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Characterization of accessibility for a class of nonlinear time-delay systems

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Abstract— **The accessibility of the class of driftless single-input nonlinear time-delay systems is fully characterized for the first time. This result is obtained through the introduction of new tools within a geometric approach recently introduced in the literature. Moreover, all those possible autonomous elements, which can depend on the variables with time-delay, are also characterized when the system is not accessible and in consequence, a canonical form of those systems is deduced.**

I. INTRODUCTION

The accessibility of nonlinear time-delay systems was considered for the first time in [8] where a suitable definition of accessibility has been proposed and a sufficient condition to test whether or not a given system is accessible has been given. However the problem to characterize completely the accessibility of nonlinear time-delay systems, by identifying the non accessible part is still an open problem.

The goal of this paper is to show how it is possible through the introduction of new tools within a new approach recently introduced in the literature [2], [3] to easily characterize the accessibility properties of driftless time–delay systems with constant commensurate delay. As an example, in such a framework it becomes immediately clear that the following simple system

$$\dot{x}(t) = \begin{pmatrix} x_2(t-D) \\ 1 \end{pmatrix} u(t),$$

with $D \geq 0$ a constant delay, is accessible whenever $D > 0$ and non accessible when $D = 0$.

The paper is organized as follows. In Section II, some fundamental notions on time-delay systems are given as well as the definition of accessibility which were introduced in [2], [8], [13]. In Section III, the new geometric notion of *closure of the module spanned by a single element* is introduced and by use of it, we propose a sufficient and necessary condition for the accessibility for this class of systems. In Section IV, when the original system is not accessible, we show how to characterize all its autonomous functions (which can depend on the variables with time-delay). Based on the computations of the autonomous functions determined in Section IV, a standard decomposition into an autonomous subsystem and an accessible subsystem is deduced in Section V. Finally, some concluding remarks are given in Section VI.

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II. PRELIMINARIES AND NOTATIONS

In this paper, we characterize the accessibility property of the driftless single-input nonlinear time-delay system

$$\Sigma: \quad \dot{x}(t) = g(\mathbf{x}_{[s]})u(t) \tag{1}$$

where $\mathbf{x}_{[s]}^T = (x^T(t), x^T(t-1), \ldots, x^T(t-s))$, $s \geq 1$, with $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ and the entries of $g(\mathbf{x}_{[s]})$ are analytic functions.

As it is well known, when $s = 0$ and $n \geq 2$, the system Σ is clearly never accessible because a single vector field in $\mathbb{R}^n$ is always involutive and thus its Lie algebra has rank 1. However, for the case $s \geq 1$, it becomes much more involved since a single vector field is not always involutive anymore [2]. Despite some sufficient conditions given in [8], [9], the characterization of accessibility is an open problem for this class of systems.

The following notations will be used [3], [7],[13]:

- $\mathscr{K}$ is the field of meromorphic functions of a finite number of variables in $\{x(t-i), u(t-i), \ldots, u^{(k)}(t-i), i, k \in \mathbb{N}\}$;
- δ represents the backward time-shift operator defined as

$$\delta(\varphi(t)\mathrm{d}f(t)) = \varphi(t-1)\mathrm{d}f(t-1), \quad \varphi(\cdot), f(\cdot) \in \mathscr{K};$$

- d is the standard differential operator;
- Given a function $f(x(t), \ldots, x(t-s))$, we will denote by $f(-l) = f(x(t-l), \ldots, x(t-s-l))$;
- $\mathscr{K}(\delta]$ is the left ring of polynomials in δ with coefficients in $\mathscr{K}$. Every element of $\mathscr{K}(\delta]$ may be written as $\alpha(\delta] = \alpha_0(t) + \alpha_1(t)\delta + \cdots + \alpha_{r_\alpha}(t)\delta^{r_\alpha}$, $\alpha_i \in \mathscr{K}$, where $r_\alpha = \deg(\alpha(\delta])$;
- We will denote by $\mathscr{X} = \mathrm{span}_{\mathscr{K}(\delta]}\{\mathrm{d}x_i, i \in [1, n]\}$. Let us recall that the elements of this space are called 1-forms;
- For convenience, we will denote by E_i the extended space $E_i = \mathbb{R}^{(i+1)n}$ with the coordinates $x(t), \ldots, x(t-i)$.
- Let $\mathscr{D}$ be a distribution defined in space $E_i = \mathbb{R}^{(i+1)n}$ with coordinates $x(t), \ldots, x(t-i)$, we denote by $\bar{\mathscr{D}}$ its involutive closure. In other words, $\bar{\mathscr{D}}$ is the smallest distribution such that $\mathscr{D} \subset \bar{\mathscr{D}}$ and for any $f \in \bar{\mathscr{D}}$, $g \in \bar{\mathscr{D}}$, we have $[f, g] \in \bar{\mathscr{D}}$.

