2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

A Truncated Prediction Approach to Stabilization of Linear Systems with Long Time-Varying Input Delay

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Abstract— **This paper studies the problem of stabilizing a linear system with long time-varying delay in the input. Under the assumption that the open-loop system is not exponentially unstable and stabilizable, a finite dimensional static time-varying linear state feedback controller is obtained by truncating the prediction based controller and by adopting the parametric Lyapunov equation based low gain feedback. As long as the time-varying delay is exactly known and bounded, explicit condition is provided to guarantee the stability of the closed-loop system. It is also shown that the proposed controller achieves semi-global stabilization of the system if its actuator is also subject to saturation. Numerical examples show the effectiveness of the proposed approach.**

I. Introduction

Time delay, which arises frequently in many engineering systems such as long transmission lines in pneumatic systems, rolling mills, nuclear reactors, hydraulic systems, manufacturing processes, digital control systems and systems that are controlled remotely ([4], [8], [16], [18]), is generally recognized as a source of performance degradation and even instability of control systems ([4]). Control problems, especially, the problems of stability analysis and stabilization, for time-delay systems have attracted much attention for several decades. Various types of time-delay systems have been investigated and a great number of results have been reported in the literature (see, *e.g.*, [4], [7], [11], [18], [22], [23] and the references cited therein). There are several categories of methods for handling stabilization of time-delay systems. The most efficient methods are probably the Lyapunov-Krasovskii functional based methods (see, *e.g.*, [5], [17], [20] and [21]). The Razumikhin Theorem based approach also falls into this category ([4]). The idea is to find a positive-definite functional such that its derivative along the trajectories of the time-delay system is negative. The results obtained by these methods can be easily recast into linear matrix inequalities, which can be efficiently solved numerically. A drawback of these methods is that in general only sufficient conditions can be obtained, which leads to conservatism. Even though such conservatism can be reduced, the resulting conditions are usually very complicated with the associated computational burden dramatically increased ([2], [3]).

Another efficient approach to dealing with time-delay systems is the predictor feedback ([1], [6], [7], [8], [12] and [13]). This approach is especially effective for input delay systems, including unstable ones. However, as noticed in [7], almost all the existing results deal with problems where the delay is constant and time-varying input delay has received very little attention. Though the predictor feedback for time-varying input delay system has been introduced by Artstein in [1], the design is not worked out in detail since the case of time-varying delay is considered only for plants that are time-varying, in which case explicit developments are not possible ([7]). Very recently, by constructing a Lyapunov functional using a backstepping transformation with time-varying kernels, and transforming the actuator state into a transport partial differential equation with a convection speed coefficient that varies with both space and time, the exponential stability of the feedback system with the predictor controller proposed in [15] for systems with time-varying input delay is proven in [7].

In this paper, inspired by the work in [7] and [15], we consider predictor based controller for linear systems with long time-varying input delay. Different from those traditional prediction based controllers, which are infinite-dimensional static feedback laws and may cause difficulties in their practical implementation (see, for example, [18] and [19]), we develop a truncated prediction based controller which only involves finite dimensional static state feedback by ignoring safely the distributed terms in the traditional prediction based feedback. It is shown that if the open-loop system is not exponentially unstable and the nominal feedback gain is designed by our recently developed parametric Lyapunov equation based low gain feedback ([25] and [26]), the asymptotic stability of the closed-loop system under the truncated prediction feedback can be established with the aid of the Razumikhin stability theorem. An explicit condition is provided for choosing the free parameter in the controller. It is also shown that the proposed truncated prediction feedback also achieves semi-global stabilization of the considered delay system when the actuator is subject to saturation. A numerical example the time-varying delays considered in [7] is worked out to illustrate the effectiveness of the proposed approach. We point out that the proposed controller reduces to the one proposed in [26] if the delay in the input is constant, exposing an underlying mechanism of the approaches given in [9] and [26]. We also point out

This work was supported in part by the National Natural Science Foundation of China under grant numbers 60904007, 61104124, 60928008, and 61074111, the China Postdoctoral Science Foundation under grant number 20100480059, the Heilongjiang Postdoctoral Foundation of China under grant number LBH-210132, and the Foundation for Innovative Research Group of the National Natural Science Foundation of China under grant 601021002, and in part by the National Science Foundation of the United States under grant number CMMI-1129752.

