_3, \dot{V}_2)$ is also a nonpositive pair Thus, (C1) holds. According to Corollary 1, it is sufficient to show that $((z_3, \dot{V}_2)^T, f)$ is OPE.

Again by (b) of Proposition 1, $((z_3,\dot{V}_2)^T,(z_3,z_2^2\kappa^2)^T)$ is a zeroing pair. It is straightforward to see that $((z_3,z_2^2\kappa^2)^T,(z_3,z_2\kappa)^T)$ is a zeroing pair. Thus, $((z_3,\dot{V}_2)^T,(z_3,z_2\kappa)^T)$ is a zeroing pair according to (a) of Proposition 1. Employing Theorem 2, the output function can then be changed as $(z_3,z_2\kappa)^T$.

• *Step 3: Changing the output function* $(z_3,z_2\kappa)^T$ *to* $(z_3,\kappa)^T$. Choosing $V_3=z_1^2/2$ and differentiating V_3 along the trajectories of (10), we get

$$\dot{V}_3=z_1\kappa(t,z_1,z_2,z_3)) \tag{12}$$

It is claimed that $((z_3,z_2\kappa)^T,\dot{V}_3)=((z_3,z_2\kappa)^T,z_1\kappa)$ is a nonpositive pair. First, it is easy to see that $(\kappa,z_1\kappa)$ is a zeroing pair and particularly, it is a nonpositive pair. Moreover,

$$((z_3,z_2)^T,z_1\kappa)=((z_3,z_2)^T,-z_1^2+\alpha(t,z_2,z_3)z_1)$$

is a nonpositive pair by (C2) and (b) of Proposition 1. Thus, $((z_3\kappa,z_2\kappa)^T,z_1\kappa)$ is a nonpositive pair using (c) of Proposition 1. By definition, it is straightforward to see that $((z_3,z_2\kappa)^T,(z_3\kappa,z_2\kappa)^T)$ is a zeroing pair. According to (a) of Proposition 1, $((z_3,z_2\kappa)^T,z_1\kappa)$ is also a nonpositive pair. So the claim is true and hence (C1) holds. By Corollary 1, it is sufficient to show that the pair $((z_3,z_2\kappa,z_1\kappa)^T,f)$ is OPE. It is easy to verify that $((z_3,z_2,z_1)^T,\kappa)$ and (κ,κ) are both zeroing pairs by (C2) and (b) of Proposition 1. Thus, $((z_3\kappa,z_2\kappa,z_1\kappa)^T,\kappa)$ is a zeroing pair in view of (c) of Proposition 1. By definition, $((z_3,z_2\kappa,z_1\kappa)^T,(z_3\kappa,z_2\kappa,z_1\kappa)^T)$ is a zeroing pair. Therefore $((z_3,z_2\kappa,z_1\kappa)^T,\kappa)$ is also a zeroing pair employing (a) of Proposition 1. This results that $((z_3,z_2\kappa,z_1\kappa)^T,(z_3,\kappa)^T)$ is a zeroing pair. Based on Theorem 2, the output function can be changed as $(z_3,\kappa)^T$.

• Step *4: Simplifying the dynamics as* $\dot{z}=0$ *and changing the output function* $(z_3,\kappa)^T$ *to* $(z_3,-z_1+\alpha(t,z_2,0))^T$. Notice that $((z_3,\kappa)^T,f)$ is a zeroing pair. By Theorem 3, we only need to show that $((z_3,\kappa)^T,0)$ is OPE. According to (C2) and (b) of Proposition 1, $((z_3,\kappa)^T,(z_3,-z_1+\alpha(t,z_2,0))^T)$ is a zeroing pair. Therefore, we may change the output function as $(z_3,-z_1+\alpha(t,z_2,0))^T$ by Theorem 2.