A. Autonomous Elements

Definition 2.1: A 1-form ω in $\mathscr{X}$ is said to be an autonomous element of the system Σ, given by (1), if there exists an integer ν and not all zero coefficients $\alpha_i \in \mathscr{K}(\delta]$, for $1 \le i \le \nu$, such that

$$\alpha_0 \omega + \cdots + \alpha_\nu \omega^{(\nu)} = 0.$$

Definition 2.2: The relative degree r of a 1-form $\omega \in \mathscr{X}$ is defined by

$$r = min\{k \in \mathbb{N} \mid \omega^{(k)} \notin \mathscr{X}\}.$$

Proposition 2.3: A 1-form ω in $\mathscr{X}$ is an autonomous element if and only if it has an infinite relative degree.

Proposition 2.4: The function $\varphi \in \mathscr{K}$ and the 1-form $\mathrm{d}\varphi$ have the same relative degree.

Definition 2.5: A nonzero function $\varphi \in \mathscr{K}$ is said to be an autonomous element of Σ if the 1-form $d\varphi$ is an autonomous element of Σ.

B. Accessibility

In this paper, we use the following definition of accessibility of nonlinear time-delay systems introduced in [8], which generalizes the accessibility of nonlinear systems without time-delay [5].

Definition 2.6: The system Σ, given by (1), is said to be accessible if there does not exist any nonconstant autonomous function.

Lemma 2.7: For any driftless system (1), the relative degree of a function $\varphi \in \mathscr{K}$ is greater than 1 if and only if it is infinite.

Proof: Let $\varphi = \varphi(x(t), x(t-1), \ldots, x(t-k))$, $k \ge 0$, be a meromorphic function depending on a finite number of variables, we have

$$\dot{\varphi} = \sum_{i=0}^{k} \frac{\partial \varphi}{\partial x(t-i)} \dot{x}(t-i) = \sum_{i=0}^{k} \frac{\partial \varphi}{\partial x(t-i)} g(-i)\, u(t-i).$$

Notice that the relative degree of φ is greater than one if and only if it satisfies $\dfrac{\partial \dot{\varphi}}{\partial u(t-i)} = 0$, $\forall i \ge 0$ which is equivalent to

$$\frac{\partial \varphi}{\partial x(t-i)} g(-i) = 0, \quad \forall i \ge 0,$$

which implies that $\dot{\varphi} = 0$ and consequently its relative degree is infinite. ■

Remark 2.8: It follows from Proposition 2.3-2.4 and Lemma 2.7 that the system Σ is accessible if there does not exist any nonzero function whose relative degree is greater than one. For the system without time-delays, i.e., $s = 0$, the problem of accessibility has a simple answer. In fact, it is well known that the module spanned by a single vector field $g(x(t))$ is always involutive and thus there always exist $n-1$ functions $\varphi_1(x(t)), \ldots, \varphi_{n-1}(x(t))$ such that

$$\frac{\partial \dot{\varphi}_l}{\partial u(t)} = \mathrm{d}\varphi_l \cdot g(x(t)) = 0, \quad 1 \le l \le n-1,$$

where $\mathrm{d}\varphi_l$ is defined as follows

$$\mathrm{d}\varphi_l = \left(\frac{\partial \varphi_l}{\partial x_1(t)}, \ldots, \frac{\partial \varphi_l}{\partial x_n(t)} \right).$$

These $n-1$ functions φ_i, $1 \le i \le n-1$, can be obtained just by solving the above equation and they satisfy $\mathrm{span}\{\mathrm{d}\varphi_1, \ldots, \mathrm{d}\varphi_{n-1}\} = g^{\perp}$. However for the nonlinear system with time-delay, this is not true any more since the module spanned by a single vector field is not always involutive any longer [2], [3].

