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Adaptive Parameter Identification and State Estimation with Partial State Information and Bounded Disturbances

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***Abstract*— In this paper, we present a joint state and adaptive parameter identification scheme for the cases when all the states of the system are measured and when only some states of the system are measured. When all the states are measured, we show that, in the presence of process and measurement noise, the state and parameter estimation errors are bounded. To this end, we show that this is possible only through the appropriate design of a virtual input which ensures that the system error signals are bounded. As a special case of all the states being measured, we show that in the case of a noise free system, the state estimation errors converge to the origin. For the case when only some states are measured, we show that for a linear system with n states, m inputs and p measurements, we can estimate at most p^2 entries of the system matrix and pm entries of the input matrix.**

I. INTRODUCTION

Having an accurate physics based mathematical model that accurately captures the behavior of a real system is most desired for estimation or control applications. However, constructing such a mathematical model is a tedious process, involving time and frequency based techniques, and often times control system engineers have to look at non-conventional methods for obtaining robustness in control design applications. A common practice is to obtain an approximate mathematical description of the real system and then augment this model with sufficient intelligence so that it can modify its behavior to adapt to any change in the environment. Such a system is an adaptive system that changes its behavior according to the change in the environment or the circumstance under which the system operates. The phrases *adaptive system*, *adaptive estimation* and *adaptive control* were used as early as 1950 [1].

Given the structure of the model, the model response is determined by the values of the model parameters [1], [2], [3], [4]. Thus the problem of parameter estimation is to identify the values of the unknown parameters of a system either online or off-line via available system measurements and inputs. In some applications these parameters may be calculated using the laws of physics, properties of materials, and such methods, while in other applications the parameters have to be calculated by observing the system's response to certain inputs. In the case of a linear time invariant, stable system, frequency or time domain based *off-line* techniques are used to deduce the unknown parameters via the available measurements. However, in cases where the system might be linear but the unknown parameters are allowed to change with time, such off-line techniques need not necessarily yield good results. The need to be able to estimate parameters that can potentially change with time has paved the way for online parameter estimation techniques. To account for modeling errors in the process adaptive parameter estimation schemes have been proposed [1], [5], [6], [7], [8].

Any control system has to meet the twin requirements of robustness to plant uncertainty and performance. However given the constraints on the sensitivity and complimentary sensitivity, by the relation $S + T = I$ (where S is the sensitivity of the closed-loop system to an infinitesimal perturbation in the system dynamics [12]), it is not possible to achieve both these requirements simultaneously. Parameter identification, reduces plant uncertainty, thereby allowing us to reduce the robustness requirements, and achieve better performance.

In this paper, we propose a methodology for a linear time invariant system whereby we jointly estimate the unknown states and the unknown parameters of the system using either full or partial state measurement. We develop the parameter adaptation laws and show boundedness of the system in the presence of adaptation via the direct method of Lyapunov. *To the best of our knowledge, the novelty in this paper is in appropriately choosing a pseudo control input,* $\mathbf{u}_2$*, to show that the state and parameter errors are bounded even in the presence of a disturbance that corrupts the available measurements.* As a special case, with noise/disturbance free measurements, we invoke Barbalat's lemma and use the fact that the system is observable to show convergence of the state estimation error to the origin on an infinite time horizon scale. In order to establish convergence to the origin for the unknown parameters, a persistently exciting (PE) input[1] is required to excite the system at a pre-computed number of frequencies as determined by the order of the system. However, since a PE input is not verifiable *apriori*, we shift our focus to bounded parameter errors and state error convergence. *Finally, we show that when only some states of a system are measured, the number of parameters of the system matrix and the input matrix that can be estimated is a function of the number of measurements.* The remainder of this paper is organized as follows: In Section II, an adaptive parameter identification problem is formulated when all states are measured, and a scheme of stability analysis for this framework is presented. In Section III, we formulate

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[1]An input is said to be *persistently exciting* over a time interval if the integral of the square of the input over that time interval is always lower bounded by a positive number [9], [10].

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the problem of adaptive parameter identification when only some states are measured. In Section IV, we present some simulation results to show the efficacy of the approach proposed and finally in Section V some concluding remarks are presented. Throughout the paper, **bold** symbols are used for **vectors**, CAPITAL letters for MATRICES, small, non-bold letters for scalars, $\mathcal{D}_{\boldsymbol{\alpha}}$ denotes a compact domain in $\boldsymbol{\alpha}$ and unless otherwise specified, $\|\cdot\|$ stands for 2 norm.

II. ALL STATES MEASURED

We consider the case where all the states of the system are available as measurements.

