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Numerical Construction of LISS Lyapunov Functions under a Small Gain Condition

Roman Geiselhart Fabian Wirth

***Abstract*— We provide a homotopy algorithm that computes a decay point of a monotone operator, i.e., a point whose image under the monotone operator is strictly smaller than the preimage. For this purpose we use a fixed point algorithm and provide a function whose fixed points correspond to decay points of the monotone operator. This decay point plays a crucial role in checking, in a semi-global fashion, the local input-to-state stability of an interconnected system numerically and in the numerical construction of local input-to-state stability (LISS) Lyapunov functions. We give some improvements of this algorithm and show the advantage to an earlier approach based on the algorithm of Eaves.**

***Index Terms*— homotopy algorithm, monotone operator, LISS Lyapunov function, interconnected system, small gain condition**

I. Introduction

For large-scale nonlinear systems it may be difficult to prove stability properties such as input-to-state stability (ISS) as introduced in [16]. If a large-scale system is defined through the interconnection of a number of smaller components, which are ISS, then there exist small-gain type conditions guaranteeing the ISS property for the interconnected system. For the case of two subsystems this result was obtained in [10], [9]. Recently, there has been a substantial effort to extend these results to the case of a greater number of subsystems, see [4], [5], [8], [11], [12], [2]. To this end the influence of the smaller components, described by comparison functions, is collected in the *gain matrix* Γ. The special structure of the interconnected system now leads to a monotone operator Γ_μ on the positive orthant $\mathbb{R}^N_+$. The available small-gain results all basically state that input-to-state stability for the overall system is equivalent to the existence of a so-called Ω-path with respect to Γ_μ. Furthermore an ISS Lyapunov function for the interconnected system can be constructed by this path and by the ISS Lyapunov functions of the subsystems. In [5] the construction of an Ω-path is outlined. The crucial ingredient that usually cannot be obtained in a straightforward manner is a *decay point* of Γ_μ. In this paper we provide numerical procedures for computing such points and thus also for local Ω-paths. The approach is semi-global as there is no a priori restriction on domain on which an ISS property can be checked numerically.

In [3] local ISS (LISS) definitions and local small gain theorems are given. Here the knowledge of a decay point leads to the local input-to-state stability of the interconnected system and to the construction of a LISS Lyapunov function. The algorithm that computes a decay point for a given monotone operator Γ_μ is a particular *simplicial fixed point* (SFP)-*algorithm* customized in a way that we obtain a decay point of Γ_μ. To ensure the convergence of the algorithm we require irreducibility of Γ_μ. This is no significant restriction because by standard graph theoretic algorithms the irreducible components of the system can be obtained efficiently. The paper is organized as follows. In Section II the necessary notions and a short introduction in comparison functions and graphs is provided. In Section III we recall the ISS Lyapunov formulation of interconnected systems, state a local small gain theorem and outline the construction of a LISS Lyapunov function for the overall systems. Section IV contains the main contribution of this paper. Some facts about homotopy algorithms and the SFP-algorithm are recalled, mainly following [17]. In subsection IV-D we state some sufficient conditions on Γ_μ and prove that the SFP-algorithm converges to a decay point of Γ_μ. In Section V an example shows that the new algorithm improves on an earlier algorithm described in [15], which was based on [6]. In particular, we reexamine a nonlinear example from [14].

Email:{roman.geiselhart,wirth}@mathematik.uni-wuerzburg.de, both authors are with Institute for Mathematics, University of Würzburg, Germany

II. Preliminaries

A. Notation and conventions

Let $\mathbb{R}_+ := [0,\infty)$, and $\mathbb{R}^N_+$ the positive orthant of nonnegative real column vectors of length N. The cone $\mathbb{R}^N_+$ induces a partial order on $\mathbb{R}^N$ as follows. We denote $v \geq w \iff v_i \geq w_i$, $v > w \iff v \geq w$ and $v \neq w$, $v \gg w \iff v_i > w_i$, each for $i = 1,\dots,N$, where v_i denotes the i^{th} component of the vector v. For $v, w \in \mathbb{R}^N_+$ the *order intervals* are given by $[v,w] := \{x \in \mathbb{R}^N_+ : v \leq x \leq w\}$ if $v \leq w$, $(v,w) := \{x \in \mathbb{R}^N_+ : v \ll x \ll w\}$ if $v \ll w$. On $\mathbb{R}^N$ we use the Euclidean norm $\|x\| = (\sum_{i=1}^N |x_i|^2)^{1/2}$. The space of measurable and essentially bounded functions is denoted by $L^\infty = L^\infty([0,\infty);\mathbb{R}^M)$ with norm $\|\cdot\|_\infty$.

B. Comparison functions and induced monotone operators

We call a function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ a *function of class* $\mathcal{K}$, if it is strictly increasing, continuous, and $\alpha(0) = 0$. If $\alpha \in \mathcal{K}$ is unbounded it is said to be of class $\mathcal{K}_\infty$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is called a *function of class* $\mathcal{KL}$, if it is of class $\mathcal{K}_\infty$ in the first argument and strictly decreasing to zero in the second argument. It is easy to see that if $\rho \in \mathcal{K}_\infty$, then $\rho^{-1} : \mathbb{R}_+ \to \mathbb{R}_+$ exists and is also of class $\mathcal{K}_\infty$.

To formulate general small gain conditions we need, cp. [5]:

Definition 1: A continuous function $\mu : \mathbb{R}^N_+ \to \mathbb{R}_+$ is called a *monotone aggregation function* if the following holds:

1) positivity: $\mu(s) \geq 0$ for all $s \in \mathbb{R}^N_+$ and $\mu(s) = 0$ if and only if $s = 0$;
2) strict increase: $\mu(s) < \mu(t)$ if $s \ll t$;

3) unboundedness: $\mu(s) \to \infty$ if $\|s\| \to \infty$.

