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# Model reduction of nonlinear systems with bounded incremental $\mathcal{L}_2$ gain

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***Abstract*— A model reduction procedure for a class of nonlinear systems is presented in the current paper. Nonlinear systems are considered that can be decomposed into a linear subsystem of high order and a nonlinear subsystem of relatively low order, allowing for an approach in which only the linear subsystem is reduced using well-developed linear model reduction techniques. For this approach, conditions for stability of the reduced-order nonlinear system are given, as well as an error bound in terms of the $\mathcal{L}_2$ signal norm. Herein, the nonlinear subsystem is assumed to have a bounded incremental $\mathcal{L}_2$ gain, which proves to be a crucial property in the derivation of the results. The method is illustrated by means of an example.**

## I. INTRODUCTION

The analysis and design of high-tech systems typically yields models of high order. To allow for fast analysis or to facilitate control design and implementation, model reduction is widely used. Herein, it is important to preserve properties of the original model in the reduced-order approximation, among which stability is the most important. In model reduction for asymptotically stable linear systems, balanced truncation [11] is a technique which preserves stability [12]. Furthermore, the availability of a bound on the reduction error [5] allows for assessing the quality of the reduced-order model. Optimal Hankel norm approximation [7] is an alternative method with the same properties.

In existing model reduction techniques for nonlinear systems, the properties of stability preservation and the existence of an error bound are typically not satisfied. Nonetheless, in the extension of balanced truncation to nonlinear systems [16], [6], local stability of the reduced-order model is guaranteed. However, application of this model reduction procedure is computationally challenging and no error bound exists. Trajectory piecewise linear approximation [13] is a model reduction procedure in which the nonlinear system is approximated as a collection of linear systems, allowing for the application of well-developed linear model reduction techniques. Results on input-output stability of reduced-order models only exists for a subclass of nonlinear systems [3] and a bound on the reduction error is not available. Finally, reduction methods based on the analysis of data generated by the high-order system, such as balancing using empirical gramians [8], [10] or proper orthogonal decomposition [1], do generally not preserve stability of the high-order model, nor exhibit an error bound.

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Hence, model reduction procedures for asymptotically stable nonlinear systems generally lack a guarantee on stability of the reduced-order model, as well as an error bound. In the current paper, a model reduction procedure for a class of nonlinear systems is presented, for which conditions for stability of the reduced-order model and an error bound are given. Nonlinear systems are considered that can be decomposed into a high-order linear subsystem and a nonlinear subsystem of relatively low order. This is motivated by the observation that, in many practical engineering problems, nonlinearities act only locally. Examples include mechanical systems with friction or hysteresis and linear systems with nonlinear actuator dynamics. For these systems, a model reduction procedure is proposed in which only the linear subsystem is reduced, allowing for the use of existing linear model reduction techniques, making the approach computationally attractive. In this setting, the nonlinear subsystem is assumed to have a bounded incremental $\mathcal{L}_2$ gain. This property will prove to be crucial in the derivation of an error bound. Namely, the incremental gain characterizes the amplifications of perturbations going through the nonlinear subsystem, where these perturbations are introduced by model reduction of the linear subsystem.

In the current paper, the nonlinear subsystem contains nonlinear *dynamics*. However, the class of *static* nonlinearities is included, leading to the subclass of Lur'e-type systems. Model reduction for Lur'e-type system was considered in [2], where conditions for absolute stability of the reduced-order model and an error bound are given, hereby using an approach based on model reduction of the linear dynamics only, as in the current paper. The current paper thus extends the results of [2] to dynamic nonlinearities. To this end, the property of a bounded incremental $\mathcal{L}_2$ gain is exploited.

This paper is organized as follows. Preliminaries regarding the incremental $\mathcal{L}_2$ gain of nonlinear systems will be given in Section II. Then, the problem setting will be discussed in Section III, whereas an approach for model reduction is given in Section IV. The main results on conditions for stability of the reduced-order model and an error bound are presented in Section V. The results are illustrated by means of an example in Section VI before conclusions are stated in Section VII.

*Notation:* The notation used in this paper is fairly standard. The field of all real numbers is denoted by $\mathbb{R}$. For a vector $x \in \mathbb{R}^n$, the Euclidian norm is denoted by $|x| = \sqrt{x^{\mathrm{T}}x}$. The space $\mathcal{L}_{2,T}^n$ consist of all vector functions $x(t) : [0,T] \to \mathbb{R}^n$, which are bounded using the norm $\|x\|_{2,T}^2 = \int_0^T |x(t)|^2 \,\mathrm{d}t$, denoted by $\|x\|_{2,T}$. For $T \to \infty$, the classical $\mathcal{L}_2$ space is obtained, where the corresponding norm is denoted as $\|x\|_2$.