In the next sections, we will study the following problem for nonlinear time-delay system Σ: how to construct the closure(s) of the module spanned by a single element $g = g(x(t), \ldots, x(t-s))$; how does the closure(s) characterize the accessibility property and all the autonomous functions of Σ.

III. CHARACTERIZATION OF THE ACCESSIBILITY

A. The Closure of $g(\mathbf{x}_{[s]})$ in E_0

To the time-delay system Σ, given by (1), we associate naturally the following infinite dimensional system

$$\tilde{\Sigma}: \quad \begin{array}{rcl} \dot{x}(t) & = & g(\mathbf{x}_{[s]})u(t) \\ \dot{x}(t-1) & = & g(\mathbf{x}_{[s]}(-1))u(t-1) \\ & \vdots & \end{array}$$

Consider the series development of $g(\mathbf{x}_{[s]})$ with respect to the parameters $\mathbf{x}(t-i)$, $i \ge 1$, around the point $(x(t), 0, 0, \ldots)$,

$$\begin{aligned} g \;=\; & g^0 + \sum_{i=1}^{s} \sum_{j=1}^{n} g^1_{ij} x_j(t-i) + \\ & \frac{1}{2} \sum_{i,k=1}^{s} \sum_{j,l=1}^{n} g^2_{ij,kl} x_j(t-i) x_l(t-k) + \cdots \end{aligned} \tag{2}$$

for $1 \le j \le n$, where

$$\begin{aligned} g^0 &= g(x(t), 0, \ldots, 0) \\ g^1_{ij} &= \left[\frac{\partial g}{\partial x_j(t-i)} \right]_{x(t-l)=0,\, l \ge 1} \\ &\;\;\vdots \end{aligned}$$

Let $\mathscr{G}$ be the distribution spanned by all the coefficient vector fields of the development (2), i.e.,

$$\mathscr{G} = \mathrm{span}\{g^0(x(t)), g^1_{ij}(x(t)), g^2_{ij,kl}(x(t)), \ldots\}. \tag{3}$$

We define the involutive distribution $\bar{\mathscr{G}}$ as the closure of $g(\mathbf{x}_{[s]})$ in space E_0.

Proposition 3.1: For the system Σ, given by (1), we have

(i) $\operatorname{rank}\bar{\mathscr{G}} = r_0 \leq n$ *if and only if there exist* $n - r_0$ *functions* $\varphi_1, \ldots, \varphi_{n-r_0}$, *depending on the* $x(t)$ *variables only, such that* $\dfrac{\partial \dot{\varphi}_l}{\partial u(t-i)} = 0$, $1 \leq l \leq n - r_0$, $\forall i \geq 0$;

(ii) $\operatorname{rank}\bar{\mathscr{G}} = r_0 = n$ *if and only if there doesn't exist any function* $\varphi = \varphi(x(t), x(t-1), \ldots, x(t-k))$, $k \geq 0$, *which satisfies* $\dfrac{\partial \dot{\varphi}}{\partial u(t-i)} = 0$, $\forall i \geq 0$.

Remark 3.2: Proposition 3.1 shows that the distribution $\bar{\mathscr{G}}$ characterizes all these autonomous functions which depend on the $x(t)$ variables only. It is important to emphasize that if $\operatorname{rank}\bar{\mathscr{G}} = r_0 < n$, besides the $n - r_0$ autonomous functions depending on the $x(t)$ variables only, there maybe exist still other ones which depend also the variables with time-delay. However, if $\operatorname{rank}\bar{\mathscr{G}} = r_0 = n$, there doesn't exist any autonomous function.

Example 3.3: Consider the following nonlinear time-delay system

$$\Sigma_1: \quad \dot{x}(t) = g(\mathbf{x}_{[1]})u(t) = \begin{pmatrix} 3x_3[x_1(t-1) - x_2^3(t-1)] \\ 1 \\ x_1(t-1) - x_2^3(t-1)) \end{pmatrix} u(t).$$

The development of $g(\mathbf{x}_{[1]})$ with respect to the parameters $\mathbf{x}(t-1)$, around the point $(x(t), 0)$, is given by

$$g(\mathbf{x}_{[1]}) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 3x_3(t) \\ 0 \\ 1 \end{pmatrix} x_1(t-1) + \begin{pmatrix} -3x_3(t) \\ 0 \\ -1 \end{pmatrix} x_2^3(t-1),$$

and thus

$$\mathscr{G} = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3x_3(t) \\ 0 \\ 1 \end{pmatrix} \right\}.$$