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that, although long time-varying input pointed delay, multiple pointed delays, and distributed delay have been respectively considered in [27], [28] and [29], the open-loop systems are only allowed to have unstable zeros at the origin. In the present paper, the unstable poles are allowed to be on the imaginary axis.

The remainder of this paper is organized as follows. The idea of the truncated prediction feedback for linear systems with time-varying input delay is introduced in Section II. In Section III we prove the stability of the closed-loop system. Numerical examples are presented in Section IV to validate the effectiveness of the proposed approach. Finally, Section V concludes the paper.

Notation: The notation used in this paper is fairly standard. For a vector $u \in \mathbf{R}^m$, we use $\|u\|_\infty$ to denote the ∞-norm of u and $\mathrm{sign}(y)$ to denote the sign function which takes value +1 if $y > 0$ and -1 if $y < 0$. The standard saturation function is defined as $\mathrm{sat}(u) = \mathrm{sign}(u)\min\{|u|, 1\}$. For a matrix $A \in \mathbf{R}^{n\times n}$, A^{T} and $\mathrm{tr}(A)$ are respectively its transpose and trace. Finally, for a positive scalar τ, let $\mathscr{C}_{n,\tau} = \mathscr{C}([-\tau, 0], \mathbf{R}^n)$ denote the Banach space of continuous vector functions mapping the interval $[-\tau, 0]$ into $\mathbf{R}^n$ with the topology of uniform convergence, and let $x_t \in \mathscr{C}_{n,\tau}$ denote the restriction of $x(t)$ to the interval $[t-\tau, t]$ translated to $[-\tau, 0]$, that is, $x_t(\theta) = x(t+\theta), \theta \in [-\tau, 0]$.

II. Truncated Prediction Approach

Consider the following linear system with input delay

$$\dot{x}(t) = Ax(t) + Bu(\phi(t)), \tag{1}$$

where $x(t) \in \mathbf{R}^n$ and $u(t) \in \mathbf{R}^m$ are respectively the state and input vectors, and $\phi(t) : \mathbf{R}^+ \to \mathbf{R}$ is a continuously differentiable function that incorporates the actuator delay. The function $\phi(t)$ can be defined in a more standard form

$$\phi(t) = t - D(t), \tag{2}$$

where $D(t) : \mathbf{R}^+ \to \mathbf{R}^+$ is the time-varying delay. However, as pointed out in [7], the formalism involving the function $\phi(t)$ turns out to be more convenient because the predictor problem we will consider later requires the inverse function of $\phi(t)$, namely, $\phi^{-1}(t)$. In this paper, we will proceed with model (1) and assume (2) whenever necessary. Some assumptions on $\phi(t)$ will be made clear as follows.

Assumption 1: The function $\phi : \mathbf{R}^+ \to \mathbf{R}$ is a continuously differentiable, invertible and exactly known function and the delay $D(t)$ is bounded, namely, there exists a finite, yet arbitrarily large, number $\overline{D} > 0$ such that

$$0 \le D(t) \le \overline{D}, \quad \forall t \in [0, \infty).$$

The main idea of predictor feedback is to design the feedback controller

$$u(\phi(t)) = Kx(t), \quad \forall \phi(t) \ge 0, \tag{3}$$

such that the closed-loop system consisting of (1) and (3) is

$$\dot{x}(t) = (A + BK)x(t), \quad \forall \phi(t) \ge 0, \tag{4}$$

where K is such that $A + BK$ is asymptotically stable. The controller (3) can also be written as

$$u(t) = Kx\left(\phi^{-1}(t)\right), \quad t \ge 0. \tag{5}$$

However, as $\phi^{-1}(t) \ge t, \forall t \ge 0$, the above controller is impossible to implement in practice. To overcome this problem, $x\left(\phi^{-1}(t)\right)$ should be predicted based on the current state. By using the system model (1) and the variation of constants formula ([4] and [7]), it can be obtained that

$$x\left(\phi^{-1}(t)\right) = \mathrm{e}^{A\left(\phi^{-1}(t)-t\right)}x(t) + \int_t^{\phi^{-1}(t)} \mathrm{e}^{A\left(\phi^{-1}(t)-s\right)}Bu(\phi(s))\,\mathrm{d}s.$$