## II. Incremental $\mathcal{L}_2$ gain

The model reduction procedure as discussed in this paper will exploit properties of systems with bounded incremental $\mathcal{L}_2$ gain. Therefore, properties of such systems are discussed in the current section. Nonlinear systems of the form

$$\mathbf{\Sigma}: \ \dot{x} = f(x,u), \quad y = h(x,u), \tag{1}$$

are considered, where $u \in \mathbb{R}^m$, $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^p$. Furthermore, is it assumed that $f(0,0)=0$ and $h(0,0)=0$, such that $x=0$ is an equilibrium point of the system (1) with zero input ($u=0$). Then, the incremental $\mathcal{L}_2$ gain can be defined by as follows [14]:

*Definition 1:* The system (1) is said to have an incremental $\mathcal{L}_2$ gain bounded by $\gamma$ if there exists a bounded function $\beta(r,s) \geq 0$ satisfying $\beta(0,0)=0$ such that the inequality

$$\|y_2 - y_1\|_{2,T}^2 \leq \gamma^2 \|u_2 - u_1\|_{2,T}^2 + \beta(x_{1,0}, x_{2,0}) \tag{2}$$

holds for all $u_1, u_2 \in \mathcal{L}_{2,T}^m$ and for all $T \geq 0$, where $y_1(t)$ and $y_2(t)$ are the (output) solutions of (1) for inputs $u_1(t)$ and $u_2(t)$ and initial conditions $x_{1,0}$ and $x_{2,0}$, respectively. The incremental $\mathcal{L}_2$ gain property as in (2) specifies an incremental input-output stability property of the nonlinear system $\mathbf{\Sigma}$. It can be characterized using the theory of dissipative systems. Thereto, the definition of dissipativity is recalled (see [17]).

*Definition 2:* A system (1) is said to be dissipative with respect to the supply rate $s$ if there exists a nonnegative function $S$, called the storage function, such that

$$S(x(t_1)) \leq S(x(t_0)) + \int_{t_0}^{t_1} s(u(t), y(t))\,\mathrm{d}t \tag{3}$$

holds for all $t_1 \geq t_0$, where $x(t)$ and $y(t)$ are the solutions of (1) for the state and output, respectively, for the input function $u(t)$.
By introduction of the auxiliary system

$$\mathbf{\Sigma}_{aux}: \begin{cases} \dot{x}_1 = f(x_1,u_1), & y_1 = h(x_1,u_1), \\ \dot{x}_2 = f(x_2,u_2), & y_2 = h(x_2,u_2) \end{cases} \tag{4}$$

and using Definition 2, it is easily checked that the following lemma holds.

*Lemma 1 ([14]):* The system (1) has a incremental $\mathcal{L}_2$ gain bounded by $\gamma$ as in Definition 1 if and only if the auxiliary system (4) is dissipative with respect to the supply rate

$$s(u_1,u_2,y_1,y_2) = \gamma^2 |u_2 - u_1|^2 - |y_2 - y_1|^2. \tag{5}$$

By noting the property $f(0,0)=0$ and $g(0,0)=0$, it can be observed that the incremental input-output stability property as in (2) directly implies ordinary (i.e. non-incremental) $\mathcal{L}_2$ input-output stability. Namely, when $S(x_1,x_2)$ is the storage function for system (4), satisfying (3) with the supply rate given in (5), the system $\mathbf{\Sigma}$ is dissipative with respect to $s(u,0,y,0)$ with storage function $S(x,0)$. Then, the characterization of (incremental) $\mathcal{L}_2$ gain using dissipativity provides a link to internal stability when the nonlinear system is zero-state detectable (see e.g. [15]).

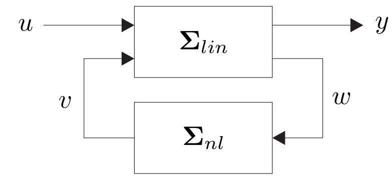


Fig. 1: Coupled system with linear and nonlinear part.

*Definition 3:* The system (1) is zero-state detectable if $u(t)=0$, $y(t)=0$, $\forall t \geq 0$ implies $\lim_{t\to\infty} x(t) = 0$.
Then, internal stability is implied by the following lemma, based on results in [9].

*Lemma 2 ([15]):* Let $S$ be a solution to (3) with $S(0)=0$ and $S(x) > 0$ for all $x \neq 0$. Furthermore, assume that the supply rate $s(u,y)$ satisfies $s(0,y) \leq 0$ for all $y$ and that the system (1) is zero-state detectable. Then $x=0$ is an asymptotically stable equilibrium of $\dot{x} = f(x,0)$. If additionally $S$ is radially unbounded, then the equilibrium is globally asymptotically stable.

## III. Problem setting

Nonlinear systems of the form as depicted in Fig. 1 are considered. Here, the system $\mathbf{\Sigma} = (\mathbf{\Sigma}_{lin}, \mathbf{\Sigma}_{nl})$ consists of a feedback configuration of a high-order linear subsystem $\mathbf{\Sigma}_{lin}$ and a nonlinear subsystem $\mathbf{\Sigma}_{nl}$ of relatively low order.