It is clear that $\operatorname{rank}\bar{\mathscr{G}} = 2$ and then there exists one function φ, which depends only the $x(t)$ variables, such that $\dfrac{\partial \dot{\varphi}}{\partial u(t-i)} = 0$, $i \geq 0$. By a simple computation, we get

$$\varphi = x_1(t) - \frac{3}{2}x_3^2(t).$$

Corollary 3.4: If there exist some function with relative degree greater than one, then there exists at least one of them which depends on the $x(t)$ variables only.

Proof: Assume that there exists no function $\varphi = \varphi(x(t))$ with relative degree greater than 1, then by (i) of Proposition 3.1, $\operatorname{rank}\bar{\mathscr{G}} = r_0 = n$. But by (ii), there doesn't exist any other function with relative degree 2, which gives a contradiction. ■

B. Main result

Theorem 3.5: The system Σ, defined by (1), is accessible if and only if $\operatorname{rank}\bar{\mathscr{G}} = n$.

Proof: Recall that for the system Σ, the relative degree of a function $\varphi \in \mathscr{K}$ is greater than 1 if and only if φ satisfies the condition $\frac{\partial \dot{\varphi}}{\partial u(t-i)} = 0$, $\forall i \geq 0$. If Σ is accessible, then there does not exist any function whose relative degree is greater than 1 and thus according to (ii) of Proposition 3.1, we get $\operatorname{rank}\bar{\mathscr{G}} = n$ which proves the necessity. The sufficiency is obvious. ■

Remark 3.6: Theorem 3.5 shows that the distribution $\bar{\mathscr{G}}$ characterizes completely the accessibility property of the original nonlinear time-delay system Σ, given by (1). Notice that the distribution $\bar{\mathscr{G}}$ is defined on space $E_0 = \mathbb{R}^n$ with coordinates $x(t)$. Therefore this result reduces the problem of verifying the accessibility of an infinite dimensional system to the computation of the rank of the distribution $\bar{\mathscr{G}}$ on $E_0 = \mathbb{R}^n$. Moreover, together with Proposition 3.1, it follows that when Σ is not accessible, the closure $\bar{\mathscr{G}}$ of $g(\mathbf{x}_{[s]})$ characterizes all the independent autonomous functions $\varphi_1, \ldots, \varphi_{n-r_0}$ with relative degree 2 depending on the $x(t)$-variables only (there maybe exist still other ones, see Section IV) and

$$(\bar{\mathscr{G}})^{\perp} = \operatorname{span}\{\mathrm{d}\varphi_1, \ldots, \mathrm{d}\varphi_{n-r_0}\}.$$

Example 3.3 cont'd. By Theorem 3.5, system Σ_1 is not accessible and possesses an autonomous function $\varphi = x_1(t) - \frac{3}{2}x_3^2(t)$.

Example 3.7: Consider the following nonlinear time-delay system

$$\Sigma_2: \quad \dot{x}(t) = g(\mathbf{x}_{[2]}) \cdot u(t) = \begin{pmatrix} x_2(t-1) \\ \sin x_1(t-2) \\ x_3(t-1)x_2(t) \end{pmatrix} u(t).$$

The development of $g(\mathbf{x}_{[2]})$ with respect to the parameters $\mathbf{x}(t-1), \mathbf{x}(t-2)$, around the point $(x(t), 0, 0)$, is given by

$$g(\mathbf{x}_{[2]}) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} x_2(t-1) + \begin{pmatrix} 0 \\ 0 \\ x_2(t) \end{pmatrix} x_3(t-1) + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} x_4(t-2) + \begin{pmatrix} 0 \\ -\frac{1}{3} \\ 0 \end{pmatrix} x_4^3(t-2) + \cdots$$

We get

$$\bar{\mathscr{G}} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

since
$$\left[\begin{pmatrix}0\\0\\x_2(t)\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right]=\begin{pmatrix}0\\0\\1\end{pmatrix}.$$
Thus $\mathrm{rank}\bar{\mathscr{G}}=3$ at any point $(x(t),0,0)$ which implies that Σ_2 is accessible at any point $(x(t),0,0)$.