A. System Dynamics

Consider the dynamics of a bounded, observable, linear, time invariant system as

$$\begin{aligned}\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{u}(t) + \mathbf{w}(t), \quad \mathbf{x}(0) = \mathbf{x}_0 \\ \mathbf{y}(t) &= \mathbf{x}(t) + \mathbf{v}(t)\end{aligned} \tag{1}$$

where, $\mathbf{x} \in \mathcal{D}_{\mathbf{x}} \subset \mathcal{R}^n$ denotes the $n-$dimensional *unknown* state vector of the system, $\mathbf{u} \in \mathcal{D}_{\mathbf{u}} \subset \mathcal{R}^m$ denotes the $m-$dimensional *known* input vector, $A \in \mathcal{R}^{n\times n}$ denotes the *unknown*, time invariant system matrix, $B \in \mathcal{R}^{n\times m}$ denotes the *unknown*, time invariant input matrix, $\mathbf{y} \in \mathcal{D}_{\mathbf{y}} \subset \mathcal{R}^n$ denotes the *known* system measurement vector, $\mathbf{w}$ denotes the $n-$dimensional zero mean random Gaussian process noise of known standard deviation and $\mathbf{v}$ denotes the $n-$dimensional zero mean random Gaussian measurement noise of known standard deviation.

B. Assumptions, Remarks and Mathematical Preliminaries

Assumption 2.1: The known input vector $\mathbf{u}$ evolves on a compact domain $\mathcal{D}_{\mathbf{u}}$ and the known system measurement vector $\mathbf{y}$ is assumed to be bounded. With system observability, this implies that the unknown state vector evolves on a compact domain which is denoted as $\mathcal{D}_{\mathbf{x}}$.

Assumption 2.2: The unknown system matrix A in Eq. (1) is bounded by its maximum singular value such that $\|A\| \leq \bar{\sigma}_A$, where $\bar{\sigma}_A = \sigma_{\max}(A)$, where $\bar{\sigma}_A$ is known.

Assumption 2.3: Typically, system-identification is not done online, but after the system outputs have been recorded in a test-phase, during which the system is excited with sufficiently rich signals. Therefore it is reasonable to assume that the recorded state estimate denoted as $\hat{\mathbf{x}}_f(t)$ differs from the true state, $\mathbf{x}(t)$, by a time-varying quantity, denoted as $\mathbf{d}(t)$, i.e., $\mathbf{d}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}_f(t)$, such that $\mathbf{d}(t)$ is uniformly bounded. Thus $\sup_{t\geq 0}\mathbf{d}(t) = \bar{d} < \infty$.

Remark 2.4: The process noise vector, $\mathbf{w}(t)$, is bounded by a value of 3 times its standard deviation, i.e., $\|\mathbf{w}\| \leq 3\sigma_{\mathbf{w}}$, where $\sigma_{\mathbf{w}}$ is the standard deviation of $\mathbf{w}$.

Remark 2.5: Denote the maximum of $\mathbf{d}(t)$, $\dot{\mathbf{d}}(t)$ and $\mathbf{w}(t)$ as $d^* = \max\left(\|\mathbf{d}\|, \|\dot{\mathbf{d}}\|, \|\mathbf{w}\|\right)$, for all $t \geq 0$, where we assume that the bound holds true uniformly.

Theorem 2.6: [11] Consider the system $\dot{\zeta}(t) = \mathcal{A}\zeta(t)$ where $\zeta \in \mathcal{R}^n$ is the $n-$dimensional state vector of the system and $\mathcal{A} \in \mathcal{R}^{n\times n}$ is the system matrix. Let $\zeta = 0$ be the equilibrium point of the above system. Then the matrix $\mathcal{A}$ is a stability matrix, i.e., all eigen values of $\mathcal{A}$ have strictly negative real parts, if and only if for any given positive definite, symmetric matrix Q, there exists a unique, positive definite, symmetric matrix P such that $P\mathcal{A} + \mathcal{A}^T P = -Q$.

Lemma 2.7: [11] Let $V : \mathcal{R} \to \mathcal{R}$ be a uniformly continuous function on $[0,\ \infty)$. Suppose that $\lim_{t\to\infty}\int_0^t V(\tau)\, d\tau$ exists and is finite. Then $V(t) \to 0$ as $t \to \infty$.