• *Step 5: Extending* $(z_3,-z_1+\alpha(t,z_2,0))^T$ *to the following output function:*

$$\hat{h}=(z_3,-z_1+\alpha(t,z_2,0),\nabla_t\alpha(t,z_2,0))^T. \tag{13}$$

Define $V_4=(z_1-\alpha(t,z_2,0))\nabla_t\alpha(t,z_2,0)$ and differentiate V_4 along with $\dot{z}=0$. Then,

$$\dot{V}_4=-\|\nabla_t\alpha(t,z_2,0)\|^2+(z_1-\alpha(t,z_2,0))\nabla_t\nabla_t\alpha(t,z_2,0).$$

Using a similar argument as in Step 2, it can be checked that $((z_3,-z_1+\alpha(t,z_2,0))^T,\dot{V}_4)$ is a nonpositive pair. By Corollary 1, Theorem 2 and changing the output function $(z_3,-z_1+\alpha(t,z_2,0),\dot{V}_4)^T$ as the function $\hat{h}$ defined in (13), we only need to show that $(\hat{h},0)$ is OPE.

• *Step 6: Using a reduced PE condition to check the original OPE condition.* In the following, let us show that the required OPE condition can be deduced by the following PE condition.

(C3) For any constants $0<a<b$, there exist a $T(a,b)>0$ and a $r(a,b)>0$ such that

$$a\le\|\zeta\|\le b\Rightarrow\int_t^{t+T}\|\nabla_t\alpha(\tau,\zeta,0)\|^2d\tau\ge r,\forall t\ge0,\forall\zeta\in R. \tag{14}$$

Indeed, for any constants $0<\hat{a}<\hat{b}$, we can choose a small positive constant $a\le\hat{a}/2$ such that $|\alpha(t,\zeta,0)|\le\hat{a}/2$, $\forall\|\zeta\|\le a$, in view of (C1). Let

$\hat{r}(\hat{a},\hat{b})=\min(r(a,\hat{b}),\hat{a}^2T(a,\hat{b})/4,\hat{a}^2(\sqrt{2}-1)^2T(a,\hat{b})/4)>0$.

It can be checked that with $\|(z_1,z_2,z_3)^T\|\le\hat{b}$, if $\|z_3\|\ge\hat{a}/2$ or $\|z_2\|\ge a$, then the following inequality holds:

$$\int_t^{t+T}\left\|\hat{h}(\tau,z_1,z_2,z_3)\right\|^2d\tau\ge\min(r(a,\hat{b}),\hat{a}^2T(a,\hat{b})/4)\ge\hat{r}(\hat{a},\hat{b}),\forall t\ge0,$$

according to (C3). So we may assume that $\|z_3\|<\hat{a}/2$ and $\|z_2\|<a\le\hat{a}/2$. In this case, if $\hat{a}\le\|(z_1,z_2,z_3)^T\|\le\hat{b}$, we have

$$\|z_1\|\ge\hat{a}\sqrt{2}/2 \text{ and } \|z_1-\alpha(\tau,z_2,0)\|\ge\hat{a}(\sqrt{2}-1)/2,$$

by the choice of the constant *a*. Consequently,

$$\int_t^{t+T}\left\|\hat{h}(\tau,z_1,z_2,z_3)\right\|^2d\tau\ge\hat{a}^2(\sqrt{2}-1)^2T(a,\hat{b})/4\ge r(\hat{a},\hat{b}),\forall t\ge0.$$

Therefore $(\hat{h},0)$ is OPE under (C3).

Hence, we have the following result.

Proposition 2. Consider the third order system (10). Suppose Conditions (C2) and (C3) hold. Then, the origin is uniformly globally asymptotically stable. □

Remark 3. The stability analysis of (10) can also be done using two approaches. One is based on the nested Matrosov theorem proposed in [14] where the main idea is to extend output functions such that they are more and more approaching to "being negative definite." Another one is applying a generalized Krasovskii-LaSalle theorem presented in [8]. This approach is to reduce "the size or dimension of the invariant set" to the zero. The former needs to employ some nontrivial Lyapunov functions (see Equations (72)-(73) in that paper), while the latter requires a limiting process (the so-called limiting system) to guarantee UGAS. Based on our approach, different tools, such as changing output functions, extending output functions and simplifying the dynamics of systems, are used to check the OPE condition. These methods combine the ideas of Krasovskii-LaSalle theorem with the classic Matrosov theorem, but in a hybrid way and unified manner. Based on this example, we demonstrate that our methods can avoid using complex Lyapunov functions and need no limiting systems to guarantee UGAS. □