The linear subsystem $\mathbf{\Sigma}_{lin}$ is given in state-space form as

$$\mathbf{\Sigma}_{lin}: \begin{cases} \dot{x} = Ax + B_u u + B_v v, \\ y = C_y x, \\ w = C_w x, \end{cases} \tag{6}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$. Here, (6) is assumed to be asymptotically stable (i.e. $A$ is Hurwitz) and a minimal realization. The linear subsystem is coupled to the nonlinear subsystem via $v \in \mathbb{R}^s$ and $w \in \mathbb{R}^q$, where the nonlinear dynamics is given as

$$\mathbf{\Sigma}_{nl}: \begin{cases} \dot{z} = g(z,w), \\ v = h(z,w), \end{cases} \tag{7}$$

with $z \in \mathbb{R}^r$ and $g(0,0)=0$, $h(0,0)=0$. It is assumed that $\mathbf{\Sigma}_{nl}$ has a bounded incremental $\mathcal{L}_2$ gain as in Definition 1, with gain $\mu$. Hence, the associated auxiliary system satisfies the differential dissipation inequality

$$\dot{S}_{nl}(z_1,z_2) \leq \mu^2 |w_2 - w_1|^2 - |v_2 - v_1|^2 \tag{8}$$

for some non-negative storage function $S_{nl}(z_1,z_2)$. It is recalled that, by exploitation of the property $g(0,0)=0$, $h(0,0)=0$, the bounded incremental $\mathcal{L}_2$ gain implies $\mathcal{L}_2$ input-output stability of the nonlinear subsystem (7). Hence, for all initial conditions $z_0$, any bounded input $v \in \mathcal{L}_2^s$ leads to a bounded output $w \in \mathcal{L}_2^q$. In addition, for the initial condition $z_0 = 0$, the bounded incremental $\mathcal{L}_2$ gain implies uniqueness of the output for all bounded inputs. For fixed initial condition $z_0=0$, this thus allows for the definition of an input-output operator $G: \mathcal{L}_2^q \to \mathcal{L}_2^s$ for the nonlinear subsystem as $v = Gw$ with $G0 = 0$. Here, the incremental $\mathcal{L}_2$ gain property implies the incremental bound on the operator $G$ as

$$\|v_2 - v_1\|_2 = \|Gw_2 - Gw_1\|_2 \leq \mu \|w_2 - w_1\|_2, \quad (9)$$

for all $w_1, w_2 \in \mathcal{L}_2^q$. Herein, (2) is used as well as the fact that zero initial conditions are considered. Besides this property of incremental $\mathcal{L}_2$ input-output stability, it is assumed that $\mathbf{\Sigma}_{nl}$ is zero-state detectable and that the storage function $S_{nl}(z, 0)$ is radially unbounded. Then, internal stability of the nonlinear subsystem $\mathbf{\Sigma}_{nl}$ is guaranteed by Lemma 2.

For linear systems, asymptotic stability directly implies a bounded incremental $\mathcal{L}_2$ gain. Hence, for zero initial condition, input-output operators can be defined for the linear subsystem $\mathbf{\Sigma}_{lin}$ as $y = F_y(u, v)$ and $w = F_w(u, v)$, where $F_y : \mathcal{L}_2^m \times \mathcal{L}_2^s \rightarrow \mathcal{L}_2^p$ and $F_w : \mathcal{L}_2^m \times \mathcal{L}_2^s \rightarrow \mathcal{L}_2^q$ satisfy the incremental bounds

$$\begin{aligned} \|F_i(u_2, v_2) - F_i(u_1, v_1)\|_2 &\leq \gamma_{iu} \|u_2 - u_1\|_2 \\ &\quad + \gamma_{iv} \|v_2 - v_1\|_2, \end{aligned} \quad (10)$$

for all $u_1, u_2 \in \mathcal{L}_2^m$, $v_1, v_2 \in \mathcal{L}_2^s$ and some bounded $\gamma_{iu}, \gamma_{iv} \geq 0$ with $i \in \{y, w\}$. It is remarked that, for asymptotically stable linear systems, the incremental $\mathcal{L}_2$ gain directly follows from the (non-incremental) $\mathcal{L}_2$ gain, such that the gains $\gamma_{ij}$, $i \in \{y, w\}$, $j \in \{u, v\}$ in (10) equal the $\mathcal{H}_\infty$ norm of the corresponding transfer function $H_{ij}(s) = C_i(sI - A)^{-1}B_j$.

Finally, it is assumed that the full nonlinear system $\mathbf{\Sigma} = (\mathbf{\Sigma}_{lin}, \mathbf{\Sigma}_{nl})$ as in Fig. 1 has a bounded incremental $\mathcal{L}_2$ gain with respect to the input $u$ and output $y$. This can be guaranteed by the following small-gain theorem, which is based on an application of the classical small-gain theorem for incrementally stable systems (see e.g. [4]).