C. Adaptive Observer Dynamics

For the system in Eq. (1), we propose the following adaptive observer

$$\begin{aligned}\dot{\hat{\mathbf{x}}}(t) &= A_H\hat{\mathbf{x}}(t) + \hat{B}(t)\mathbf{u}(t) + (\hat{A}(t) - A_H)\hat{\mathbf{y}}_f(t) - \mathbf{u}_2 \\ \hat{\mathbf{y}}(t) &= \hat{\mathbf{x}}(t)\end{aligned} \tag{2}$$

where, $\hat{A}(t)$ and $\hat{B}(t)$ are respectively the adaptive estimates of the unknown matrices A and B in Eq. (1), $\hat{\mathbf{y}}_f(t) = \hat{\mathbf{x}}_f(t)$ and $\hat{\mathbf{x}}_f(t)$ is obtained based on Assumption 2.3, $\hat{\mathbf{x}}_0$ is the value at which the adaptive estimator is initialized and $\mathbf{u}_2(t)$ is a pseudo control input (which will be defined subsequently) fed to the estimator so as to ensure that the system error signals are bounded outside a compact domain. The matrix A_H is chosen to be a stability matrix or a Hurwitz[2] matrix. Furthermore, according to Theorem 2.6, with a $Q > 0$, there exists a unique $P = P^T > 0$ such that

$$PA_H + A_H^T P \;=\; -Q \tag{3}$$

Remark 2.8: Denote the maximum singular value of P as $\bar{\sigma}_P = \sigma_{\max}(P)$, such that $\|P\| \leq \bar{\sigma}_P$.

D. Adaptation Update Laws

We propose the following adaptation update laws for the unknown matrices A and B as:

$$\dot{\hat{A}}(t) = \Gamma_A P\hat{\mathbf{e}}(t)\hat{\mathbf{x}}_f^T(t), \qquad \dot{\hat{B}}(t) = \Gamma_B P\hat{\mathbf{e}}(t)\mathbf{u}^T(t) \tag{4}$$

where, $\hat{\mathbf{e}}(t) \equiv \hat{\mathbf{x}}(t) - \hat{\mathbf{x}}_f(t)$, $\Gamma_A = \Gamma_A^T > 0$ and $\Gamma_B = \Gamma_B^T > 0$ are respectively the learning rates of the adaptive laws in Eqs. (4) and P is the solution of Eq. (3).

E. Error Dynamics

Denote the following error signals as $\mathbf{e}(t) \equiv \mathbf{x}(t) - \hat{\mathbf{x}}(t)$, $\tilde{A}(t) \equiv A - \hat{A}(t)$ and $\tilde{B}(t) \equiv B - \hat{B}(t)$. Thus, taking the difference between Eqs. (1) and (2), we obtain

$$\begin{aligned}\dot{\mathbf{e}}(t) &= A_H\mathbf{e}(t) + \tilde{B}(t)\mathbf{u}(t) + \tilde{A}(t)\mathbf{x}(t) + \mathbf{w}(t) \\ &\quad + \left(\hat{A}(t) - A_H\right)\mathbf{d}(t) + \mathbf{u}_2(t)\end{aligned} \tag{5}$$

Furthermore, denoting $\hat{\mathbf{e}}(t) \equiv \mathbf{e}(t) - \mathbf{d}(t)$, we arrive at

$$\begin{aligned}\dot{\hat{\mathbf{e}}}(t) &= A_H\hat{\mathbf{e}}(t) + \tilde{B}(t)\mathbf{u}(t) + \tilde{A}(t)\hat{\mathbf{x}}_f(t) \\ &\quad + A\mathbf{d}(t) + \mathbf{u}_2(t) - \dot{\mathbf{d}}(t) + \mathbf{w}(t)\end{aligned} \tag{6}$$

The adaptation error dynamics can be obtained as follows:

$$\dot{\tilde{A}}(t) = -\Gamma_A P\hat{\mathbf{e}}(t)\hat{\mathbf{x}}_f^T(t), \quad \dot{\tilde{B}}(t) = -\Gamma_B P\hat{\mathbf{e}}(t)\mathbf{u}^T(t) \tag{7}$$

[2] A square matrix is called a Hurwitz matrix if every eigenvalue of the square matrix has a strictly negative real part.

F. Lyapunov Analysis

We ignore the argument t for ease of analysis. Consider the following candidate Lyapunov function for the dynamics in Eqs. (6) and (7) as

$$V\left(\hat{\mathbf{e}},\tilde{A},\tilde{B}\right)=\hat{\mathbf{e}}^TP\hat{\mathbf{e}}+tr\left(\tilde{A}^T\Gamma_A^{-1}\tilde{A}\right)+tr\left(\tilde{B}^T\Gamma_B^{-1}\tilde{B}\right) \quad (8)$$

where, tr denotes the trace operator. Taking the derivative of the candidate Lyapunov function in Eq. (8) along Eqs. (6) and (7), we obtain