Substituting the above relation into (5) gives the following predicator feedback

$$u(t) = K\left(\mathrm{e}^{A\left(\phi^{-1}(t)-t\right)}x(t) + \int_t^{\phi^{-1}(t)} \mathrm{e}^{A\left(\phi^{-1}(t)-s\right)}Bu(\phi(s))\,\mathrm{d}s\right). \tag{6}$$

For easy reference, the first term

$$u_{\mathrm{f}}(t) = K\mathrm{e}^{A\left(\phi^{-1}(t)-t\right)}x(t), \tag{7}$$

is referred to as the finite dimensional prediction while the second term

$$u_{\mathrm{i}}(t) = K\int_t^{\phi^{-1}(t)} \mathrm{e}^{A\left(\phi^{-1}(t)-s\right)}Bu(\phi(s))\,\mathrm{d}s, \tag{8}$$

is referred to as the infinite dimensional prediction.

The prediction based controller (6) is conceptually appealing as it ensures that the closed-loop system (4) has finite spectrum (this is why this method is also referred to as the finite spectrum assignment method). However, the controller in (6) is implicit since u is present on both sides of equation (6) and under an integral sign, which makes the implementation hard even when the delay $D(t)$ is a constant one ([18], [24]). As explained in [12], obtaining this integral term as the solution to a differential equation must be discarded because it involves unstable pole-zero cancellations when A is unstable. An alternative is to approximate the integral term with a sum of point-wise delays by using a numerical quadrature rule such as rectangular, trapezoidal and Simpson's rules. During the past several decades, the effect of such a semi-discretization on the stability of the closed-loop system has been examined thoroughly. It is demonstrated in [14] and [19] with a scalar example that for some prescribed system parameters, the approximated control law may not stabilize the delay system no matter how precise the approximation is. Considerable attention has been paid to overcome this problem in the past several decades (see [14], [18], [24] and the references therein).

In this paper, we will show that the distributed term in the prediction based controller (6) is not required if some additional requirements are imposed on the system. Consequently, the implementation problem for such type of controllers is avoided entirely. To this end, we first notice that, since $D(t)$ is bounded, the function $\phi^{-1}(t)-t$, which was referred to as the prediction time, is also bounded. In fact,

$$0 \le \phi^{-1}(t) - t \le \overline{D}. \quad (9)$$

Let the nominal feedback gain K be parameterized as $K = K(\gamma) : \gamma \in (0,1]$. If $K(\gamma)$ is of order 1 with respect to γ, namely,

$$\lim_{\gamma\to 0^+} \frac{1}{\gamma} \|K(\gamma)\| < \infty, \quad (10)$$

then the finite dimensional prediction term $u_{\mathrm{f}}(t)$ in the predictor feedback law (6) is also "of order 1" with respect to γ in view of (9). Consequently, control $u(t)$ itself is "of order 1" with respect to γ, namely,

$$\lim_{\gamma\to 0^+} \frac{1}{\gamma} \|u(t)\| < \infty, \quad t \ge 0.$$

As a result, by virtue of (9),

$$\begin{aligned}
&\lim_{\gamma\to 0^+} \frac{1}{\gamma^2} \left\| K(\gamma) \int_t^{\phi^{-1}(t)} \mathrm{e}^{A\left(\phi^{-1}(t)-s\right)} B u(\phi(s))\,\mathrm{d}s \right\| \\
&= \lim_{\gamma\to 0^+} \left\| \left(\frac{1}{\gamma} K(\gamma)\right) \right. \\
&\quad \left. \cdot \int_t^{\phi^{-1}(t)} \mathrm{e}^{A\left(\phi^{-1}(t)-s\right)} B \left(\frac{u(\phi(s))}{\gamma}\right) \mathrm{d}s \right\| \\
&\le \lim_{\gamma\to 0^+} \frac{1}{\gamma} \|K(\gamma)\| \int_t^{\phi^{-1}(t)} \left(\left\| \mathrm{e}^{A\left(\phi^{-1}(t)-s\right)} B \right\| \right. \\
&\quad \left. \cdot \lim_{\gamma\to 0^+} \frac{1}{\gamma} \|u(s)\| \right) \mathrm{d}s, \\
&< \infty,
\end{aligned}$$