The space of monotone aggregation functions is denoted by MAF_N. The properties in Definition 1 can be extended to vectors in the sense that $\mu = (\mu_1, \dots, \mu_N)^\top \in \mathrm{MAF}_N^N$, $\mu_i \in \mathrm{MAF}_N, i = 1, \dots, N$, defines a mapping from $\mathbb{R}^{N\times N}$ to $\mathbb{R}^N$ by $\mu(A)_i = \mu_i(a_{i1}, \dots, a_{iN})$ for $A = (a_{ij})_{i,j=1}^N \in \mathbb{R}_+^{N\times N}$. We want to generalize this to matrices of the form $\Gamma = (\gamma_{ij})_{i,j=1}^N \in (\mathcal{K}_\infty \cup \{0\})^{N\times N}$, where 0 denotes the zero function. This leads to an operator $\Gamma_\mu : \mathbb{R}_+^N \to \mathbb{R}_+^N$ with

$$\Gamma_\mu(s) := \begin{pmatrix} \mu_1(\gamma_{11}(s_1), \dots, \gamma_{1N}(s_N)) \\ \vdots \\ \mu_N(\gamma_{N1}(s_1), \dots, \gamma_{NN}(s_N)) \end{pmatrix} \in \mathbb{R}_+^N \quad (1)$$

for $s \in \mathbb{R}_+^N$. For the k-times composition of this operator we write Γ_μ^k. We call the operator Γ_μ

1) *monotone*, if $\Gamma_\mu(v) \le \Gamma_\mu(w)$ for all $v, w \in \mathbb{R}_+^N$ with $v \le w$;
2) *strictly increasing*, if $\Gamma_\mu(v) \ll \Gamma_\mu(w)$ for all $v, w \in \mathbb{R}_+^N$ with $v \ll w$.

Remark 2: Note that if $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{N\times N}$ and $\mu \in \mathrm{MAF}_N^N$, then Γ_μ is monotone and satisfies $\Gamma_\mu(0) = 0$.

The next definition is fundamental in the following.

Definition 3: For a given function $T : \mathbb{R}_+^N \to \mathbb{R}_+^N$ we define the *set of decay* Ω by $\Omega(T) := \{s \in \mathbb{R}_+^N : T(s) \ll s\}$. For short we just write Ω, if the reference to T is clear from the context. Points in Ω are called *decay points*.

C. Graphs and matrices

A directed graph $G(V, E)$ consists of a finite set of vertices V and a set of edges $E \subset V \times V$. If $G(V, E)$ consists of N vertices, then we may identify $V = \{1, \dots, N\}$. So if $(j, i) \in E$ then there is an edge from j to i. The *adjacency matrix* $A_G = (a_{ij})$ of this graph is defined by $a_{ij} = 1$ if $(j, i) \in E$ and $a_{ij} = 0$ else. We call the graph $G(V, E)$ *strongly connected* if for each pair (i, j) there exists a *path* $(e_{i_0,i_1}, e_{i_1,i_2}, \dots, e_{i_{k-1},i_k})$ with $i = i_0, j = i_k$ such that $e_{i_{l-1},i_l} \in E$ for all $i = 1, \dots, k$. It is well known that the graph $G(V, E)$ is strongly connected iff the adjacency matrix A_G is irreducible, i.e., there exists no permutation matrix P such that $A = P^\top \left(\begin{smallmatrix} B & C \\ 0 & D \end{smallmatrix}\right) P$ for suitable, square matrices B and D. These definitions can be carried over to matrices $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{N\times N}$. To this end we define the matrix $A_\Gamma = (a_{ij})_{i,j=1}^N$ by $a_{ij} = 1$ if $\gamma_{ij} \in \mathcal{K}_\infty$ and $a_{ij} = 0$ if $\gamma_{ij} \equiv 0$. We call Γ irreducible, if the matrix A_Γ is.

III. Input-to-state stability and small gain theorems

Consider the $N \in \mathbb{N}$ interconnected systems given by

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, \dots, x_N, u) \\ &\vdots \\ \dot{x}_N &= f_N(x_1, \dots, x_N, u) \end{aligned} \quad (2)$$

Assume that $x_i \in \mathbb{R}^{n_i}, u \in \mathbb{R}^m$ and the functions $f_i : \mathbb{R}^{\sum_{j=1}^N n_j + m} \to \mathbb{R}^{n_i}$ are continuous and locally Lipschitz in $x = (x_1^\top, \dots, x_N^\top)^\top$ uniformly for u in compacts. Let x_i denote the state of the i^{th} subsystem and assume u as an external control variable. Without loss of generality we may assume to have the same input for all systems.

If we consider individual systems, we treat the state $x_j, j \ne i$, as an independent input for x_i. Assume that for each subsystem $i \in \{1, \dots, N\}$ there exists a continuous and locally Lipschitz continuous function $V_i : \mathbb{R}^{n_i} \to \mathbb{R}_+$ such that for suitable $\alpha_{1i}, \alpha_{2i} \in \mathcal{K}_\infty$

$$\alpha_{1i}(\|x_i\|) \le V_i(x_i) \le \alpha_{2i}(\|x_i\|) \quad \forall x_i \in \mathbb{R}^{n_i}. \quad (3)$$