$$\begin{aligned}\dot{V} &= \dot{\hat{\mathbf{e}}}^TP\hat{\mathbf{e}}+\hat{\mathbf{e}}^TP\dot{\hat{\mathbf{e}}}+2tr\left(\tilde{A}^T\Gamma_A^{-1}\dot{\tilde{A}}\right)+2tr\left(\tilde{B}^T\Gamma_B^{-1}\dot{\tilde{B}}\right)\\ &= \Big(\hat{\mathbf{e}}^TA_H^T+\hat{\mathbf{x}}_f^T\tilde{A}^T+\mathbf{u}^T\tilde{B}^T+\mathbf{d}^TA^T+\mathbf{u}_2^T-\dot{\mathbf{d}}^T\\ &\quad +\mathbf{w}^T\Big)P\hat{\mathbf{e}}+\hat{\mathbf{e}}^TP\Big(A_H\hat{\mathbf{e}}+\tilde{A}\hat{\mathbf{x}}_f+\tilde{B}\mathbf{u}+A\mathbf{d}+\mathbf{u}_2\\ &\quad -\dot{\mathbf{d}}+\mathbf{w}\Big)-2tr\left(\tilde{A}^TP\hat{\mathbf{e}}\hat{\mathbf{x}}_f^T\right)-2tr\left(\tilde{B}^TP\hat{\mathbf{e}}\mathbf{u}^T\right)\\ &= \hat{\mathbf{e}}^T\left(A_H^TP+PA_H\right)\hat{\mathbf{e}}+\underbrace{2\hat{\mathbf{x}}_f^T\tilde{A}^TP\hat{\mathbf{e}}}_{uncty.\ term\ 1}\\ &\quad +\underbrace{2\mathbf{u}^T\tilde{B}^TP\hat{\mathbf{e}}}_{uncty.\ term\ 2}+2\mathbf{d}^TA^TP\hat{\mathbf{e}}+2\mathbf{u}_2^TP\hat{\mathbf{e}}-2\dot{\mathbf{d}}^TP\hat{\mathbf{e}}\\ &\quad +2\mathbf{w}^TP\hat{\mathbf{e}}-\underbrace{2tr\left(\tilde{A}^TP\hat{\mathbf{e}}\hat{\mathbf{x}}_f^T\right)}_{adap.\ term\ 1}-\underbrace{2tr\left(\tilde{B}^TP\hat{\mathbf{e}}\mathbf{u}^T\right)}_{adap.\ term\ 2} \quad (9)\end{aligned}$$

Since the term marked as $uncty.\ term$ 1 in Eq. (9) is a scalar, it can be written as $2tr\left(\hat{\mathbf{x}}_f^T\tilde{A}^TP\hat{\mathbf{e}}\right)=2tr\left(\tilde{A}^TP\hat{\mathbf{e}}\hat{\mathbf{x}}_f^T\right)$. Similarly, the term marked as $uncty.\ term$ 2 in Eq. (9) can be written as $2tr\left(\mathbf{u}^T\tilde{B}^TP\hat{\mathbf{e}}\right)=2tr\left(\tilde{B}^TP\hat{\mathbf{e}}\mathbf{u}^T\right)$. Thus, $uncty.\ term$ 1 cancels with $adap.\ term$ 1 and $uncty.\ term$ 2 cancels with $adap.\ term$ 2 in Eq. (9) and with Eq. (3), Eq. (9) reduces to

$$\dot{V}=-\hat{\mathbf{e}}^TQ\hat{\mathbf{e}}+2\mathbf{d}^TA^TP\hat{\mathbf{e}}+2\mathbf{u}_2^TP\hat{\mathbf{e}}-2\dot{\mathbf{d}}^TP\hat{\mathbf{e}}+2\mathbf{w}^TP\hat{\mathbf{e}}$$

Further, the Lyapunov derivative can be upper bounded as

$$\begin{aligned}\dot{V} &\leq -\lambda_{\min}(Q)\|\hat{\mathbf{e}}\|^2+2\|\mathbf{d}\|\|A\|\|P\|\|\hat{\mathbf{e}}\| \quad (10)\\ &\quad +2\|\dot{\mathbf{d}}\|\|P\|\|\hat{\mathbf{e}}\|+2\|\mathbf{w}\|\|P\|\|\hat{\mathbf{e}}\|+2\mathbf{u}_2^TP\hat{\mathbf{e}}\end{aligned}$$

Using Assumption 2.2 and Remark 2.8, the Lyapunov derivative in Eq. (10) can be further upper bounded as

$$\begin{aligned}\dot{V} &\leq -\lambda_{\min}(Q)\|\hat{\mathbf{e}}\|^2+2\|\mathbf{d}\|\bar{\sigma}_A\bar{\sigma}_P\|\hat{\mathbf{e}}\| \quad (11)\\ &\quad +2\|\dot{\mathbf{d}}\|\bar{\sigma}_P\|\hat{\mathbf{e}}\|+2\|\mathbf{w}\|\bar{\sigma}_P\|\hat{\mathbf{e}}\|+2\mathbf{u}_2^TP\hat{\mathbf{e}}\end{aligned}$$