Example 3.8: **(The chained form)** Consider the family of nonlinear time-delay systems
$$\Sigma_c:\quad \dot{x}(t)=g(\mathbf{x}_{[1]})\cdot u(t)=\begin{pmatrix}x_2(t)\\ \vdots\\ x_k(t-1)\\ \vdots\\ x_n(t)\\ 1\end{pmatrix}\cdot u(t),$$
where $x_k(t-1)$ denotes the $(k-1)$-th element of $g(\mathbf{x}_{[1]})$. If $k=n$, it is easy to check that $\mathrm{rank}\bar{\mathscr{G}}=n$ and moreover the distribution $\mathscr{G}$ satisfies the conditions $\mathrm{rank}\mathscr{G}^{(i)}=i+2$, for $0\le i\le n-2$, where $\mathscr{G}^{(i)}$ is defined by $\mathscr{G}^{(0)}=\mathscr{G}$ et $\mathscr{G}^{(i)}=\mathscr{G}^{(i-1)}+[\mathscr{G}^{(i-1)},\mathscr{G}^{(i-1)}]$.

If $k<n$, then this system is not accessible since $\mathrm{rank}\bar{\mathscr{G}}=k$ and thus there exist $n-k$ autonomous functions ($\varphi=x_{n-1}(t)-\frac{1}{2}x_n^2(t)$, etc.).

Remark 3.9: Note that the system defined in $\mathbb{R}^n$
$$\Sigma_3:\quad \dot{x}(t)=g(\mathbf{x}(t))\cdot u(t)=\begin{pmatrix}x_2(t)\\ \vdots\\ x_n(t)\\ 1\end{pmatrix}u(t).$$
is never accessible. Example 3.8 shows an interesting phenomenon, that is Σ_3 becomes completely accessible when we add a time-delay to the $(n-1)$-th element of g and it becomes partially accessible if we add a time-delay to the k-th element of g, for $1\le k\le n-2$.

IV. Autonomous functions in E_i, $i\ge 1$

Example 4.1: Consider the following nonlinear time-delay system
$$\Sigma_4:\quad \dot{x}(t)=\begin{pmatrix}1\\0\\x_2(t-1))\end{pmatrix}u(t).$$
It is easy to calculate that
$$\bar{\mathscr{G}}=\mathrm{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}.$$
Since $\mathrm{rank}\bar{\mathscr{G}}=2$, this system is not accessible and there exists one autonomous function depending on the $x(t)$ variables only that is given by $\varphi_1=x_2(t)$.

It is interesting to note that besides $\varphi_1=x_2(t)$, the system Σ_4 possesses still other autonomous functions. On one hand, obviously all the functions with time-delay $\varphi_1(-l)=x_2(t-l)$, $\forall l\ge 0$, satisfy the conditions $\dfrac{\partial\dot{\varphi}}{\partial u(t-i)}=0$, $\forall i\ge 0$, in other words, they are autonomous functions of Σ_4. But they can be seen as trivial ones since for any autonomous function φ of system Σ, given by (1), it is obvious that all the functions $\varphi(-l)$, $l\ge 0$, are also autonomous ones. On the other hand, there exists still another independent non trivial autonomous function of Σ_4 given by $\varphi_2=x_1(t)x_2(t-1)-x_3(t)$ which can not be characterized by the closure $\bar{\mathscr{G}}$ in E_0. One question arises: which distribution can determine this new autonomous function? More generally, when $r_0<n$, i.e ., the considered system Σ is not accessible, how to find all its autonomous functions $\varphi=\varphi(x(t),\dots,x(t-k))$ such that $\dfrac{\partial\dot{\varphi}}{\partial u(t-i)}=0$, $i\ge 0$? In this section, we will construct the closures of $g(\mathbf{x}_{[s]})$ in the extended space E_i, $i\ge 0$, which can characterize all the autonomous functions.