We call V_i an *ISS Lyapunov function* for the subsystem i, if there exist $\mu_i \in \mathrm{MAF}_{N+1}$, $\gamma_{ij} \in (\mathcal{K}_\infty \cup \{0\}), j \ne i, \gamma_{iu} \in (\mathcal{K} \cup \{0\})$ and a positive definite function α_i such that

$$\begin{aligned} V_i(x_i) \ge \mu_i\big(\gamma_{i1}(V_1(x_1)), \dots, \gamma_{iN}(V_N(x_N)), \gamma_{iu}(\|u\|)\big) \\ \Rightarrow \ \nabla V_i(x_i) f_i(x, u) \le -\alpha_i(\|x_i\|). \end{aligned} \quad (4)$$

The functions γ_{ij} and γ_{iu} are called *ISS Lyapunov gains*. We distinguish between the *internal inputs* x_j and the *external input* u of the i^{th} subsystem. These gains indicate the influence of the inputs on the state. This is why we set $\gamma_{ij} \equiv 0$, if f_i does not depend on x_j and we collect the internal inputs into the *gain matrix* $\Gamma := (\gamma_{ij})_{i,j=1}^N$. Note that Γ and the μ_i define a monotone operator $\Gamma_\mu : \mathbb{R}_+^N \to \mathbb{R}_+^N$ as in (1) (cf. Remark 2).

A. A local small gain theorem

In this section we assume that the interconnected system (2) satisfies an ISS condition of the form (4) for ISS Lyapunov functions V_i, $i = 1, \dots, N$. Denote the corresponding gain operator by Γ_μ as in (1). We assume that Γ is irreducible, so that Γ_μ is strictly increasing (cf. [13, Lemma 2.7]). A local ISS Lyapunov function for the overall system given by

$$\dot{x} = f(x, u) \quad (5)$$

and $x = (x_1^\top, \dots, x_N^\top)^\top, f = (f_1^\top, \dots, f_N^\top)^\top$ may now be constructed as follows.

Assume there exists a decay point $w \gg 0$ with

$$\Gamma_\mu(w) \ll w. \quad (6)$$

With Γ_μ even the sequence $\{\Gamma_\mu^k(w)\}_{k\in\mathbb{N}}$ is strictly decreasing and so $\lim_{k\to\infty} \Gamma_\mu^k(w)$ exists. If

$$\lim_{k\to\infty} \Gamma_\mu^k(w) = 0, \quad (7)$$

then we define $\sigma : [0, 1] \to \mathbb{R}_+^N$ by the linear interpolation of the points $\sigma(\frac{1}{k}) := \Gamma_\mu^k(w)$, $k \in \mathbb{N}$ and $\sigma(0) = 0$.

Note that σ is continuous on $[0, 1]$ by (7) and strictly increasing in all component functions. With this construction local ISS Lyapunov functions can be constructed using the following summary of existing results (cf. [3, Theorem 6.5]).

Theorem 4: Assume that system (2) satisfies an ISS condition of the form (4) and that the gain matrix Γ is irreducible. If there exists an $w \gg 0$ so that (6) and (7) hold then a local ISS Lyapunov function for the overall system (5) is

$$V(x) = \max_{i=1,\dots,N} \sigma_i^{-1}(V_i(x_i)), \quad (8)$$

In particular, the implication

$$V(x) \geq \gamma(\|u\|) \quad \Rightarrow \quad \nabla V(x) \cdot f(x,u) \leq -\alpha(V(x)) \quad (9)$$

holds locally with $\gamma \in \mathcal{K}_\infty$ given by [3, Proposition 4.3].

Remark 5: 1) By "local" we mean "in an open neighborhood of the origin" (cf. [3] for more details).
2) Note that by (6) and (7) it follows that the *small gain condition* $\Gamma_\mu(s) \not\geq s$ for all $s \in [0,w]$ holds.
3) Note that $\sigma(r) \in \Omega(\Gamma_\mu)$ for all $r \in [0,1]$ and σ belongs to the class of Ω-paths (cf. [5, Definition 5.1]).

Remark 6: In the linear case with $\mu = \Sigma$ we have $\Gamma_\Sigma(s) = \Gamma s$ with $\Gamma \in \mathbb{R}_+^{N\times N}$. Here the existence of a decay point $w \gg 0$ with $\Gamma w \ll w$ is equivalent to the spectral radius of Γ being less than one, i.e., $1 > \rho(\Gamma) = \{|\lambda| : \lambda \text{ is an eigenvalue of } \Gamma\}$ (cf. [13, Lemma 1.1]). So finding a decay point is just an eigenvalue problem. That's why we assume Γ_μ to be nonlinear.

IV. A HOMOTOPY ALGORITHM FOR COMPUTING A DECAY POINT $\mathbf{w} \in \mathbf{\Omega}(\mathbf{\Gamma}_\mu)$

In this section we want to develop an algorithm that computes a decay point $w \in \Omega(\Gamma_\mu)$ for a given continuous and monotone operator $\Gamma_\mu : \mathbb{R}_+^N \to \mathbb{R}_+^N$. We know that such a point exists for any norm if the *small gain condition*

$$\Gamma_\mu(s) \not\geq s \quad \text{for all } s \in \mathbb{R}_+^N \backslash \{0\} \qquad (10)$$

is satisfied (cf. [4, Proposition 5.3]). To find such a point we will extend a homotopy algorithm that was also used by Merrill (cf. [17]) to compute fixed points of upper-semicontinuous (u.s.c.) point-to-set mappings. Note that since a continuous single-valued function is in particular an u.s.c. point-to-set mapping the homotopy algorithm can clearly be used in our case. The idea is to construct a function $\phi : \mathbb{R}_+^N \to \mathbb{R}^N$ which has the property that fixed points of ϕ are decay points of Γ_μ and show that the homotopy algorithm will converge to approximate fixed points of ϕ which are also decay points of Γ_μ. This algorithm is semi-global since by choosing design variables appropriately we end up in a decay point with arbitrarily large norm.