With $\|\hat{\mathbf{e}}\|>\max\left(\|\mathbf{d}\|,\|\dot{\mathbf{d}}\|,\|\mathbf{w}\|\right)$, Eq. (11) reduces to

$$\dot{V}\leq-\lambda_{\min}(Q)\|\hat{\mathbf{e}}\|^2+2\left(\bar{\sigma}_A\bar{\sigma}_P+2\bar{\sigma}_P\right)\|\hat{\mathbf{e}}\|^2+2\mathbf{u}_2^TP\hat{\mathbf{e}} \quad (12)$$

Furthermore, if the pseudo control input is chosen as $\mathbf{u}_2=-\left(\bar{\sigma}_A\bar{\sigma}_P+2\bar{\sigma}_P\right)P^{-1}\hat{\mathbf{e}}$, Eq. (12) simplifies to

$$\dot{V} \leq -\lambda_{\min}(Q)\|\hat{\mathbf{e}}\|^2+\sigma^*\|\hat{\mathbf{e}}\|^2-\sigma^*\hat{\mathbf{e}}^T\hat{\mathbf{e}} \quad (13)$$

which implies that $\dot{V}\left(\hat{\mathbf{e}},\tilde{A},\tilde{B}\right)\leq-\lambda_{\min}(Q)\|\hat{\mathbf{e}}\|^2\leq0$ for all $\|\hat{\mathbf{e}}\|>d^*$, where $\sigma^*=2\bar{\sigma}_P\left(\bar{\sigma}_A+2\right)$. Thus, the error signals $\hat{\mathbf{e}}(t)$, $\tilde{A}(t)$ and $\tilde{B}(t)$ are uniformly bounded for $t\geq0$ [1]. Further, with $\hat{\mathbf{e}}$ uniformly bounded and Assumption 2.3, $\mathbf{e}$ is uniformly bounded. This further implies that with $\mathbf{e}$ uniformly bounded and with Assumption 2.1, $\hat{\mathbf{x}}$ is uniformly bounded. Thus we have proved that with the adaptation laws as given in Eq. (4) and the pseudo control as chosen below Eq. (12), the estimate $\hat{\mathbf{x}}$ of the unknown state vector in Eq. (1) is bounded. Furthermore, the estimates of the unknown parameters are also bounded.

G. Lyapunov analysis for a noise free system

We specialize the results in Section II-F for the case when the process and the measurements are noise free, i.e., $\mathbf{w}(t)=0,\ \mathbf{v}(t)=0,\ \forall\ t\geq0$. For such a case, we assume that $\mathbf{d}(t)=0$, for all $t\geq0$. Thus, $\hat{\mathbf{e}}=\mathbf{e}$. for this case, $\mathbf{x}(t)$ can be used in place of $\hat{\mathbf{x}}_f(t)$. Furthermore, a pseudo control input need not be designed since we are interested in showing that the state estimation error converges to the origin. By observing Eq. (10), it can be seen that with $\mathbf{d}=0$, $\dot{\mathbf{d}}=0$, $\mathbf{w}=0$ and $\mathbf{u}_2=0$, $\dot{V}\left(\mathbf{e},\tilde{A},\tilde{B}\right)\leq0$. Thus, the equilibrium point defined by $\left(\mathbf{e},\tilde{A},\tilde{B}\right)=0$ is uniformly stable. However, to show convergence of $\mathbf{e}(t)$ to 0 as $t\to\infty$, we invoke Lemma 2.7. We need to first show that $\dot{V}\left(\mathbf{e},\tilde{A},\tilde{B}\right)$ is uniformly continuous on $[0,\ \infty)$. To show this, we need to show that $\ddot{V}\left(\mathbf{e},\tilde{A},\tilde{B}\right)$ is finite. This is evident from $\ddot{V}\left(\mathbf{e},\tilde{A},\tilde{B}\right)=-2\mathbf{e}^TQ\dot{\mathbf{e}}$. Now with $\mathbf{e}$ bounded and from Assumption 2.1, notice that $\dot{\mathbf{e}}$ in Eq. (5) is also bounded. Thus $\ddot{V}\left(\mathbf{e},\tilde{A},\tilde{B}\right)$ is finite. Furthermore, notice that $\lim_{t\to\infty}\int_0^t\dot{V}\left(\mathbf{e}(\tau),\tilde{A}(\tau),\tilde{B}(\tau)\right)d\tau$ is given by $\lim_{t\to\infty}\left(V(0)-V(\infty)\right)$. This limit exists and is finite because $V\left(\mathbf{e},\tilde{A},\tilde{B}\right)$ is a non-increasing function of time. Thus from Lemma 2.7, $\dot{V}\left(\mathbf{e},\tilde{A},\tilde{B}\right)\to0$ as $t\to\infty$, which implies that $\mathbf{e}\to0$ as $t\to\infty$. Further, if the input $\mathbf{u}$ is persistently exciting then $\tilde{A}$, $\tilde{B}$ converge to 0 as $t\to\infty$ [1].