B. The second example

Next, the following switched time-varying system is considered:

$$\begin{aligned}\Sigma_1: \quad & \dot{z}_1 = \alpha_1(t,z_1,z_2)z_2 - z_1 \\ & \dot{z}_2 = -\alpha_1(t,z_1,z_2)z_1 \\ \Sigma_2: \quad & \dot{z}_1 = -\alpha_2(t,z_1,z_2)z_2 \\ & \dot{z}_2 = \alpha_2(t,z_1,z_2)z_1 - z_2\end{aligned} \tag{15}$$

where $\alpha_i : R_+ \times R^2 \to R$, $i = 1,2$, are continuously differentiable functions with $(\alpha_i, \nabla_t \alpha_i, \nabla_{z_1}\alpha_i, \nabla_{z_2}\alpha_i)$ being uniformly bounded. First consider the following condition.

(C4) There is a continuously differentiable function $\gamma : R_+ \times R^2 \to R$, with $(\gamma, \nabla_t\gamma, \nabla_{z_1}\gamma, \nabla_{z_2}\gamma)$ being uniformly bounded and such that $\gamma\alpha_i$, $i = 1,2$, are nonpositive functions, and (γ, α_i), $i = 1,2,$ are zeroing pairs.

If (C4) holds, let Φ denote the set of all pairs (z, λ) where z is a complete solution of (15) w. r. t. λ with $\lambda : [t_0(z), \infty) \to \{1,2\}$ being a piecewise constant function. Otherwise, let Φ denote the set of all pairs (z, λ) where $z = (z_1, z_2)^T$ is a complete solution of (15) w.r.t. λ with $\lambda : [t_0(z), \infty) \to \{1,2\}$ being a piecewise constant function satisfying $z_1(t)z_2(t) = 0$ at any jump time t. Since arbitrary switching signals are considered, it is reasonable to assume that the individual systems are stable [11]. Notice that $\Sigma_i, i = 1,2,$ are well-studied systems. For such systems, a necessary and sufficient condition of achieving uniform global asymptotic stability is the following PE condition.

(C5) Let $\hat{\alpha}_i, i = 1,2,$ be defined as

$$\hat{\alpha}_i(t,\zeta) = \begin{cases}\alpha_1(t,0,\zeta), \forall t \in R_+, \forall \zeta \in R, if\ i = 1, \\ \alpha_2(t,\zeta,0), \forall t \in R_+, \forall \zeta \in R, if\ i = 2.\end{cases} \tag{16}$$

Assume that $\hat{\alpha}_i, i = 1,2,$ satisfy the following PE condition: For any constants $0 < a < b$, there exist a $T(a,b) > 0$ and a $r(a,b) > 0$ such that

$$a \le \|\zeta\| \le b \Rightarrow \int_t^{t+T} \|\hat{\alpha}_i(\tau,\zeta)\|^2 d\tau \ge r, \forall t \in R_+, \forall \zeta \in R. \tag{17}$$

In the following, we would like to show that the origin is UGAS w. r. t. Φ under (C5). First, let us denote the system function $f : R_+ \times R^2 \times \{1,2\} \to R^2$ as

$$f(t,z_1,z_2,1) = (\alpha_1(t,z_1,z_2)z_2 - z_1, -\alpha_1(t,z_1,z_2)z_1)^T,$$
$$f(t,z_1,z_2,2) = (-\alpha_2(t,z_1,z_2)z_2, \alpha_2(t,z_1,z_2)z_1 - z_2)^T,$$