*Theorem 3:* Let $\mathbf{\Sigma} = (\mathbf{\Sigma}_{lin}, \mathbf{\Sigma}_{nl})$ be given as in (6) and (7), where the subsystems $\mathbf{\Sigma}_{lin}$ and $\mathbf{\Sigma}_{nl}$ have a bounded incremental $\mathcal{L}_2$ gain such that the corresponding input-output operators satisfy (10) and (9), respectively. Then, the bidirectionally coupled system configuration $\mathbf{\Sigma}$ has a bounded incremental $\mathcal{L}_2$ gain if the small-gain condition

$$\gamma_{wv} \mu < 1 \quad (11)$$

holds. If (11) is satisfied, the incremental $\mathcal{L}_2$ gain of $\mathbf{\Sigma}$ is bounded by

$$\bar{\gamma}_{yu} = \gamma_{yu} + \frac{\gamma_{yv} \mu \gamma_{wu}}{1 - \gamma_{wv} \mu}. \quad (12)$$

*Proof:* Substitution of (9) in (10) for $i = w$ yields

$$\|w_2 - w_1\|_2 \leq \gamma_{wu} \|u_2 - u_1\|_2 + \gamma_{wv} \mu \|w_2 - w_1\|_2, \quad (13)$$

from which is it easily observed that the small-gain condition (11) guarantees boundedness of $\|w_2 - w_1\|_2$. Namely,

$$\|w_2 - w_1\|_2 \leq \frac{\gamma_{wu}}{1 - \gamma_{wv} \mu} \|u_2 - u_1\|_2 \quad (14)$$

holds. Substitution of (14) in (10) for $i = y$ by using (9) gives (12). ∎

Even though Theorem 3 provides a result on input-output stability of the coupled configuration $\mathbf{\Sigma} = (\mathbf{\Sigma}_{lin}, \mathbf{\Sigma}_{nl})$ for zero initial condition, it is noted that internal stability is guaranteed under the same small-gain condition (11). Namely, $\mathcal{L}_2$ input-output stability of the linear subsystem guarantees the existence of a (quadratic) storage function $S_{lin}(x)$ which, for zero external input $u(t) = 0$, satisfies

$$\dot{S}_{lin}(x) \leq \gamma_{wv}^2 |v|^2 - |w|^2. \quad (15)$$

It is assumed that $S_{lin}(x)$ is positive definite. Positive-definiteness is not guaranteed, since the linear subsystem $\mathbf{\Sigma}_{lin}$ is not necessary minimal with respect to the input $v$ and output $w$. The positive semi-definite case can be treated by using results in [15]. Furthermore, it is recalled that the bounded incremental $\mathcal{L}_2$ gain of the nonlinear subsystem $\mathbf{\Sigma}_{nl}$, as characterized by (8), implies that $S_{nl}(z, 0)$ is a storage function for the supply rate $\mu^2 |w|^2 - |v|^2$. Hence, the $\mathcal{L}_2$ gain of the nonlinear subsystem is bounded by $\mu$. Then, introduction of $S(x, z) = S_{lin}(x) + \alpha^2 S_{nl}(z, 0)$ leads to

$$\dot{S}(x, z) \leq (\gamma_{wv}^2 - \alpha^2)|v|^2 + (\alpha^2 \mu^2 - 1)|w|^2. \quad (16)$$

Hence, the small-gain condition (11) allows for choosing $\alpha$ such that it satisfies $\gamma_{wv} < \alpha < \mu^{-1}$, such that the right-hand side of (16) is nonpositive (see e.g. [15]). Hence, $S(x, z)$ is a Lyapunov function for the coupled system $\mathbf{\Sigma} = (\mathbf{\Sigma}_{lin}, \mathbf{\Sigma}_{nl})$. Then, zero-state detectability of $\mathbf{\Sigma}_{lin}$ (as implied by assymptotic stability) and $\mathbf{\Sigma}_{nl}$ yield global asymptotic stability of the equilibrium $x = 0$, $z = 0$ for zero input, via Lemma 2.

## IV. Model reduction

In the configuration $\mathbf{\Sigma} = (\mathbf{\Sigma}_{lin}, \mathbf{\Sigma}_{nl})$ as in Fig. 1, the linear subsystem is assumed to be of high order, whereas the nonlinear subsystem is of relatively low order. A natural approach for model reduction is thus to perform reduction on the linear subsystem only. This approach allows for the application of well-developed model reduction techniques for linear systems, making the reduction computationally attractive. Thus, reduction of the linear subsystem $\mathbf{\Sigma}_{lin}$ to obtain the reduced-order linear subsystem $\hat{\mathbf{\Sigma}}_{lin}$, hereby taking into account the inputs $u$ and $v$ and outputs $y$ and $w$, leads to the reduced-order nonlinear system $\hat{\mathbf{\Sigma}} = (\hat{\mathbf{\Sigma}}_{lin}, \mathbf{\Sigma}_{nl})$.