III. SOME STATES MEASURED

We consider the case where some states of the system are available as measurements.

A. System Dynamics

Consider the dynamics of a bounded, observable, linear, time invariant system as

$$\begin{aligned}\dot{\mathbf{x}}(t) &= A\mathbf{x}(t)+B\mathbf{u}(t)+\mathbf{w}(t), \quad \mathbf{x}(0)=\mathbf{x}_0\\ \mathbf{y}(t) &= C\mathbf{x}(t) \quad (14)\end{aligned}$$

where, $\mathbf{x}\in\mathcal{D}_{\mathbf{x}}\subset\mathcal{R}^n$ denotes the $n-$dimensional *unknown* state vector of the system, $\mathbf{u}\in\mathcal{D}_{\mathbf{u}}\subset\mathcal{R}^m$ denotes the $m-$dimensional *known* input vector, $A\in\mathcal{R}^{n\times n}$ denotes the *unknown*, time invariant system matrix, $B\in\mathcal{R}^{n\times m}$ denotes the *unknown*, time invariant input matrix, $\mathbf{y}\in\mathcal{D}_{\mathbf{y}}\subset\mathcal{R}^p$ denotes the *known* system measurement, $C\in\mathcal{R}^{p\times n}$ denotes the measurement matrix $\mathbf{w}\in\mathcal{R}^n$ denotes the $n-$dimensional zero mean random Gaussian process noise of known standard deviation. Furthermore, Assumption 2.1 and Remark 2.4 hold true here.

Assumption 3.1: We assume that the unknown system matrix, A, in Eq. (14) is in the form $A = \begin{bmatrix} A^*_{1_{p\times p}} & A_{2_{p\times n-p}} \\ A_{3_{n-p\times p}} & A_{4_{n-p\times n-p}} \end{bmatrix}$, and $B = \begin{bmatrix} B^*_{1_{p\times m}} \\ B_{2_{n-p\times m}} \end{bmatrix}$, where n denotes the number of system states, m denotes the number of inputs and p denotes the number of measurements. Furthermore, $A_1^* \in \mathcal{R}^{p\times p}$ and $B_1^* \in \mathcal{R}^{p\times m}$ are the only *unknowns*. As the number of measurements increase, i.e., as $p \to n$, the sizes of the unknown matrices also increase.

B. Mathematical Preliminaries

Fact 3.2: In this paper we only consider a measurement matrix of the form $C = \begin{bmatrix} I_{p\times p} & 0_{p\times n-p} \end{bmatrix} \in \mathcal{R}^{p\times n}$, where n denotes the number states of the continuous, linear time invariant dynamical system, p denotes the number of measurements, $I_{p\times p}$ denotes an identity matrix of size $p\times p$ and $0_{p\times n-p}$ denotes a zero matrix of size $p \times n-p$. From the form of C we see that $CC^T = I_{p\times p}$.

Fact 3.3: A real symmetric matrix (or Hermitian matrix) $M \in \mathcal{R}^{n\times n}$ is positive definite if and only if the determinant of M is positive and the successive principal minors of the determinant of M are positive [13]. This is also called as Sylvester's criterion for positive definiteness. Thus $M = \begin{bmatrix} m_{11} & \dots & m_{1n} \\ \vdots & \ddots & \vdots \\ m_{n1} & \dots & m_{nn} \end{bmatrix}$ is positive definite if and only if

$$m_{11} > 0, \begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{vmatrix} > 0, \dots\dots, |M| > 0 \quad (15)$$

Fact 3.4: Consider a positive definite, symmetric matrix $M \in \mathcal{R}^{n\times n}$. Let $C \in \mathcal{R}^{p\times n}$ be a matrix as suggested in Fact 3.2. Then CMC^T is a positive definite matrix.

Proof: Let $M \in \mathcal{R}^{n\times n}$ be represented as $M = \begin{bmatrix} M_{1_{p\times p}} & M_{2_{p\times n-p}} \\ M_{3_{n-p\times p}} & M_{4_{n-p\times n-p}} \end{bmatrix}$. From Fact 3.2 $C = \begin{bmatrix} I_{p\times p} & 0_{p\times n-p} \end{bmatrix}$. Furthermore, $CMC^T = M_{1_{p\times p}}$. Given that M is positive definite and from Fact 3.3, $M_{1_{p\times p}}$ should also be positive $\Rightarrow CMC^T$ is a positive definite matrix. ∎