Now consider the following elements defined in the extended space $E_i=\mathbb{R}^{n(i+1)}$
$$\begin{pmatrix}g\\0\\ \vdots\\ \vdots\\0\\0\end{pmatrix},\begin{pmatrix}0\\g(-1)\\ \vdots\\ \vdots\\0\\0\end{pmatrix},\dots,\begin{pmatrix}0\\ \vdots\\ \vdots\\ \vdots\\0\\g(-i)\end{pmatrix}$$
and express them by their development with respect to the parameters $\mathbf{x}(t-l)$, $l\ge i+1$, around $x(t-i-1)=x(t-i-2)=\cdots=0$. Denote by $\mathscr{G}_{E_i}$ the distribution spanned by all the coefficient vector fields, which depend on obviously the $(x(t),\dots,x(t-i))$ variables only. Clearly, $\mathscr{G}_{E_0}$ coincides with $\mathscr{G}$, defined by (3). We define the distribution $\bar{\mathscr{G}}_{E_i}$ as the *extended closure* of $g(\mathbf{x}_{[s]})$ in space E_i.

Proposition 4.2: Assume that the system Σ, given by (1), is not accessible, i.e., it satisfies $\mathrm{rank}\bar{\mathscr{G}}_{E_0}=r_0<n$*, then*

(i) *The extended closures $\bar{\mathscr{G}}_{E_k}$, $k\ge 0$, characterizes completely all those autonomous functions, which depend on the $x(t),\dots,x(t-k)$ variables only, by the equation* $\mathrm{d}\varphi\cdot\bar{\mathscr{G}}_{E_k}=0$.

(ii) *Assume that* $(\bar{\mathscr{G}}_{E_0})^{\perp}=\mathrm{span}\{\mathrm{d}\varphi_l(x(t)),\ 1\le l\le n-r_0\}$ *and for arbitrary $k\ge 1$,*
$$(\bar{\mathscr{G}}_{E_k})^{\perp}=(\bar{\mathscr{G}}_{E_{k-1}})^{\perp}+\mathrm{span}\{\mathrm{d}\psi_j(\mathbf{x}_{[k]}),\ j\ge 0\},$$
then we have
$$\begin{aligned}\mathrm{d}\psi_j(\mathbf{x}_{[k]})\ \in\ &\mathrm{span}\{\mathrm{d}x(t),\dots,\mathrm{d}x(t-k-1),\\ &\mathrm{d}\varphi_l(x(t-k)),\quad 1\le l\le n-r_0\}.\end{aligned}$$

(iii) *The system Σ possesses at most $n-1$ independent (on $\mathscr{K}(\delta]$) autonomous functions.*

(iv) *There exists an index $\gamma \geq 0$ such that the extended closure $\bar{\mathscr{G}}_{E_\gamma} = 0$ characterizes completely all the independent autonomous functions of Σ by the equation $\mathrm{d}\varphi \cdot \bar{\mathscr{G}}_{E_\gamma} = 0$. Moreover*

$$\bar{\mathscr{G}}_{E_\gamma} = \begin{pmatrix} \mathscr{G}_{\gamma,0} & & \\ & \ddots & \\ & & \mathscr{G}_{\gamma,\gamma} \end{pmatrix}, \tag{4}$$

and denoting by $r_\gamma = rank(\mathscr{G}_{\gamma,0})$ there exist $n - r_\gamma$ independent autonomous functions which satisfy $\mathrm{d}\varphi_j \cdot \mathscr{G}_{\gamma,i} = 0$, $i = 0, \cdots, \gamma$, $j \in [1, \cdots n - r_\gamma]$, and any other autonomous function φ of Σ satisfies $d\varphi \in \mathrm{span}_{\mathscr{K}(\delta]}\{d\varphi_1, \cdots, d\varphi_{n-r_0}\}$.

Remark 4.3: The rank of $\bar{\mathscr{G}}_{E_\gamma}$ is a constant and thus due to the structure of $\bar{\mathscr{G}}_{E_\gamma}$, there always exist distributions $\mathscr{G}_{\gamma,0}, \mathscr{G}_{\gamma,1}, \ldots, \mathscr{G}_{\gamma,\gamma}$ that are of constant rank and satisfy (4).

Example 4.4: Consider the following nonlinear time-delay system

$$\Sigma_5 : \quad \dot{x}(t) = \begin{pmatrix} 1 \\ x_3(t-2) \\ 0 \\ x_3(t-1) \end{pmatrix} u(t).$$

The involutive closure $\bar{\mathscr{G}}_{E_0}$ of $g(\mathbf{x}_{[2]})$ is given by

$$\bar{\mathscr{G}}_{E_0} = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\},$$

which gives an autonomous function depending on the $x(t)$ variables only, that is

$$\varphi_{11} = x_3(t)$$

since $(\bar{\mathscr{G}}_{E_0})^\perp = \mathrm{span}\{\mathrm{d}x_3(t)\}$.