for all $t \ge 0$, $z_1, z_2 \in R$. Choose the Lyapunov function $V = (z_1^2 + z_2^2)/2$. Then, we have $\dot{V}\big|_{\Sigma_1} = -z_1^2 \le 0$ and $\dot{V}\big|_{\Sigma_2} = -z_2^2 \le 0$. This implies that the origin is UGS w. r. t. Φ and simultaneously, Assumption 1 holds with $h : R_+ \times R^2 \times \{1,2\} \to R^2$ being defined as $h(t,z_1,z_2,i) = z_i$ for all $t \ge 0$, $z_1, z_2 \in R$, $i = 1,2$ [9]. According to Theorem 1, it remains to show that (h, f) is OPE w. r. t. Φ . In the following, we divide the discussion into two cases to prove this fact. First consider the case that (C4) holds. Let $W = z_1\gamma(t,z_1,z_2)z_2$. Then, $\dot{W}\big|_{\Sigma_i} = \gamma\alpha_i z_{3-i}^2 + \phi_i z_i$ for some uniformly bounded functions $\phi_i, i = 1,2$. It is not difficult to check that for each $i \in \{1,2\}$, $(z_i, \dot{W}\big|_{\Sigma_i})$ is a nonpositive pair. Moreover, W is continuous and independent to the switching signals. Thus, (C1) holds. According to Corollary 1, it is sufficient to check that $((h, \dot{W})^T, f)$ is OPE w. r. t. Φ . Since (γ, α_i) , $i = 1,2,$ are zeroing pairs, we have $((z_i, \gamma\alpha_i z_{3-i}^2 + \phi_i z_i)^T, (z_i, \alpha_i z_{3-i})^T))$, $i = 1,2,$ are zeroing pairs in view of (C4). Hence, we can change the output function $(h, \dot{W})^T$ as

$$\tilde{h}(t,z_1,z_2,i) = (z_i, \alpha_i(t,z_1,z_2)z_{3-i})^T, \forall i = 1,2 , \tag{18}$$

for all $t \ge 0$, $z_1, z_2 \in R$, based on Theorem 2. When (C4) does not hold, let $W_i = -z_1 z_2 \alpha_i(t,z_1,z_2)$, $\forall i = 1,2,$ $\forall t \ge 0$, $\forall z_1, z_2 \in R$. Then, $\dot{W}_i\big|_{\Sigma_i} = -\alpha_i^2 z_{3-i}^2 + \psi_i z_i$ for some uniformly bounded functions ψ_i , $i = 1,2$. It is easy to see that for each $i \in \{1,2\}$, $(z_i, \dot{W}_i\big|_{\Sigma_i})$ is a nonpositive pair. Since $z_1(t)z_2(t) = 0$ at any jump time t, (C1) also holds. Based on Theorem 4 and changing the output function as the function $\tilde{h}$ defined in (18), it remains to show that $(\tilde{h}, f)$ is OPE w. r. t. Φ for both cases. Notice that for each $i \in \{1,2\}$, $(\tilde{h}_i, f_i)$ is a zeroing pair. By Theorem 3, we only need to check that $(\tilde{h}, 0)$ is OPE w. r.,t. $\overline{\Phi}$ where

$$\overline{\Phi} = \{(c,\lambda) |\ c \in R^2, \lambda \in \Phi^{sw}\}.$$

Since $\nabla_{z_i}\alpha_i, i = 1,2,$ are uniformly bounded, $((z_i, \alpha_i(t,z_1,z_2)z_{3-i})^T), (z_i, \hat{\alpha}_i(t, z_{3-i})z_{3-i})^T))$, $i = 1,2,$ are zeroing pairs by (b) of Proposition 1 where $\hat{\alpha}_i, i = 1,2,$ are the functions defined in (16). Employing Theorem 2, it is sufficient to show the OPE condition of $(\hat{h}, 0)$ where

$$\hat{h}(t,z_1,z_2,i) = (z_i, \hat{\alpha}_i(t,z_{3-i})z_{3-i})^T, \ i = 1,2 . \tag{19}$$

According to (C5), for any $0 < a < b$, there exist a $T(a,b) > 0$ and a $r(a,b) > 0$ such that (17) holds. Since α_i , $i = 1,2$ are uniformly bounded, there is a constant $M(b) > 1$ such that

$$\|\alpha_i(t,z_1,z_2)\| \le M , \forall i = 1,2 , \forall t \ge 0 , \forall \|(z_1,z_2)^T\| \le b .$$