Remark 3.5: The system in Eq. (14) can be written as

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \bar{A}\mathbf{x}(t) + \bar{B}\mathbf{u}(t) + A_0^*\mathbf{y}(t) + B_0^*\mathbf{u}(t) \\ &\quad + \mathbf{w}(t) + \boldsymbol{\phi}(t), \ \mathbf{x}(0) = \mathbf{x}_0 \\ \mathbf{y}(t) &= C\mathbf{x}(t) \end{aligned} \quad (16)$$

where, $\mathbf{x} = \begin{bmatrix} \mathbf{x}_{1_{p\times 1}} \\ \mathbf{x}_{2_{n-p\times 1}} \end{bmatrix}$, $\bar{A} = \begin{bmatrix} A_{1_{p\times p}} & 0_{p\times n-p} \\ A_{3_{n-p\times p}} & A_{4_{n-p\times n-p}} \end{bmatrix}$, $\bar{B} = \begin{bmatrix} B_{1_{p\times m}} \\ B_{2_{n-p\times m}} \end{bmatrix}$ are known matrices, $A_0^* = \begin{bmatrix} A^*_{1_{p\times p}} - A_{1_{p\times p}} \\ 0_{n-p\times p} \end{bmatrix}$, $B_0^* = \begin{bmatrix} B^*_{1_{p\times m}} - B_{1_{p\times m}} \\ 0_{n-p\times m} \end{bmatrix}$ are unknown matrices that need to identified, $\boldsymbol{\phi}(t) = \begin{bmatrix} A_{2_{p\times n-p}}\mathbf{x}_2 \\ 0_{n-p\times 1} \end{bmatrix}$ and C takes the form shown in Fact 3.2.

Assumption 3.6: We modify Assumption 2.3 such that $\mathbf{y}(t) = \hat{\mathbf{y}}_f(t) + \mathbf{d}(t)$, where $\mathbf{d} \in \mathcal{D}_{\mathbf{d}} \subset \mathcal{R}^{p\times 1}$ and $\hat{\mathbf{y}}_f(t)$ is the estimate of the available measurement.

Assumption 3.7: The time varying quantity $\mathbf{d}(t)$ is uniformly bounded implying that $\sup_{t\geq 0} \mathbf{d}(t) = \bar{d} < \infty$.

Remark 3.8: Denote the maximum of $\mathbf{d}(t)$, $\mathbf{w}(t)$ and $\boldsymbol{\phi}(t)$ as $d^* = \max(\|\mathbf{d}\|, \|\mathbf{w}\|, \|\boldsymbol{\phi}\|)$, for all $t \geq 0$, where we assume that the bound holds true uniformly.

Definition 3.9: Consider a convex compact set with a smooth boundary as $\mathcal{D}_c \equiv \{\boldsymbol{\theta} \in \mathcal{R}^n | \mathcal{G}(\boldsymbol{\theta}) \leq c\}$, $0 \leq c \leq 1$ and $\mathcal{G}(\boldsymbol{\theta}) : (\mathcal{R}^n \to \mathcal{R}) = \frac{\boldsymbol{\theta}^T\boldsymbol{\theta} - \theta^2_{\max}}{\epsilon_{\boldsymbol{\theta}}}$, where ϵ_θ is a user defined convergence tolerance. The projection operator is defined as $Proj(\boldsymbol{\theta}, \boldsymbol{\xi}) = \boldsymbol{\xi}$ if $\mathcal{G}(\boldsymbol{\theta}) < 0$ or $(\mathcal{G}(\boldsymbol{\theta}) \geq 0, \ \& \ \nabla\mathcal{G}^T(\boldsymbol{\theta})\boldsymbol{\xi} \leq 0)$ and $Proj(\boldsymbol{\theta}, \boldsymbol{\xi}) = \boldsymbol{\xi} - \frac{\nabla\mathcal{G}(\boldsymbol{\theta})\nabla\mathcal{G}^T(\boldsymbol{\theta})\boldsymbol{\xi}}{\|\nabla\mathcal{G}(\boldsymbol{\theta})\|^2}\mathcal{G}(\boldsymbol{\theta})$ if $\mathcal{G}(\boldsymbol{\theta}) < 0$ or $(\mathcal{G}(\boldsymbol{\theta}) \geq 0, \ \& \ \nabla\mathcal{G}^T(\boldsymbol{\theta})\boldsymbol{\xi} > 0)$, where $\nabla\mathcal{G}(\boldsymbol{\theta}) = \begin{bmatrix} \frac{\partial\mathcal{G}}{\partial\theta_1} & \cdots & \frac{\partial\mathcal{G}}{\partial\theta_n} \end{bmatrix}^T$.

Property 3.10: Definition 3.9 means that $\boldsymbol{\theta}(0) \in \mathcal{D}_c \Rightarrow \boldsymbol{\theta}(t) \in \mathcal{D}_c, \ \forall\, t \geq 0$. Furthermore, given $\mathcal{Y}, \Theta \in \mathcal{R}^{q_1\times q_2}$, we have $tr((\Theta - \Theta^*)(Proj(\Theta, Y) - Y)) \leq 0$.