The extended closure $\bar{\mathscr{G}}_{E_1}$ of $g(\mathbf{x}_{[2]})$ is given by

$$\bar{\mathscr{G}}_{E_1} = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ x_3(t-1) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Observe that $(\bar{\mathscr{G}}_{E_1})^\perp = \mathrm{span}\{\mathrm{d}x_3(t), \mathrm{d}x_3(t-1), x_3(t-1)\mathrm{d}x_1(t) - \mathrm{d}x_4(t)\}$, and thus we get 3 autonomous functions depending on the $(x(t), x(t-1))$ variables

$$\begin{aligned} \varphi_{11} &= x_3(t) \\ \varphi_{12} &= x_3(t-1) \\ \varphi_{21} &= x_1(t)x_3(t-1) - x_4(t). \end{aligned}$$

Clearly, $(\bar{\mathscr{G}}_{E_1})^\perp = (\bar{\mathscr{G}}_{E_0})^\perp + \mathrm{span}\{\mathrm{d}\varphi_{12}, \mathrm{d}\varphi_{21}\}$ and $\mathrm{d}\varphi_{12}$, $\mathrm{d}\varphi_{21}$ satisfy that

$$\mathrm{d}\varphi_{12}, \mathrm{d}\varphi_{21} \in \mathrm{span}\{\mathrm{d}x(t), \mathrm{d}\varphi_{11}(x(t-1))\},$$

and thus satisfy item (i) and (ii) of Proposition 4.2. Now consider the extended closure $\bar{\mathscr{G}}_{E_2}$ of $g(\mathbf{x}_{[2]})$ in E_2,

$$\bar{\mathscr{G}}_{E_2} = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ x_3(t-2) \\ 0 \\ x_3(t-1) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ x_3(t-2) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

A simple calculation shows that $(\bar{\mathscr{G}}_{E_2})^\perp = (\bar{\mathscr{G}}_{E_1})^\perp + \mathrm{span}\{\mathrm{d}x_3(t-2), x_3(t-2)\mathrm{d}x_1(t-1) - \mathrm{d}x_4(t-1)$ which implies that we have 6 autonomous functions depending on the $(x(t), x(t-1), x(t-2))$ variables

$$\begin{aligned} \varphi_{11} &= x_3(t) \\ \varphi_{12} &= x_3(t-1) \\ \varphi_{13} &= x_3(t-2) \\ \varphi_{21} &= x_1(t)x_3(t-1) - x_4(t) \\ \varphi_{22} &= x_1(t-1)x_3(t-2) - x_4(t-1) \\ \varphi_{31} &= x_1(t)x_3(t-2) - x_2(t). \end{aligned}$$

However, only three autonomous functions φ_{11}, φ_{21}, φ_{31} are independent on $\mathscr{K}(\delta]$. By item (iii) of Proposition 4.2, they span a basis of all the non trivial autonomous functions of

Σ_5. Moreover, it is easy seen that φ_{11}, φ_{21}, φ_{31} are exactly the solutions of the equation $\mathrm{d}\varphi \cdot \mathscr{G}_{20} = 0$, where

$$\mathscr{G}_{20} = \begin{pmatrix} 1 \\ x_3(t-2) \\ 0 \\ x_3(t-1) \end{pmatrix},$$

which corresponds the result of item (iv) of Proposition 4.2.