Notice that for any $t \in R_+$, any $b > 0$ and any $\|(z_1,z_2)^T\| \le b$, we have

$$M^2\left\|\hat{h}(t,z_1,z_2,\lambda(t))\right\|^2 = M^2\left\|\hat{h}(t,z_1,z_2,i)\right\|^2 \geq \left\|\hat{\alpha}_{3-i}(t,z_i)\right\|^2 z_i^2$$

$$+\left\|\hat{\alpha}_i(t,z_{3-i})\right\|^2 z_{3-i}^2 = \left\|\hat{\alpha}_1(t,z_2)\right\|^2 z_2^2 + \left\|\hat{\alpha}_2(t,z_1)\right\|^2 z_1^2,$$

where $i=\lambda(t)$. This leads to

$$\int_t^{t+T}\left\|\hat{h}(\tau,z_1,z_2,\lambda(\tau))\right\|^2 d\tau \geq \frac{z_2^2}{M^2(b)}\int_t^{t+T}\left\|\hat{\alpha}_1(\tau,z_2)\right\|^2 d\tau$$

$$+\frac{z_1^2}{M^2(b)}\int_t^{t+T}\left\|\hat{\alpha}_2(\tau,z_1)\right\|^2 d\tau \geq r(a/2,b)a^2/4M^2(b) > 0,$$

for any $t\in R_+$, any $b>a>0$ and any $a\leq\left\|(z_1,z_2)^T\right\|\leq b$. Therefore $(\hat{h},0)$ is OPE w. r. t. $\overline{\Phi}$, under (C5). The following result can then be proposed.

Proposition 3. Consider the switched system (15). Suppose (C5) holds. Then, the origin is uniformly globally asymptotically stable w. r. t. Φ where under (C4), Φ is the set of all pairs (z,λ) with z being a complete solution of (15) w. r. t. a piecewise constant function $\lambda:[t_0(z),\infty)\to\{1,2\}$, and otherwise, Φ is the set of all pairs (z,λ) with $z=(z_1,z_2)^T$ being a complete solution of (15) w. r. t. a piecewise constant function $\lambda:[t_0(z),\infty)\to\{1,2\}$ satisfying $z_1(t)z_2(t)=0$ at any jump time *t*. □

Remark 4. There are many pairs of functions satisfying condition (C4). For instance, $\alpha_1=-\gamma$ and $\alpha_2=-c\gamma$ for some constant $c>0$ and some continuously differentiable function $\gamma: R_+\times R^2\to R$ with $(\gamma,\nabla_t\gamma,\nabla_{z_1}\gamma,\nabla_{z_2}\gamma)$ being uniformly bounded. Another example is $\alpha_1=\sin t$ and $\alpha_2=\sin^3 t$. On the other hand, if (C4) does not hold, Φ^{sw} at least includes the following state-driven switching law: When $x_1x_2>0$, Σ_1 is switched on, and Σ_2 is switched on in case of $x_1x_2<0$. □

Remark 5. The previous discussion showed that the origin is UGS w. r. t. arbitrary switching signals. Thus, every solution is complete [9]. When (C4) holds, Proposition 3 tells us that the origin of (15) is UGAS w. r. t. any switching signals. To the best of our knowledge, such results can only be proven using the approach of common Lyapunov functions [11]. Particularly, LaSalle-type theorem cannot be used here. However, it is nontrivial how to find a common Lyapunov function for the studied switched systems in general. In fact, for the simple case with $\alpha_1=1$ and $\alpha_2=-1$, it is not difficult to show that there is no common quadratic Lyapunov function for (15). Thus, it is also hard to use the approach of common Lyapunov functions to study (15). In contrast to these methods, we have shown that our results can be applied to check UGAS. □

V. Conclusions

In this paper, three tools have been presented to help the verification of the OPE condition. From the proposed examples, it can be seen that our approach affords more flexibility comparing with the existing results. Particularly, for some well-studied systems illustrated in Example 1, our methods can use simple Lyapunov functions to guarantee uniform global asymptotic stability without employing limiting systems. A class of switched systems was studied in Example 2 to demonstrate how the proposed methods can be applied to the arbitrary switching case without founding a common Lyapunov function that is not an easy job in general. Future work may involve providing more tools to help the verification of OPE conditions and extending the derived results to other types of dynamic systems such as hybrid systems and time-delay systems.

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