In subsection IV-A we offer the triangulation which we need for the computation of the fixed points. Before we introduce the homotopy algorithm in subsection IV-C we first give some facts about homotopy algorithms in subsection IV-B. In subsection IV-C we mainly follow [17, Section 4.3]. In subsection IV-D we will give the function ϕ mentioned above and show the convergence of the SFP-algorithm. For reasons of space proofs and suggestions for the choice of the design variables are omitted and can be found in [7].

A. Simplices and triangulations

A set $C \subset \mathbb{R}^N$ is called *convex* if for all $a, b \in C$ it holds $S_{a,b} := \{\lambda a + (1-\lambda)b : \lambda \in [0,1]\} \in C$. The *dimension* of a convex set is equal to the dimension of the smallest affine subspace $U \subset \mathbb{R}^N$ containing C.

Definition 7: An *N-simplex* S is an N-dimensional, convex polytope spanned by $N+1$ vectors $v^1, \ldots, v^{N+1}$ in $\mathbb{R}^M$, $M \geq N$, i.e., $S := \text{co}\{v^i : i \in \{1, \ldots, N+1\}\}$, where co denotes the convex hull of the vectors v^i. A *subsimplex* ς is a simplex spanned by a subset of the set of vertices of S, i.e., $\varsigma = \{v^i : i \in I_\varsigma\}$ with $I_\varsigma \subset \{1, \ldots, N+1\}$. Zero-dimensional subsimplices are just the vertices of the simplex and $(N-1)$-subsimplices are called *facets*. The subsimplex $S(j) = \text{co}\{v^i : i \neq j\}$ is called the *facet opposite* v^j.

Clearly, since any N-simplex is N-dimensional, N of the $N+1$ vertices are linearly independent and it holds $v^i \neq v^j$ for $i \neq j$. Simplices can be used to cover convex sets in $\mathbb{R}^N$.

Definition 8: Let C be an m-dimensional convex set in $\mathbb{R}^N$. A set $\mathcal{T}$ of m-simplices is called a *triangulation* of C, if
1) C is the union of all simplices in $\mathcal{T}$;
2) for any $\eta_1, \eta_2 \in \mathcal{T}$, the intersection $\eta_1 \cap \eta_2$ is either the empty set or a common facet of both;
3) each element x in C contains a neighborhood intersecting only a finite number of simplices in $\mathcal{T}$.

By $\mathcal{T}^k$ we denote the set of all k-subsimplices of $\mathcal{T}$. It is easy to see that $\mathcal{T}^N = \mathcal{T}$ and $\mathcal{T}^0$ describes the set of the vertices of the simplices in $\mathcal{T}$. To distinguish simplices, or triangulations, we introduce the *diameter of a simplex* $\eta \in \mathcal{T}$ by $\text{diam}(\eta) = \max\{\|x - y\| : x, y \in \eta\}$ and the *mesh size of a triangulation* $\mathcal{T}$ by $\text{mesh}(\mathcal{T}) = \sup\{\text{diam}(\eta) : \eta \in \mathcal{T}\}$. There is one special triangulation of $\mathbb{R}^N$, that will be used to compute decay points. The K_1-triangulation is defined as the set of all N-simplices with vertices $x^1, \ldots, x^{N+1}$ such that

$$x^1 \in \mathbb{Z}^N \text{ and } x^{i+1} = x^i + e_{\pi_N(i)} \text{ for all } i \in \{1, \ldots, N\},$$

where $\pi_N = (\pi_N(1), \ldots, \pi_N(N))$ is a permutation of the elements of the set $\{1, \ldots, N\}$. See ([17, Theorem 1.4.8]) for a proof that K_1 is a triangulation in the sense of Definition 8. Defining $\delta C = \{\delta x : x \in C\}$ for $C \subset \mathbb{R}^N$, $\delta > 0$, and $\delta F = \{\delta C : C \in F\}$ for a family F of subsets of $\mathbb{R}^N$ we obtain that if $\mathcal{T}$ is a triangulation of C and $\delta > 0$ then $\delta\mathcal{T}$ is a triangulation of δC. In this way we get the δK_1-triangulation of $\mathbb{R}^N$ for which $\text{mesh}(\delta K_1) = \delta\sqrt{N}$ for $\delta > 0$.

Let $\mathcal{T}$ be a triangulation of $\mathbb{R}^N \times [0,1]$ with the restriction $\mathcal{T}^0 \subseteq \mathbb{R}^N \times \{0,1\}$, i.e., the vertices only lie in $\mathbb{R}^N \times \{0,1\}$. Then we call this triangulation *two-layered*. Let $\tilde{K}_1$ denote the restriction of the K_1-triangulation of $\mathbb{R}^{N+1}$ to $\mathbb{R}^N \times [0,1]$. Then $\tilde{K}_1$ is two-layered. Further define the $(N+1)\times(N+1)$-matrix $P = [\delta e_1, \ldots, \delta e_N, e_{N+1}]$ for given $\delta > 0$, where e_i denotes the i^{th} unit vector in $\mathbb{R}^{N+1}$. Define

$$\tilde{K}_1(\delta) = \{\langle Py^1, \ldots, Py^{N+2}\rangle : \langle y^1, \ldots, y^{N+2}\rangle \in \tilde{K}_1\},$$

then $\tilde{K}_1(\delta)$ is a two-layered triangulation of $\mathbb{R}^N \times [0,1]$. In Figure 1 the $\tilde{K}_1(\delta)$-triangulation is illustrated.

B. Some facts about homotopy algorithms

Now we state the basic principles of homotopy algorithms.