The reduced-order linear subsystem is given as

$$\hat{\mathbf{\Sigma}}_{lin} : \begin{cases} \dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}_u u + \hat{B}_v \hat{v}, \\ \hat{y} = \hat{C}_y \hat{x}, \\ \hat{w} = \hat{C}_w \hat{x}, \end{cases} \quad (17)$$

with $\hat{x} \in \mathbb{R}^k$, $k < n$, where it is assumed that asymptotic stability is preserved, i.e. $\hat{A}$ is Hurwitz. Similar to the high-order linear subsystem, this allows for the definition of the input-output operators $\hat{y} = \hat{F}_y(u, \hat{v})$ and $\hat{w} = \hat{F}_w(u, \hat{v})$, which satisfy incremental bounds as in (10) with gains $\hat{\gamma}_{ij}$. To characterize the quality of the reduction of the linear subsystem, the error operators $E_i$ are defined:

$$E_i(u, v) = F_i(u, v) - \hat{F}_i(u, v), \quad i \in \{y, w\}. \quad (18)$$

Here, it is noted that, due to asymptotic stability of the high-order and reduced-order linear subsystems, the error operators as defined in (18) indeed exist. Then, it is assumed that the error induced by the reduction of the linear subsystem is incrementally bounded as

$$\|E_i(u_2, v_2) - E_i(u_2, v_2)\|_2 \leq \varepsilon_{iu}\|u_2 - u_1\|_2 + \varepsilon_{iv}\|v_2 - v_1\|_2 \qquad (19)$$

with $\varepsilon_{iu}, \varepsilon_{iv} > 0$ and $i \in \{y, w\}$. Even though the assumption on the error bound as in (19) might seem restrictive at first sight, it is remarked that this incremental form is directly implied by an ordinary (i.e. non-incremental) error bound, due to linearity.

Furthermore, model reduction techniques for linear systems satisfying the assumptions on stability of the reduced-order model and an error bound using the $\mathcal{L}_2$ signal norm exist. Herein, balanced truncation [11] is the most popular one. This procedure preserves stability [12] and satisfies a bound on the error [5]. An alternative model reduction procedure, which has the same properties, is given by optimal Hankel norm approximation [7]. Finally, it is noted that these methods yield a single error bound $\varepsilon_{lin}$, such that no distinction is made between different input-output combinations as in (19). In this case, the relation $\varepsilon_{ij} \leq \varepsilon_{lin}$ holds with $i \in \{y, w\}$, $j \in \{u, v\}$.

## V. Stability and error bound

For the reduced-order nonlinear system obtained by the procedure in Section IV, conditions for stability as well as an error bound are given in the following theorem.

*Theorem 4:* Let $\mathbf{\Sigma} = (\mathbf{\Sigma}_{lin}, \mathbf{\Sigma}_{nl})$ be a nonlinear system of the form (6-7), where $\mathbf{\Sigma}_{lin}$ is asymptotically stable and $\mathbf{\Sigma}_{nl}$ satisfies the incremental dissipation inequality (8) for some bounded $\mu > 0$ and non-negative incremental storage function $S_{nl}$, such that the input-output operators of both $\mathbf{\Sigma}_{nl}$ and $\mathbf{\Sigma}_{lin}$ have a bounded incremental $\mathcal{L}_2$ gain as in (9) and (10), respectively. Furthermore, assume that the small-gain condition (11) holds. When $\hat{\mathbf{\Sigma}} = (\hat{\mathbf{\Sigma}}_{lin}, \mathbf{\Sigma}_{nl})$ is a reduced-order approximation of the same form where the reduced-order linear subsystem $\hat{\mathbf{\Sigma}}_{lin}$ is asymptotically stable and satisfies the error bound (19), the following statements hold:

1) The reduced-order nonlinear system $\hat{\mathbf{\Sigma}} = (\hat{\mathbf{\Sigma}}_{lin}, \mathbf{\Sigma}_{nl})$ is incrementally $\mathcal{L}_2$ input-output stable if

$$(\gamma_{wv} + \varepsilon_{wv})\mu < 1. \qquad (20)$$

2) When the reduced-order nonlinear system is incrementally $\mathcal{L}_2$ input-output stable, the output error $\delta y(t) = y(t) - \hat{y}(t)$ is bounded as $\|\delta y\|_2 \leq \varepsilon \|u\|_2$ with

$$\varepsilon = \varepsilon_{yu} + \frac{\varepsilon_{yv}\mu\gamma_{wu}}{1 - \gamma_{wv}\mu} + \frac{(\gamma_{yv} + \varepsilon_{yv})\mu}{1 - (\gamma_{wv} + \varepsilon_{wv})\mu}\left(\varepsilon_{wu} + \frac{\varepsilon_{wv}\mu\gamma_{wu}}{1 - \gamma_{wv}\mu}\right). \qquad (21)$$

*Proof:* The two statements are proven separately.

*1. Incremental $\mathcal{L}_2$ stability of $\hat{\mathbf{\Sigma}} = (\hat{\mathbf{\Sigma}}_{lin}, \mathbf{\Sigma}_{nl})$.* Theorem 3 directly guarantees incremental $\mathcal{L}_2$ stability of the reduced-order system if the small-gain condition $\hat{\gamma}_{wv}\mu < 1$ holds. However, the incremental gain $\hat{\gamma}_{wv}$ of the reduced-order linear system is not known a priori. Nonetheless, an upper bound for $\hat{\gamma}_{wv}$ can be obtained by considering

$$\hat{F}_w(u_2, v_2) - \hat{F}_w(u_1, v_1) = F_w(u_2, v_2) - F_w(u_1, v_1) - E_w(u_2, v_2) + E_w(u_1, v_1), \qquad (22)$$

where (18) is used, leading to the bound

$$\begin{aligned} &\|\hat{F}_w(u, v_2) - \hat{F}_w(u, v_1)\|_2 \\ &\leq \|F_w(u, v_2) - F_w(u, v_1)\|_2 + \|E_w(u, v_2) - E_w(u, v_1)\|_2, \\ &\leq (\gamma_{wv} + \varepsilon_{wv})\|v_2 - v_1\|_2. \end{aligned} \qquad (23)$$