V. Decomposition of non-accessible driftless single-input systems

When the system Σ, given by (1), is not accessible, i.e., it satisfies $\mathrm{rank}\bar{\mathscr{G}} = r_0 < n$, it can be decomposed into two parts and one of them being nonaccessible. Let $\mathrm{rank}\mathscr{G}_{\gamma,0} = r_\gamma \geq r_0 \geq 1$, by item (iv) of Proposition 4.2, there exists $n - r_\gamma$ independent autonomous functions, denoted by $\varphi_1, \ldots, \varphi_{n-r_\gamma}$, which are determined by the equation $\mathrm{d}\varphi \cdot \mathscr{G}_{\gamma,i} = 0$, $i \in [0, \gamma]$ and such that any other autonomous function $\varphi(x(t), \cdots, x(t-l))$ satisfies $\mathrm{d}\varphi \in \mathrm{span}_{\mathscr{K}(\delta]}\{\mathrm{d}\varphi_1, \cdots, \mathrm{d}\varphi_{n-r_0}\}$. Obviously we have

$$\frac{\partial(\varphi_1, \ldots, \varphi_{n-r_\gamma})}{\partial(x_1(t), \ldots, x_n(t))} = n - r_\gamma.$$

A consequence of the previous consideration is the possibility of finding an appropriate bicausal change of coordinates which decomposes the system into an accessible part and an autonomous part. This result is stated in the next theorem while its proof is omitted for space reasons.

Theorem 5.1: Let $r_\gamma = \mathrm{rank}(\mathscr{G}_{\gamma,0})$, and let $\varphi_1, \ldots, \varphi_{n-r_\gamma}$ be the $n - r_\gamma$ independent functions such that $\mathrm{d}\varphi_i g(\cdot) = 0$ and any other autonomous function φ for Σ, satisfies $\mathrm{d}\varphi \in \mathrm{span}_{\mathscr{K}(\delta]}\{\mathrm{d}\varphi_1, \cdots, \mathrm{d}\varphi_{n-r_0}\}$. Then there exists r_γ independent functions $\varphi_{n-r_\gamma+1}, \ldots, \varphi_n)$ such that the change of coordinates $z = (\varphi_1, \ldots, \varphi_{n-r_\gamma}, \varphi_{n-r_\gamma+1}, \ldots, \varphi_n)^T$ is bicausal and in the new coordinates the system reads

$$\begin{cases} \dot{z}_1(t) & = & 0 \\ & \vdots & \\ \dot{z}_{n-r_\gamma}(t) & = & 0 \\ \dot{z}_{n-r_{\gamma+1}}(t) & = & \tilde{g}_{n-r_{\gamma+1}}(\mathbf{z}_{[\bar{s}]}) \cdot u(t) \\ & \vdots & \\ \dot{z}_n(t) & = & \tilde{g}_n(\mathbf{z}_{[\bar{s}]}) \cdot u(t). \end{cases}$$

Example 4.4 cont'd. By Proposition 4.2 any autonomous function φ for Σ_5 satisfies $\mathrm{d}\varphi \in \mathrm{span}_{\mathscr{K}(\delta]}\{\mathrm{d}\varphi_{11}, \mathrm{d}\varphi_{21}, \mathrm{d}\varphi_{31}\}$. Set now $z_1(t) = \varphi_{11} = x_3(t)$, $z_2(t) = \varphi_{21} = x_1(t)x_3(t-1) - x_4(t)$,$z_3(t) = \varphi_{31} = x_1(t)x_3(t-2) - x_2(t)$ and note that setting $z_4(t) = x_1(t)$ the change of coordinates

$$z(t) = \begin{pmatrix} x_3(t) \\ x_1(t)x_3(t-1) - x_4(t) \\ x_1(t)x_3(t-2) - x_2(t) \\ x_1(t) \end{pmatrix}$$

which is characterized by the differential representation

$$\mathrm{d}z = \begin{pmatrix} 0 & 0 & 1 & 0 \\ x_3(t-1) & 0 & x_1(t)\delta & -1 \\ x_3(t-2) & -1 & x_1(t)\delta^2 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} dx$$

is bicausal, being $T(x_{[2]}, \delta)$ unimodular. In the new coordinates the system reads

$$\begin{aligned} z_1(t) &= 0 \\ z_2(t) &= 0 \\ z_3(t) &= 0 \\ z_4(t) &= u(t) \end{aligned}$$

VI. Conclusions

A class of driftless single-input nonlinear time-delay systems has been considered. Surprisingly, it has been shown that they may be fully accessible. More generally, thanks to appropriate geometric tools, it has been possible to characterize completely the accessibility of those systems. Moreover, whenever the system is not accessible, then it was shown that a standard decomposition into an autonomous subsystem and an accessible subsystem always exists. The latter generalizes a canonical decomposition which is well known for linear and nonlinear delay free systems. Further research is required for more general systems which include a drift term and delayed input terms.

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