Definition 9: Let $f, g : C \to D$ be two continuous mappings from the topological space C to the topological space D. We call f, g *homotopic*, if there exists a continuous mapping $\vartheta : C \times [0,1] \to D$, $(s,t) \mapsto \vartheta(s,t)$ with $\vartheta(s,0) = f(s)$ and $\vartheta(s,1) = g(s)$ for all $s \in C$. We call ϑ the *homotopy* from f to g.

Let C be a nonempty, compact and convex subset of $\mathbb{R}^N$ and assume that $f : C \to C$ is continuous. Then it follows

by Kakutani's fixed point theorem (cf. [1, p.174]) that there exists at least one fixed point of f. To determine any fixed point we use the idea of the classical homotopy. Define the continuous mapping $f_t : C \to C$ by

$$f_t(s) := (1-t)s_0 + tf(s), \qquad t \in [0,1]$$

with $s_0 \in C$. Then there exists a fixed point of f_t for every $t \in [0,1]$. We start with the constant mapping $f_0(s) = s_0$ and its fixed point s_0. Assume that $t_k \to 1$ for $k \to \infty$, then the sequence of functions $\{f_{t_k}(\cdot)\}_{k\in\mathbb{N}}$ converges even uniformly to $f_1(\cdot) = f(\cdot)$. Now one can show that the cluster points of the set of fixed points s_{t_k} of f_{t_k} are just the fixed points of f. Note that in this approach we have to extend the dimension of this problem, i.e., we now work in $C \times [0,1]$. The numerical procedure for nonempty, compact and convex $C \subseteq \mathbb{R}^N$ is the following. We split the space $C \times [0,1]$ in simplices using a suitable triangulation $\mathcal{T}$. Under certain conditions there exists a path in this triangulation from an N-simplex $\tau^0 \in C \times \{0\}$ to an N-simplex $\tau^* \in C \times \{1\}$ which yields an approximate fixed point of the function f. The *simplicial fixed point algorithm* that we use here, follows the path by using the lexicographic pivoting rule from linear programming. The advantage is that the so-called degeneration problem (i.e., the path ends up in a circuit) cannot occur. We don't want to enlarge on that fact and will only give the definition of lexicographically positive matrices.

Definition 10: A row vector is called *lexicographically positive* if its first nonzero entry is positive. A matrix W is called *lexicographically positive* denoted by $W \succ 0$ if every row vector is lexicographically positive.

C. The SFP-algorithm

To compute a fixed point of a continuous function $\phi : \mathbb{R}^N \to \mathbb{R}^N$ the SFP-algorithm uses a suitable homotopy $\vartheta : \mathbb{R}^N \times [0,1] \to \mathbb{R}^N$ and a pivoting method to get from an N-simplex $\tau^0 \subset \mathbb{R}^N \times \{0\}$ to an N-simplex $\tau^1 \subset \mathbb{R}^N \times \{1\}$ which yields an approximate fixed point of ϕ. For this purpose we have to triangulate the set $\mathbb{R}^N \times [0,1]$ suitably. Let $\mathcal{T}$ be a triangulation of $\mathbb{R}^N \times [0,1]$ with the restriction $\mathcal{T}^0 \subset \mathbb{R}^N \times \{0,1\}$, i.e., $\mathcal{T}$ is two-layered. We denote elements of $\mathbb{R}^N \times [0,1]$ by $y = (v_1, \ldots, v_N, t)$ with $v \in \mathbb{R}^N$ and $t \in [0,1]$ and define the *projection onto the first factor* $p_1 : \mathbb{R}^N \times [0,1] \to \mathbb{R}^N$, $p_1(v,t) = v$. Suppose that the N-simplex $\tau = \langle y^1, y^2, \ldots, y^{N+1}\rangle \in \mathcal{T}^N$. We define the diameter of the projection of τ by $\operatorname{diam}_p(\tau) := \max\{\|p_1(y^i) - p_1(y^j)\| : i,j \in \{1,\ldots,N+1\}\}$, and the mesh size of the projection of $\mathcal{T}$ is defined by $\operatorname{mesh}_p(\mathcal{T}) := \sup\{\operatorname{diam}_p(\tau) : \tau \in \mathcal{T}^N\}$. If $\tau = \langle y^1, \ldots, y^{N+1}\rangle \in \mathcal{T}^N$ and $\tau \subset \mathbb{R}^N \times \{i\}$, $i \in \{0,1\}$, then $\tau_p := \langle p_1(y^1), \ldots, p_1(y^{N+1})\rangle$ is an N-simplex in $\mathbb{R}^N$. The collection of all such simplices τ_p is denoted by $\mathcal{T}_i$.

We choose an arbitrary point $(c,0) \in \mathbb{R}^N \times [0,1]$ such that $(c,0)$ lies in the interior of an N-simplex $\tau^0 \in \mathcal{T}_0$. Define the homotopy mapping $\vartheta : \mathbb{R}^N \times [0,1] \to \mathbb{R}^N$

$$\vartheta(v,t) = (1-t)c + t\phi(v). \tag{11}$$

A point y is called a *fixed point* of ϑ if $p_1(y) = \vartheta(y)$. Clearly, $(c,0)$ is the only fixed point of ϑ in $\mathbb{R}^N \times \{0\}$ and any fixed point y of ϑ in $\mathbb{R}^N \times \{1\}$ projects to a fixed point of ϕ, i.e., $p_1(y) = \phi(p_1(y))$. The concept of labelings establishes a way of studying the relation of the triangulation with approximate fixed points of ϕ.