namely, the infinite dimensional prediction term $u_{\mathrm{i}}(t)$ is at least "of order 2" with respect to γ. This indicates that, no matter how large the value of $\overline{D}$ is, the infinite dimensional prediction term $u_{\mathrm{i}}(t)$ in (8) is dominated by the finite dimensional prediction term $u_{\mathrm{f}}(t)$ in (7) and thus might be safely neglected in $u(t)$ when γ is sufficiently small. As a result, the predictor feedback law (6) can be truncated as

$$u(t) = u_{\mathrm{f}}(t) = K(\gamma)\, \mathrm{e}^{A\left(\phi^{-1}(t)-t\right)} x(t), \quad (11)$$

which we refer to as the "truncated predictor feedback". The main advantages of the truncated predictor feedback (11) over the predictor feedback (6) is that the numerical problems encountered in the implementation of the integral prediction (distributed) term (8) is entirely avoided.

However, to ensure that the truncated prediction approach is indeed possible, two problems should be solved. On the one hand, we need to identify what type of systems can be stabilized by a parameterized feedback gain $K = K(\gamma) : \gamma \in (0,1]$ such that (10) is satisfied. On the other hand, we need to verify that the truncated prediction feedback (11) can indeed stabilize the time-delay system (1). For the first problem, it is well-known that such a parameterized feedback gain exists if and only if (A, B) is stabilizable and all the eigenvalues of A are on the closed left-half plane (see, for example, [10] and [25]). Since the stable eigenvalues of A does not affect the stabilization of the system, for simplicity, we impose the following assumption on the system.

Assumption 2: The matrix pair $(A, B) \in \mathbf{R}^{n\times n} \times \mathbf{R}^{n\times m}$ is controllable with all the eigenvalues of A being on the imaginary axis.

The main purpose of this paper is to give a positive answer to the second problem mentioned above.

Remark 1: When the delay in the time-delay system (1) is constant, say, $\phi(t) = t - D$ where D is a constant, then $\phi^{-1}(t) = t + D$, and the truncated prediction controller (11) becomes

$$u(t) = K(\gamma)\, \mathrm{e}^{AD} x(t).$$

In this case, it has been proven in [9] and [26] that, if $K(\gamma)$ is properly designed, such a controller can indeed globally stabilize the time-delay system (1). This also explains why we have designed such a controller in [9] and [26].

III. Stability Analysis of the Closed-Loop System

In this section, we first prove that the truncated prediction controller (11) can indeed stabilize the time-delay system (1) under Assumptions 1 and 2.

Theorem 1: Consider the linear system (1) with time-varying delay. Let Assumptions 1 and 2 be satisfied and $n \ge 2$. Then the truncated prediction feedback

$$u(t) = -B^{\mathrm{T}} P(\gamma)\, \mathrm{e}^{A\left(\phi^{-1}(t)-t\right)} x(t), \quad \forall \gamma \in \left(0, \frac{\delta^*}{\overline{D}(n-1)}\right), \quad \forall t \ge 0, \quad (12)$$

globally stabilizes system (1), where $u(t) = 0, \forall t \in [\phi(0), 0)$, the matrix $P(\gamma)$ is the unique positive definite solution to the parametric ARE

$$A^{\mathrm{T}} P + PA - PBB^{\mathrm{T}} P = -\gamma P, \quad (13)$$

and δ^* is the unique positive root of the following equation

$$\frac{(n-1)^2}{n^3} = \delta \mathrm{e}^{\delta} \left(\mathrm{e}^{\delta} - 1\right). \quad (14)$$

Proof: (Sketch) Let $K = -B^{\mathrm{T}} P$. The closed-loop system can be written as

$$\dot{x}(t) = Ax(t) + BK\mathrm{e}^{A(t-\phi(t))} x(\phi(t)). \quad (15)$$

It can be readily shown that we need only to consider the stability of system (15) with $t \ge \phi^{-1}\left(\phi^{-1}(0)\right)$. Notice that with the help of the above model (15) and the variation of constants formula ([4] and [7]), we can compute

$$\begin{aligned}
x(t) &= \mathrm{e}^{A(t-\phi(t))} x(\phi(t)) \\
&\quad + \int_{\phi(t)}^t \mathrm{e}^{A(t-s)} BK\mathrm{e}^{A(s-\phi(s))} x(\phi(s))\,\mathrm{d}s.
\end{aligned}$$