Here, the latter inequality follows from the incremental bound on the input-output operator of the high-order linear subsystem (10) and the error bound (19). Clearly, $\gamma_{wv} + \varepsilon_{wv}$ provides an upper bound to the incremental $\mathcal{L}_2$ gain $\hat{\gamma}_{wv}$ of the reduced-order linear subsystem. Hence, (20) implies $\hat{\gamma}_{wv}\mu < 1$, such that Theorem 3 guarantees that the reduced-order system $\hat{\mathbf{\Sigma}} = (\hat{\mathbf{\Sigma}}_{lin}, \mathbf{\Sigma}_{nl})$ is incrementally $\mathcal{L}_2$ stable.

*2. Error bound.* As a first step in error analysis, bounds on the magnitude of the signals $w$ and $v$ will be derived. Here, by using the fact that $w = 0$ is the unique solution of $\mathbf{\Sigma} = (\mathbf{\Sigma}_{lin}, \mathbf{\Sigma}_{nl})$ to $u = 0$ (for zero initial condition), (14) in the proof of Theorem 3 directly leads to

$$\|w\|_2 \leq \frac{\gamma_{wu}}{1 - \gamma_{wv}\mu}\|u\|_2. \qquad (24)$$

Substitution of (24) in (9) and exploiting the property $G0 = 0$ gives a bound on $\|v\|_2$ as

$$\|v\|_2 \leq \frac{\mu\gamma_{wu}}{1 - \gamma_{wv}\mu}\|u\|_2. \qquad (25)$$

Next, the error $\delta w(t) = w(t) - \hat{w}(t)$ is considered, which can be expressed as

$$\begin{aligned} \delta w &= F_w(u, v) - \hat{F}_w(u, \hat{v}), &(26)\\ &= F_w(u, v) - \hat{F}_w(u, v) + \hat{F}_w(u, v) - \hat{F}_w(u, \hat{v}), &(27) \end{aligned}$$

such that $\|\delta w\|_2$ can be bounded as

$$\|\delta w\|_2 \leq \|F_w(u, v) - \hat{F}_w(u, v)\|_2 + \|\hat{F}_w(u, v) - \hat{F}_w(u, \hat{v})\|_2. \qquad (28)$$

Here, the first term is related to the error bound on the linear subsystem, which is bounded by (19). The second term can be related to the incremental $\mathcal{L}_2$ gain of the reduced-order linear subsystem, which yields

$$\|\delta w\|_2 \leq \varepsilon_{wu}\|u\|_2 + \varepsilon_{wv}\|v\|_2 + \hat{\gamma}_{wv}\|\delta v\|_2. \qquad (29)$$

The gain $\hat{\gamma}_{wv}$ is unknown a priori, but can be bounded as $\hat{\gamma}_{wv} \leq \gamma_{wv} + \varepsilon_{wv}$, as is shown in the proof of the first part of this theorem. Furthermore, the bound on the incremental $\mathcal{L}_2$ gain for the nonlinear system (9) implies $\|\delta v\|_2 \leq \mu\|\delta w\|_2$. Exploiting this in (29) leads to

$$\|\delta w\|_2 \leq \frac{\varepsilon_{wu}\|u\|_2 + \varepsilon_{wv}\|v\|_2}{1 - (\gamma_{wv} + \varepsilon_{wv})\mu}, \qquad (30)$$

where it is noted that the small-gain condition (20) guarantees boundedness of (30). The bound on $\|v\|_2$ in (25) can be substituted in (30), such that the error in $w$ is only dependent on the energy of the input $u(t)$. Furthermore, application of this result in (9) leads to a bound on the error $\delta v(t) = v(t) - \hat{v}(t)$ as

$$\|\delta v\|_2 \leq \frac{\mu}{1-(\gamma_{wv}+\varepsilon_{wv})\mu}\left(\varepsilon_{wu}+\frac{\varepsilon_{wv}\mu\gamma_{wu}}{1-\gamma_{wv}\mu}\right)\|u\|_2. \quad (31)$$

By construction, (31) provides a bound on $\delta v(t)$ in the coupled configuration. This result will be exploited to obtain the final error bound. Hereto, the output error $\delta y(t) = y(t) - \hat{y}(t)$ is introduced, which is given by

$$\begin{aligned}\delta y &= F_y(u,v) - \hat{F}_y(u,\hat{v}), &(32)\\ &= F_y(u,v) - \hat{F}_y(u,v) + \hat{F}_y(u,v) - \hat{F}_y(u,\hat{v}). &(33)\end{aligned}$$