Definition 11: Let $\mathcal{T}$ be a two-layered triangulation of $\mathbb{R}^N \times [0,1]$. We define the *labeling rule* $l : \mathbb{R}^N \times [0,1] \to \mathbb{R}^N$ by

$$l(y) = \vartheta(y) - p_1(y). \tag{12}$$

Let the N-simplex $\tau = \langle y^1, \ldots, y^{N+1}\rangle \in \mathcal{T}^N$ be given. Then we call the $(N+1)\times(N+1)$ matrix

$$L(\tau) := \begin{pmatrix} 1 & \ldots & 1 \\ l(y^1) & \ldots & l(y^{N+1}) \end{pmatrix} \tag{13}$$

the *label matrix* of τ.

The simplex τ is called *complete* if the system

$$L(\sigma)W = I_{N+1}, \quad W \succ 0 \tag{14}$$

has a solution $W^* \in \mathbb{R}^{(N+1)\times(N+1)}$. The aim in the following will be to find a complete N-simplex $\tau \in \mathbb{R}^N \times \{1\}$, since it contains an approximate fixed point of ϕ. By choosing the mesh size of the triangulation small enough we achieve any accuracy of the approximate fixed point.

Proposition 12: Let $D \subset \mathbb{R}^N$ be compact and $\phi : D \to \mathbb{R}^N$ be continuous. For $\varepsilon > 0$ let $\delta > 0$ be such that for all $x, y \in D$ we have the implication $\|x - y\| < \delta \Rightarrow \|\phi(x) - \phi(y)\| < \varepsilon$. Let $\mathcal{T}$ be a two-layered triangulation of $\mathbb{R}^N \times [0,1]$ with $\operatorname{mesh}(\mathcal{T}) < \delta$ and $\tau = \langle y^1, \ldots, y^{N+1}\rangle \in \mathbb{R}^N \times \{1\}$ a complete simplex in $\mathcal{T}$ with $y^j = (v^j, t_j)$ for all $j = 1, \ldots, N+1$. Let $\lambda \in \mathbb{R}^N_+$ satisfy $L(\tau)\lambda = e_1$. Then $v^* := \sum_{j=1}^{N+1} \lambda_j v^j$ is an approximate fixed point of ϕ, i.e., $\|\phi(v^*) - v^*\| < \varepsilon$.

To get to a complete simplex in $\mathbb{R}^N \times \{1\}$ we first characterize the complete simplices. To this end we define the graph $G(V,E)$ of all complete simplices as follows. An $(N+1)$-simplex η of $\mathcal{T}$ is a *node*, if it has at least one complete facet τ. Two nodes are *adjacent* and connected by an edge if they share a common complete facet. The *degree* $\deg(\eta)$ *of a node* η is the number of nodes adjacent to η.

Recall that τ^0 is the N-simplex lying on $\mathbb{R}^N \times \{0\}$ and containing $(c,0)$ in its interior. Let η^0 be the unique $(N+1)$-simplex of $\mathcal{T}$ having τ^0 as its facet. Then we have (cf. [17, Lemma 4.3.3, Lemma 4.3.4 and Theorem 4.3.5]).

Lemma 13: The N-simplex τ^0 is the only complete simplex on $\mathbb{R}^N \times \{0\}$.

Lemma 14: Given the graph $G(V,E)$ defined as above, for each node η of $G(V,E)$, we have

1) if η has a complete facet lying on $\mathbb{R}^N \times \{0\}$ or $\mathbb{R}^N \times \{1\}$, then $\deg(\eta) = 1$;
2) in all other cases, $\deg(\eta) = 2$.

Theorem 15: For the graph $G(V,E)$ defined as above, each connected component of $G(V,E)$ has one of the following five forms

1) a simple circuit;
2) a finite simple path whose two end nodes all have a complete facet lying on $\mathbb{R}^N \times \{1\}$;
3) an infinite simple path starting with an $(N+1)$-simplex

which has a complete facet lying on $\mathbb{R}^N \times \{1\}$;

4) a finite simple path which starts with the $(N+1)$-simplex η^0 and ends with another $(N+1)$-simplex having a complete facet on $\mathbb{R}^N \times \{1\}$;
5) an infinite simple path starting with η^0.

From the point of view of computation we are interested in case 4). In this case we can algorithmically go from η^0 to a simplex $\eta^* \in \mathbb{R}^N \times \{1\}$ containing an approximate fixed point of ϕ by Proposition 12.

D. Using the SFP-algorithm for computing decay points

Now we want to use the SFP-algorithm to compute a decay point $w \in \Omega(\Gamma_\mu)$ of the monotone operator $\Gamma_\mu : \mathbb{R}_+^N \to \mathbb{R}_+^N$. In the following the aim is to find a suitable function ϕ whose fixed points $w = \phi(w)$ correspond to decay points $w \in \Omega(\Gamma_\mu)$, and to show that the SFP-algorithm converges for this choice of ϕ. Since $\Gamma_\mu(0) = 0$ we have to exclude 0 to be a fixed point of ϕ.

Consider the function $\phi : \mathbb{R}_+^N \to \mathbb{R}^N$ defined by

$$\phi(v) = \Gamma_\mu(v)\left(1 + \min\left\{0, \tfrac{\kappa_\Gamma - 2\|v\|}{\|v\| + \kappa_0}\right\}\right) + \max\{0, \kappa_h - 2\|v\|\}\, e. \tag{15}$$

Here let $\kappa_0 > 0$, $\kappa_\Gamma > \kappa_h > 0$ and e the N-dimensional vector of ones. We note some properties of ϕ:

1) With Γ_μ also ϕ is continuous on $\mathbb{R}_+^N$.
2) For large $v \in \mathbb{R}_+^N$ it holds $\phi(v) < 0$.
3) It holds $\phi(0) = \kappa_h e \gg 0$, i.e., the origin cannot be a fixed point of ϕ.