Then the closed-loop system (15) can be rewritten as

$$\begin{aligned}
\dot{x}(t) &= (A + BK)\, x(t) \\
&\quad - BK \int_{\phi(t)}^t \mathrm{e}^{A(t-s)} BK\mathrm{e}^{A(s-\phi(s))} x(\phi(s))\,\mathrm{d}s \\
&\triangleq (A + BK)\, x(t) - BK\lambda(t). \quad (16)
\end{aligned}$$

By virtue of (13) and Lemma 1, and after some computation, the time derivative of $V(x(t)) = x^{\mathrm{T}}(t) P x(t)$ along the trajectories of the system in (16) satisfies

$$\dot{V}(x(t)) \leq -\gamma V(x(t)) + n\gamma\lambda^{\mathrm{T}}(t) P \lambda(t). \tag{17}$$

By using the Jensen inequality and Lemma 1, and after some intricate computation, we get

$$\lambda^{\mathrm{T}}(t) P \lambda(t) = (n\gamma)^2 \overline{D} \mathrm{e}^{\omega\gamma\overline{D}} \int_{\phi(t)}^{t} \mathrm{e}^{\omega\gamma(t-s)} V(x(\phi(s))) \, \mathrm{d}s,$$

where $\omega = n - 1$. Substituting the above inequality into (17) gives

$$\begin{aligned} \dot{V}(x(t)) \quad \leq \quad & -\gamma V(x(t)) + (n\gamma)^3 \overline{D} \mathrm{e}^{\omega\gamma\overline{D}} \\ & \cdot \int_{\phi(t)}^{t} \mathrm{e}^{\omega\gamma(t-s)} V(x(\phi(s))) \, \mathrm{d}s. \end{aligned} \tag{18}$$

Notice that

$$\phi(\phi(t)) = t - D(t) - D((t - D(t))) \triangleq t - D'(t).$$

Clearly, we have $|D'(t)| \leq 2\overline{D}$. Hence, under the condition that

$$V(x(t+\theta)) < \eta V(x(t)), \quad \forall \theta \in \left[-2\overline{D}, 0\right],$$

where $\eta > 1$ is any given scalar, the inequality in (18) can be continued as

$$\begin{aligned} & \dot{V}(x(t)) \\ & \leq -\gamma V(x(t)) + \eta (n\gamma)^3 \overline{D} \mathrm{e}^{\omega\gamma\overline{D}} \int_{t-\overline{D}}^{t} \mathrm{e}^{\omega\gamma(t-s)} V(x(t)) \, \mathrm{d}s \\ & = -\gamma \frac{\eta n^3}{\omega^2} \left(\frac{\omega^2}{\eta n^3} - \delta \mathrm{e}^{\delta} \left(\mathrm{e}^{\delta} - 1 \right) \right) V(x(t)), \end{aligned} \tag{19}$$

where $\delta = \omega\gamma\overline{D}$. Notice that $f(\delta) = \delta \mathrm{e}^{\delta} \left(\mathrm{e}^{\delta} - 1 \right)$ is a strictly increasing function. Therefore we deduce from equation (14) that there exists a number $\eta > 1$ and a sufficiently small number $\varepsilon > 0$ such that

$$\frac{\omega^2}{\eta n^3} - \delta \mathrm{e}^{\delta} \left(\mathrm{e}^{\delta} - 1 \right) > \varepsilon, \quad \forall \delta \in (0, \delta^*).$$

With the above inequality we get from (19) that

$$\dot{V}(x(t)) \leq -\gamma \frac{\eta n^3 \varepsilon}{\omega^2} V(x(t)), \quad \forall \gamma \in \left(0, \frac{\delta^*}{\overline{D}\omega}\right).$$

The closed-loop system (15) is thus asymptotically stable by virtue of the Razumikhin stability theorem. ■

Remark 2: In the above we have assumed that $n \geq 2$. If $n = 1$, say, the system is of the form $\dot{x} = -u(\phi(t))$, then we get from the proof of Theorem 1 that

$$\dot{V}(x(t)) \leq -\gamma \left(1 - \eta\gamma^2\overline{D}^2\right) V(x(t)).$$

Therefore the stability of the closed-loop system is guaranteed provided $\gamma \in \left(0, 1/\overline{D}\right)$.