Now, (33) can be bounded as

$$\begin{aligned}\|\delta y\|_2 \leq\ & \|F_y(u,v) - \hat{F}_y(u,v)\|_2\\ &+ \|\hat{F}_y(u,v) - \hat{F}_y(u,\hat{v})\|_2, \quad (34)\end{aligned}$$

where the first term is related to the error introduced by reduction of the linear subsystem, which is bounded by (19). The second term can be related to the incremental gain of the reduced-order linear subsystem, such that

$$\|\delta y\|_2 \leq \varepsilon_{yu}\|u\|_2 + \varepsilon_{yv}\|v\|_2 + \hat{\gamma}_{yv}\|\delta v\|_2. \quad (35)$$

Again, the gain $\hat{\gamma}_{yv}$ is unknown a priori, but can be bounded as $\hat{\gamma}_{yv} \leq \gamma_{yv} + \varepsilon_{yv}$. Then, substitution of the bound on $v(t)$ (25) and the bound on $\delta v(t)$ (31) in (35) leads to the output error bound (21), proving the second statement. ∎

In the proof of the error bound in Theorem 4, the incremental gain property of the input-output operators of the linear and nonlinear subsystem play an important role. Namely, the incremental gains characterize the amplification of perturbations going through the subsystems, where these perturbations are introduced by model reduction of the linear subsystem. The small-gain theorem then guarantees boundedness of the perturbations in the bidirectionally coupled configuration as in Fig. 1.

The result in Theorem 4 is based on the availability of the error bounds $\varepsilon_{ij}$ with $i \in \{y, w\}$, $j \in \{u, v\}$, providing bounds on all relevant input-output pairs. However, existing model reduction techniques for linear systems generally provide a single error bound $\varepsilon_{lin}$. When this error bound is exploited as $\varepsilon_{ij} \leq \varepsilon_{lin}$ for $i \in \{y, w\}$, $j \in \{u, v\}$, the error bound (21) reduces to

$$\varepsilon = \varepsilon_{lin}\left(1+\frac{\mu\gamma_{wu}}{1-\gamma_{wv}\mu}\right)\left(1+\frac{(\gamma_{yv}+\varepsilon_{lin})\mu}{1-(\gamma_{wv}+\varepsilon_{lin})\mu}\right). \quad (36)$$

*Remark 1:* The condition for stability (20) and the error bound (21) depend only on properties of the high-order system and the error bound on the linear subsystems and can therefore be evaluated a priori. However, a tighter error bound can be obtained when the gains $\hat{\gamma}_{wv}$ and $\hat{\gamma}_{yv}$ of the reduced-order linear subsystem are computed a posteriori (i.e. after the reduction has been employed). Namely, these gains can directly be used in (29) and (35), respectively. Additionally, availability of the gain $\hat{\gamma}_{wv}$ will allow for the direct evaluation of stability via $\hat{\gamma}_{wv}\mu < 1$.

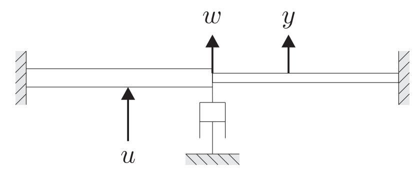


Fig. 2: Flexible beam with nonlinear damping.

*Remark 2:* The dynamics of the nonlinear subsystem are not taken into account explicitly. Instead, only an upper bound on the incremental $\mathcal{L}_2$ gain is used. Thus, the results hold for all nonlinearities satisfying the same incremental gain. This is particularly useful in practice, since nonlinearities are typically hard to model and subject to uncertainty.

Even though the focus of Theorem 4 is on input-output stability, it is noted that the condition (20) also guarantees internal stability of the reduced-order nonlinear model. Namely, the reasoning as presented below Theorem 3 can be repeated.

## VI. Illustrative Example

The flexible beam example in Fig. 2 is considered to illustrate the results outlined in Section V. The beam, which consists out of two sections with different heights, is modeled using Euler beam elements, leading to an asymptotically stable linear model $\mathbf{\Sigma}_{lin}$ of the form (6) with $x \in \mathbb{R}^{80}$. The input $u \in \mathbb{R}$ is an external force acting on the beam, whereas the vertical deflection $y \in \mathbb{R}$ is taken as an output. In its center, the beam is supported by a damping element, which is modeled by the nonlinear dynamics

$$\mathbf{\Sigma}_{nl} : \dot{z} = -z - \sigma(z) + \kappa w, \quad v = z. \quad (37)$$

Here, $w \in \mathbb{R}$ is the vertical velocity of the center of the beam, $v \in \mathbb{R}$ is the damping force and $z \in \mathbb{R}$ is the internal state. Additionally, $\sigma(z)$ is an arbitrary nondecreasing continuous function. Then, by using the storage function $S(z_1, z_2) = \frac{1}{2}(z_1 - z_2)^2$, it can be shown that the incremental $\mathcal{L}_2$ gain of (37) is bounded by $\kappa$, i.e. $\mu = \kappa$.