We partition the positive orthant into five regions:

$$\begin{aligned}
\mathbf{I} &= \{v \in \mathbb{R}_+^N : \|v\| \in [0, \kappa_h/2)\} & \mathbf{I}' &= \mathbf{I} \times [0,1] \\
\mathbf{II} &= \{v \in \mathbb{R}_+^N : \|v\| \in [\kappa_h/2, \kappa_\Gamma/2)\} & \mathbf{II}' &= \mathbf{II} \times [0,1] \\
\mathbf{III} &= \{v \in \mathbb{R}_+^N : \|v\| \in [\kappa_\Gamma/2, \kappa_\Gamma + \kappa_0)\} & \mathbf{III}' &= \mathbf{III} \times [0,1] \\
\mathbf{IV} &= \{v \in \mathbb{R}_+^N : \|v\| \in [\kappa_\Gamma + \kappa_0, \kappa_\Gamma + \kappa_0 + \delta]\} & \mathbf{IV}' &= \mathbf{IV} \times [0,1] \\
\mathbf{V} &= \{v \in \mathbb{R}_+^N : \|v\| \in (\kappa_\Gamma + \kappa_0 + \delta, \infty)\} & \mathbf{V}' &= \mathbf{V} \times [0,1]
\end{aligned}$$

The next proposition indicates the relation between fixed points of ϕ and decay points of Γ_μ.

Proposition 16: Let $\phi : \mathbb{R}_+^N \to \mathbb{R}^N$ be defined as in (15) and assume that $\Gamma_\mu : \mathbb{R}_+^N \to \mathbb{R}_+^N$ is monotone and satisfies the small gain condition (10). Let $s \in \mathbb{R}_+^N$ be a fixed point of the function. Then $s \in \Omega(\Gamma_\mu)$. Moreover, $s \in \mathbf{I}$.

In the following we will always use the $\tilde{K}_1(\delta)$-triangulation. This triangulation has the essential advantage that the vertices of an N-simplex $\tau = \langle y^1, \ldots, y^{N+1} \rangle$ are in the order of $\mathbb{R}_+^{N+1}$, i.e., $y^1 < \ldots < y^{N+1}$ with $y = (v,t) \in \mathbb{R}_+^N \times \{0,1\}$. Again, the SFP-algorithm starts with the $(N+1)$-simplex η^0 which has the N-simplex $\tau^0 \in \mathbb{R}^N \times \{0\}$ as a facet containing $(0, c)$ in its interior. Here we choose $c \in \mathbf{I} \cup \mathbf{II}$ and any approximate fixed point c' will also lie in $\mathbf{I} \cup \mathbf{II}$, see Theorem 19. Then the algorithm follows the path of complete N-simplices. If we can show that this path is finite and inside of the positive orthant, then we get, by Theorem 15, that the SFP-algorithm ends up with a $(N+1)$-simplex containing a complete facet on $\mathbb{R}_+^N \times \{1\}$. Proposition 12 now tells us that this simplex contains an approximate fixed point of ϕ.

A first rough estimation where the path of complete simplices can lead to is given in the next proposition.

Proposition 17: Let $\phi : \mathbb{R}_+^N \to \mathbb{R}^N$ be defined as in (15) and assume that $\Gamma_\mu : \mathbb{R}_+^N \to \mathbb{R}_+^N$ is monotone. Assume that $c \in \mathbf{I} \cup \mathbf{II}$ and let $\tau = \langle y^1, \ldots, y^{N+1} \rangle$ be an N-simplex in $\mathbf{V}'$. Then τ is not complete.

Note, this does not show that the path is inside the positive orthant. To prove this we have to look at the boundary of the positive orthant. Here we need some additional assumptions.

Theorem 18: Let $\phi : \mathbb{R}_+^N \to \mathbb{R}^N$ be defined as in (15) and assume that the underlying gain matrix Γ is irreducible. Let $\tau = \langle y^1, \ldots, y^{N+1} \rangle$ be an N-simplex on the boundary of the positive orthant. If $\|v^{N+1}\| = \|p_1(y^{N+1})\| < \kappa_0 + \kappa_\Gamma$ then τ is not complete.

In other words Theorem 18 provides that no N-simplex $\tau \in \mathbf{I}' \cup \mathbf{II}' \cup \mathbf{III}'$ lying on the boundary of the positive orthant can be complete. It remains to show that the path cannot enter the set $\mathbf{IV}'$. For this purpose we show in the next theorem that the path of complete simplices runs inside of the region which is painted dark grey in Figure 1. To prove this we require an upper bound for the feasible size of δ.

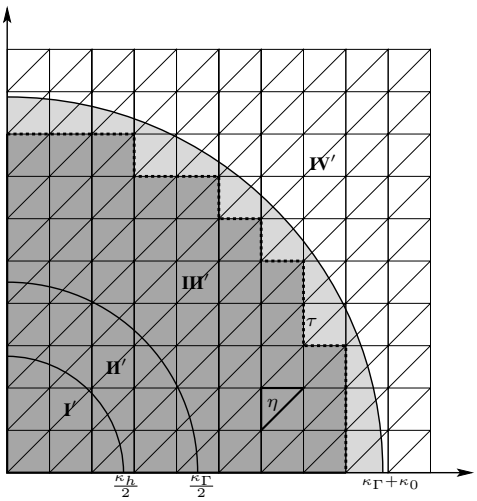

Fig. 1. $\tilde{K}_1(\delta)$-triangulation and the maximum region of the path (τ-simplices are 1-dimensional and η-simplices are 2-dimensional)