We next briefly discuss about the output feedback stabilization of system (1) by a truncated prediction based controller. Let us assume that the time-delay system (1) has an output

$$y(t) = Cx(t), \quad C \in \mathbf{R}^{p\times n}, \tag{20}$$

where (A, C) is detectable. We construct the following observer based controller

$$\left\{ \begin{array}{l} \dot{\widehat{x}}(t) = A\widehat{x}(t) + Bu(\phi(t)) + L(y(t) - C\widehat{x}(t)), \\ u(t) = -B^{\mathrm{T}} P(\gamma) \mathrm{e}^{A\left(\phi^{-1}(t) - t\right)} \widehat{x}(t), \forall t \geq 0, \end{array} \right. \tag{21}$$

where $P(\gamma)$ is the unique positive definite solution to the parametric ARE (13) and $L \in \mathbf{R}^{n\times p}$ is such that $A - LC$ is asymptotically stable.

Theorem 2: Let Assumptions 1 and 2 be satisfied and $n \geq 2$. Then there exists a $\gamma^* > 0$ such that the observer based truncated prediction feedback (21) stabilizes system (1) for any $\gamma \in (0, \gamma^*]$.

In the event that the time-delay system in (1) is also subject to input saturation, the system becomes

$$\dot{x}(t) = Ax(t) + B\mathrm{sat}(u(\phi(t))), \tag{22}$$

where

$$\mathrm{sat}(u) = \left[\begin{array}{cccc} \mathrm{sat}(u_1) & \mathrm{sat}(u_2) & \cdots & \mathrm{sat}(u_m) \end{array} \right]^{\mathrm{T}}, \tag{23}$$

with $\mathrm{sat}(u_j)$ being the standard scalar saturation function. We have the following result regarding semi-global stabilization of the time-delay system. The proof is omitted due to space limitation. Output feedback results can also be obtained accordingly and will not be presented here for brevity.

Theorem 3: Let Assumptions 1 and 2 be all satisfied. Then the truncated prediction feedback laws (12) with $P(\gamma)$ being the unique positive definite solution to the parametric ARE (13), semi-globally stabilize system (22), *i.e.*, for any *a priori* given bounded set $\Omega \subset \mathscr{C}_{n,\overline{D}}$, there exists a $\gamma^* > 0$ such that, for an arbitrary $\gamma \in (0, \gamma^*]$, the closed-loop system is asymptotically stable at the origin with Ω contained in the domain of attraction.

IV. Numerical Examples

We consider a delayed double oscillator system characterized by (1) in which A and B are given by

$$A = \left[\begin{array}{cccc} 0 & \omega & 0 & 0 \\ -\omega & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega \\ 0 & 0 & -\omega & 0 \end{array} \right], \quad B = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \end{array} \right], \tag{24}$$

where ω is an positive number. For this system, the unique solution to the parametric ARE can be computed as

$$P = \left[\begin{array}{cccc} p_{11} & \frac{3\gamma^6}{\omega^3} - \frac{4\gamma^4}{\omega} & \frac{3\gamma^5}{\omega^3} - \frac{8\gamma^3}{\omega} & \frac{\gamma^4}{\omega^2} - 4\gamma^2 \\ \frac{3\gamma^6}{\omega^3} - \frac{4\gamma^4}{\omega} & \frac{10\gamma^5}{\omega^2} + 8\gamma^3 & \frac{11\gamma^4}{\omega^2} + 4\gamma^2 & \frac{4\gamma^3}{\omega} \\ \frac{3\gamma^5}{\omega^3} - \frac{8\gamma^3}{\omega} & \frac{11\gamma^4}{\omega^2} + 4\gamma^2 & \frac{14\gamma^3}{\omega^2} + 4\gamma & \frac{6\gamma^2}{\omega} \\ -4\gamma^2 & \frac{4\gamma^3}{\omega} & \frac{6\gamma^2}{\omega} & 4\gamma \end{array} \right],$$

where $p_{11} = \frac{\gamma^7}{\omega^4} - \frac{2\gamma^5}{\omega^2} + 8\gamma^3$. We consider two cases of delay function $\phi(t)$ in the system:

- In the first case, the delay function is (Example 5.3 in [7])

$$\phi(t) = t - \frac{t+1}{2t+1}, \quad t \geq 0.$$

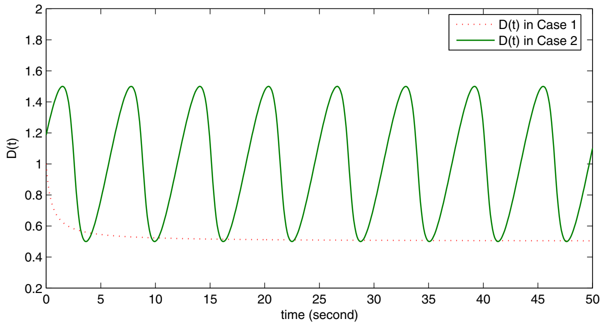

Fig. 1. The time-delay function $D(t)$ in two cases.

It follows that $\overline{D} = 1$ and the inverse function of $\phi(t)$ is

$$\phi^{-1}(t) = \frac{t + \sqrt{(t+2)^2 + 1}}{2}.$$

The delay function $D(t) = t - \phi(t)$ is shown in Fig. 1. Hence, according to Theorem 1, the truncated prediction based (time-varying) controller is given as

$$u = -B^{\mathrm{T}} P \exp\left(A\left(\frac{t+1}{\sqrt{(t+1)^2+1}+t}\right)\right)x.$$

- In the second case the delay function is oscillatory and is given by (Example 5.4 in [7]),

$$\phi(t) = \rho^{-1}(t), \quad \rho(t) = t + 1 + \frac{1}{2}\cos(t).$$

For an illustration of this function, see Fig. 2 in [7]. The delay function $D(t) = t - \phi(t)$ is also shown in Fig. 1. For this function, we obtain $\overline{D} = \frac{3}{2}$. Again, according to Theorem 1, the truncated prediction based (time-varying) controller is given as

$$u = -B^{\mathrm{T}} P \exp\left(A\left(1 + \frac{1}{2}\cos(t)\right)\right)x. \tag{25}$$

For these two cases, with a given initial condition $x(\theta) = [-1\ 2\ 2\ -1]^{\mathrm{T}}, \forall \theta \in [-\overline{D}, 0]$, and by setting $\omega = 2$ and $\gamma = 0.3$, the state responses and control signals $u(\phi(t))$ are shown in Fig. 2 and Fig. 3, respectively. It is clear that the systems are indeed stabilized by these two truncated predication controllers. Finally, for such a given initial condition, with different values of γ, the control signals $u(\phi(t))$ for the above two different kinds of delay functions are recorded in Fig. 4, from which we see that the peak values in the control signals decrease as γ decreases, which indicates semi-global stabilization in the presence of input saturation.

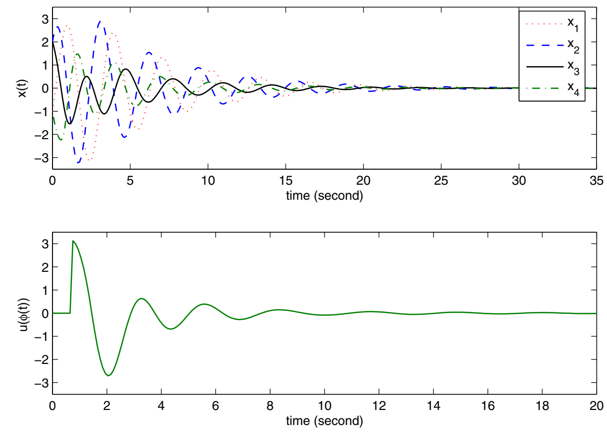

Fig. 2. State evolution and control signal of the closed-loop system for the first case of delay function.

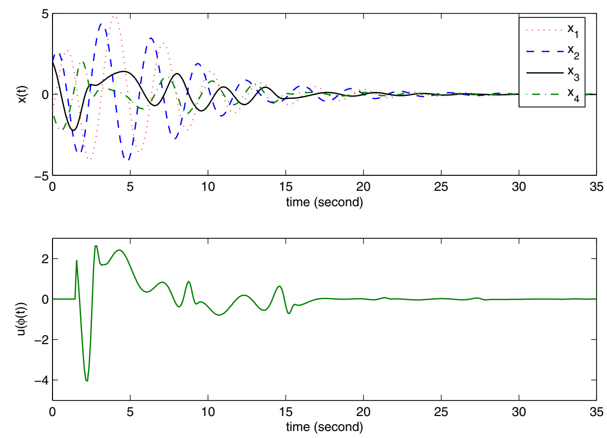

Fig. 3. State evolution and control signal of the closed-loop system for the second case of delay function.