Balanced truncation is applied to the linear beam model to obtain an asymptotically stable reduced-order approximate $\hat{\mathbf{\Sigma}}_{lin}$ with $\hat{x} \in \mathbb{R}^4$, of which the frequency response function $\hat{H}_{wv}$ is depicted in Fig. 3, along with the frequency response function $H_{wv}$ of the high-order linear dynamics. Herein, the line $\mu^{-1}$ indicates that both the original and reduced-order nonlinear systems are stable (see Theorem 3). However, stability of the reduced-order model is guaranteed a priori, which follows from the observation that $\gamma_{wv} < \mu^{-1} - \varepsilon_{lin}$ and Theorem 4. Here, $\varepsilon_{lin}$ denotes the a priori error bound on reduction of the linear subsystem, as results from balanced truncation. Finally, a less conservative guarantee on stability of the reduced-order nonlinear model is obtained by computing the error bounds (on the linear subsystem) after reduction, such that the relevant error $\varepsilon_{wv}$ (satisfying $\varepsilon_{wv} \leq \varepsilon_{lin}$) can be used.

Error bounds on the nonlinear system are computed using Theorem 4 and can be found in Table I, for several values

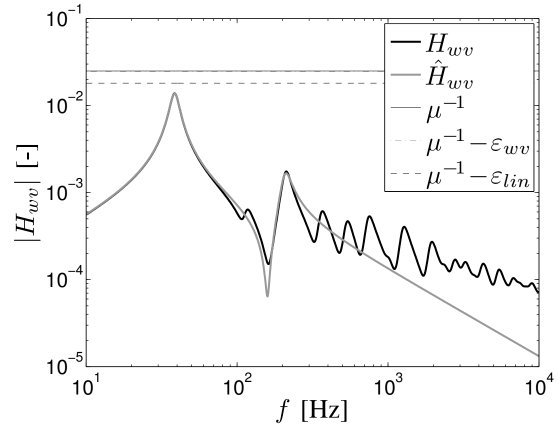


Fig. 3: Frequency response function $H_{wv}$ for the high-order and reduced-order linear subsystems $\mathbf{\Sigma}_{lin}$ and $\hat{\mathbf{\Sigma}}_{lin}$, including stability bounds for $\mu = 40$.

TABLE I: A priori and a posteriori error bound by model reduction using balanced truncation.

| $\kappa$ | $\varepsilon_{lin}$ | $\varepsilon$ (a priori (36)) | $\varepsilon$ (a posteriori (21)) |
|---|---|---|---|
| 20 | $0.687 \cdot 10^{-2}$ | $0.903 \cdot 10^{-2}$ | $0.530 \cdot 10^{-5}$ |
| 40 | $0.687 \cdot 10^{-2}$ | $1.406 \cdot 10^{-2}$ | $0.658 \cdot 10^{-5}$ |
| 60 | $0.687 \cdot 10^{-2}$ | – | $1.239 \cdot 10^{-5}$ |

of $\kappa$. Here, error bounds are computed twice. First, an a priori error bound is computed using the bounds $\varepsilon_{ij} \leq \varepsilon_{lin}$ (with $i \in \{y, w\}$, $j \in \{u, v\}$), leading to (36). Secondly, the error bounds on the linear subsystem $\varepsilon_{ij}$ are computed after the reduced-order linear subsystem is obtained, such that (21) can be evaluated. This, as can be seen in the rightmost column of Table I, clearly leads to a much tighter error bound. However, it is noted that the conservativeness in the a priori error bound is largely due to the large bound on the linear subsystem rather than the result of Theorem 4, as can be concluded from the first two columns of Table I. Finally, it is remarked that stability of the reduced-order nonlinear system can not be guaranteed a priori (i.e. using $\varepsilon_{lin}$) for $\kappa = 60$, such that no a priori error bound can be computed.

In Fig. 4, a comparison of the outputs of the high-order and reduced-order system is depicted for two input signals. Here, it can be observed that the output of the reduced-order model $\hat{\mathbf{\Sigma}}$ matches the output of the high-order system $\mathbf{\Sigma}$ well.

## VII. Conclusions

A model reduction procedure is presented for systems that can be decomposed into a linear and nonlinear subsystem, which are bidirectionally coupled. Since model reduction is performed on the linear subsystem only, this approach is computationally attractive. Furthermore, conditions for internal and input-output stability of the reduced-order nonlinear system are given, as well as an error bound. Herein, the property of a bounded incremental $\mathcal{L}_2$ gain on the subsystems is used extensively.

Future work will focus on the application of this result for the design of low-order controllers for classes of nonlinear systems.

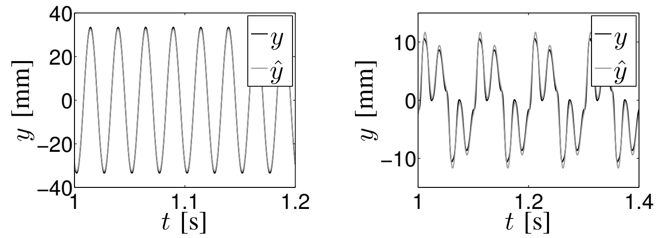


Fig. 4: Outputs of the high-order and reduced order nonlinear system for input functions $u(t) = 10^2 \sin(2\pi 40 t)$ (left) and $u(t) = 10^2 \,\mathrm{sign}(\sin(2\pi 10 t))$ (right) and $\kappa = 40$.

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