For the next Theorem we choose $k \in \mathbb{N}$ such that $\frac{1}{2k-1}\Gamma_\mu((\kappa_\Gamma + \kappa_0)e) < c$ and set $\delta > 0$ such that

$$\delta < \min\left\{\frac{\kappa_\Gamma - \kappa_h}{2\sqrt{N}}, \frac{\kappa_\Gamma + 2\kappa_0}{2k\sqrt{N}}\right\}. \tag{16}$$

Theorem 19: Let $\phi : \mathbb{R}_+^N \to \mathbb{R}^N$ be defined as in (15) and assume that $\Gamma_\mu : \mathbb{R}_+^N \to \mathbb{R}_+^N$ satisfies the small gain condition (10). Furthermore assume that the underlying gain matrix Γ does not contain any zero row. Let $\tilde{K}(\delta)$ be the triangulation of $\mathbb{R}_+^N \times [0,1]$ with $\delta > 0$ as in (16). Then for all simplices $\tau = \langle y^1, \ldots, y^{N+1} \rangle \in \mathbf{III}'$ with $y^{N+2} := y^1 + [\delta, \ldots, \delta, 1]^\top \in \mathbf{IV}'$ it follows that τ cannot be complete. In particular, any approximate solution c' satisfies $c' \in \mathbf{I} \cup \mathbf{II}$. Now we can deduce the following main theorem.

Theorem 20: Let ϕ be defined as in (15) and assume that Γ is irreducible and that the operator $\Gamma_\mu : \mathbb{R}_+^N \to \mathbb{R}_+^N$, deduced from the gain matrix Γ, satisfies the small gain condition (10). Let $\delta > 0$ be chosen as in (16). Then the simple path starting with η^0 is finite and the SFP-algorithm converges to a decay point $s \in \Omega(\Gamma_\mu)$.

Proof: The dark grey painted region in Figure 1 is compact. The path of complete simplices starts in the interior of this region. Theorem 18 and Theorem 19 now show that under the above assumptions the path starting with η^0 cannot leave this region. So the path remains in this region. Since the region is compact there exist only finitely many simplices in this region and we are in the situation of Theorem 15 case 4). So the path is finite and ends up in a simplex $\tau \in \mathbb{R}^N_+ \times \{1\}$ which contains an approximate fixed point of ϕ by Proposition 12. So refining the triangulation leads to the convergence of the SFP-algorithm to a fixed point s of ϕ. Since Γ_μ satisfies the small gain condition (10) it follows by Proposition 16 that the fixed point s of ϕ lies in the set of decay $\Omega(\Gamma_\mu)$. So the SFP-algorithm converges to a decay point $s \in \Omega(\Gamma_\mu)$. ∎

V. Example

In [15] an algorithm due to Eaves [6] is modified to compute decay points. This algorithm is used in [14] to compute decay points for an example of a nonlinear system. For this purpose a nonnegative matrix $P \in \mathbb{R}^{N\times N}$ with spectral radius $\rho(P) < 1$ is constructed for given dimension N. By Perron-Frobenius theory it follows that the matrix $A := -I_N + P$ then has spectral abscissa $\alpha(A) := \max\{\mathtt{Re}\lambda : \lambda \text{ is an eigenvalue of } A\} = -1+\rho(P) < 0$. So the matrix A is Hurwitz with negative diagonal entries and nonnegative off-diagonal entries. Now we define a smooth coordinate transformation $S : \mathbb{R}^N \to \mathbb{R}^N$ by

$$S(v)_i = \begin{cases} e^{v_i-1} & \text{if } v_i > 1 \\ v_i & \text{if } v_i \in [-1,1] \\ -e^{-v_i-1} & \text{if } v_i < -1 \end{cases} .$$

It holds $S(0) = 0$ and $S(\mathbb{R}^N_+) = \mathbb{R}^N_+$. The mapping $S : \mathbb{R}^N_+ \to \mathbb{R}^N_+$ is a monotone operator. Therefore the systems

$$\dot{v} = S'(S^{-1}(v))AS^{-1}(v) =: g(v) \tag{17}$$

$$\dot{z} = Az =: h(z) \tag{18}$$

are equivalent under a nonlinear change of coordinates. Let v^* be any decay point for the function g in equation (17). With it $z^* := S^{-1}(v^*)$ is a decay point for the function h in equation (18). We want to pick up the associated run times and numbers of iterations and compare them with those of the SFP-algorithm.

The following results correspond to matrices $P \in \mathbb{R}^{N\times N}_+$ with uniformly distributed positive entries, and 30% of those are set to zero. Then $\alpha(A) = -0.2$. The numbers are averages over 100 simulations. The design variables are chosen such that the expected decay point w has $\|w\| \approx 10$.

In Table I(i) and (ii) we compare the results of [14] with those of the SFP-algorithm. In addition, we tested the SFP-algorithm for large N. Note that the run times and iterations can only be compared relatively since the simulations are executed on different computers. Nevertheless, our run times are considerably lower and even for relatively large dimensions we are able to compute decay points in a quite acceptable run time.

	(i) Eaves		(ii) SFP-algorithm	
N	time	# iter.	time	# iter.
5	0.11s	268	0.03s	21
10	0.65s	2060	0.04s	35
15	1.78s	5506	0.06s	72
25	7.99s	19743	0.17s	188
50			1.2s	688
100			13.2s	2712
150			78.4s	6614
200			273.6s	11244

TABLE I

Results (i) of Eaves K1-algorithm from [14] and (ii) of the SFP-algorithm

VI. Conclusion

In this paper we presented an algorithm to compute decay points which are crucial in checking the local input-to-state stability and showed the advantage to an earlier algorithm due to Eaves [6], [14], [15].

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