V. Conclusions

In this paper, we proposed a new design approach, referred to as truncated prediction feedback, for linear systems with long time-varying input delay. By adopting the idea of prediction based feedback and the recently developed parametric Lyapunov equation based low gain feedback, a finite dimensional static linear time-varying state feedback was proposed that stabilizes the system as long as the open-loop system is not exponentially unstable and the delay is bounded. An explicit condition on the free parameter in the controller was obtained to guarantee the stability of the closed-loop system. It was also shown that the proposed parameterized controller semi-globally stabilizes the system in the presence of actuator saturation. In comparison with the prediction based controllers which are infinite dimensional state feedback, the proposed new controllers are more convenient to implement. Numerical examples have demonstrated the effectiveness of the proposed approach.

The research in this paper opened several future research topics. For example, it would be interesting to consider linear systems with long multiple and distributed *time-varying* delays in the input by combining the truncated predication approach and those ideas found in our early studies [28] and [29]. However, our initial study indicates that the controllers

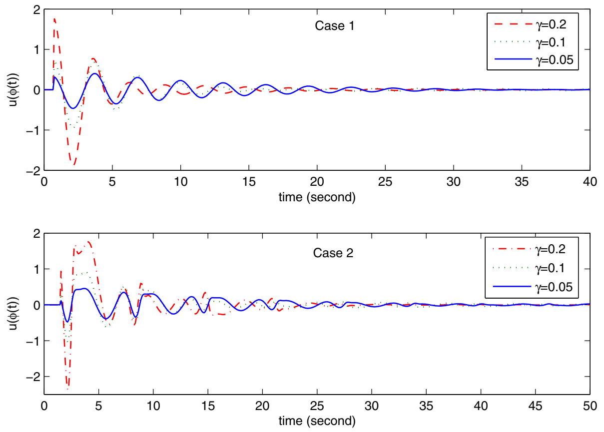

Fig. 4. Control signals for different values of γ with two different kinds of delay functions.

involve some nonlinear differential equations leading to some technical difficulties. Also, it is expected that the proposed truncated approach can be adopted to handle input delay systems that are exponentially unstable. In that case, though it cannot be expected that the delay can be as large as possible, it is reasonable to anticipate that less conservative results concerning the maximal allowable delay in the system can be obtained than those approaches without any predication term can since the finite dimensional prediction term in the form of (7) can partially compensate the delay effect ([8]). Further study along these lines are now under way.

Appendix

In this appendix, we recall the following results from [25] and [26] regarding properties of solutions to the parametric Riccati equation (13).

Lemma 1: Assume that the matrix pair $(A, B) \in (\mathbf{R}^{n\times n}, \mathbf{R}^{n\times m})$ is controllable and all the poles of A are on the imaginary axis. Then the parametric ARE

$$A^{\mathrm{T}}P + PA - PBB^{\mathrm{T}}P = -\gamma P,$$

has a unique positive definite solution $P(\gamma) = W^{-1}(\gamma)$, where $W(\gamma)$ is the unique positive definite solution to the following Lyapunov equation

$$W\left(A + \frac{\gamma}{2}I_n\right)^{\mathrm{T}} + \left(A + \frac{\gamma}{2}I_n\right)W = BB^{\mathrm{T}}.$$

Moreover, 1). $\lim_{\gamma\to 0^+} P(\gamma) = 0$; 2). $\frac{\mathrm{d}}{\mathrm{d}\gamma}P(\gamma) > 0, \forall\gamma > 0$; 3). $\mathrm{tr}\left(B^{\mathrm{T}}P(\gamma)B\right) = n\gamma$; 4). $P(\gamma)BB^{\mathrm{T}}P(\gamma) \le n\gamma P(\gamma)$; and 5). $\mathrm{e}^{A^{\mathrm{T}}t}P(\gamma)\mathrm{e}^{At} \le \mathrm{e}^{\omega\gamma t}P(\gamma)$, where $t \ge 0$ and $\omega \ge n-1$.

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