C. Adaptive Observer Dynamics

For the system in Eq. (16), we propose the following adaptive observer

$$\begin{aligned} \dot{\hat{\mathbf{x}}}(t) &= \bar{A}\hat{\mathbf{x}}(t) + \bar{B}\mathbf{u}(t) + \hat{A}_0(t)\hat{\mathbf{y}}_f(t) + \hat{B}_0(t)\mathbf{u}(t) \\ &\quad + L(\hat{\mathbf{y}}(t) - \hat{\mathbf{y}}_f(t)) - \mathbf{u}_2(t), \qquad \hat{\mathbf{x}}(0) = \hat{\mathbf{x}}_0 \\ \hat{\mathbf{y}}(t) &= C\hat{\mathbf{x}}(t) \end{aligned} \quad (17)$$

where, $\hat{A}_0(t)$ and $\hat{B}_0(t)$ are respectively the adaptive estimates of the unknown matrices A_0^* and B_0^* in Eq. (16), $\hat{\mathbf{x}}_0$ is the value at which the adaptive estimator is initialized, $\mathbf{u}_2(t)$ is a pseudo control input (which will be defined subsequently) fed to the estimator so as to ensure that the system error signals are bounded outside a compact domain and $L \in \mathcal{R}^{n\times p}$ is a gain matrix such that $A_H = \bar{A} + LC$ is Hurwitz. Furthermore, according to Theorem 2.6, with a $Q > 0$ and a Hurwitx matrix A_H, there exists a unique $\tilde{P} = \tilde{P}^T > 0$ such that

$$\tilde{P}A_H + A_H^T\tilde{P} = -Q \quad (18)$$

Remark 3.11: Denote the maximum singular value of $\tilde{P}$ as $\bar{\sigma}_{\tilde{P}} = \sigma_{\max}\left(\tilde{P}\right)$, such that $\|\tilde{P}\| \leq \bar{\sigma}_{\tilde{P}}$.

Fact 3.12: If the gain matrix L in Eq. (17) is chosen such that $L = \begin{bmatrix} L_{1_{p\times p}} \\ -A_{3_{n-p\times p}} \end{bmatrix}$, then the matrix $A_H = \bar{A} + LC$ is a block diagonal matrix, where C is given by Fact 3.2. Furthermore, the solution matrix $\tilde{P}$ is block diagonal.

Proof: Notice that A_H is formed as follows:

$$\begin{aligned} A_H &= \begin{bmatrix} A_{1_{p\times p}} & 0_{p\times n-p} \\ A_{3_{n-p\times p}} & A_{4_{n-p\times n-p}} \end{bmatrix} + \begin{bmatrix} L_{1_{p\times p}} \\ -A_{3_{n-p\times p}} \end{bmatrix} C \\ &= \begin{bmatrix} A_{1_{p\times p}} & 0_{p\times n-p} \\ 0_{n-p\times p} & A_{4_{n-p\times n-p}} \end{bmatrix} \end{aligned} \quad (19)$$

which proves that $A_H = \bar{A} + LC$ is a block diagonal matrix. Furthermore, pre and post multiply Eq. (18) with $e^{A_H^T t}$ and $e^{A_H t}$ respectively to obtain $e^{A_H^T t}\left(\tilde{P}A_H + A_H^T\tilde{P}\right)e^{A_H t} = -e^{A_H^T t}Qe^{A_H t}$. This can also be written as $d\left(e^{A_H^T t}\tilde{P}e^{A_H t}\right) = -e^{A_H^T t}Qe^{A_H t}dt$, which upon integration yields $\int_0^\infty d\left(e^{A_H^T t}\tilde{P}e^{A_H t}\right) = -\int_0^\infty e^{A_H^T t}Qe^{A_H t}dt$. Since A_H is Hurwitz, we obtain

$$\tilde{P} = \int_0^\infty e^{A_H^T t}Qe^{A_H t}dt, \quad \texttt{since } e^{A_H^T \infty} = 0 \quad (20)$$

Notice that since A_H is block diagonal, $e^{A_H^T t}$ and $e^{A_H t}$ are also diagonal. Choosing a diagonal Q matrix ensures that $\tilde{P}$ is also block diagonal and can be represented in the form $\tilde{P} = \texttt{blkdiag}\left[\begin{array}{cc} \tilde{P}_{1_{p\times p}} & P_{4_{n-p\times n-p}} \end{array}\right]$. ■

D. Adaptation Update Laws

We propose the following projection based adaptation laws [14] for the unknown matrices A_0^* and B_0^* in Eq. (16) as:

$$\dot{\hat{A}}_0(t) = \bar{\Gamma}_A Proj\left(\hat{A}_0(t), C^T P\tilde{\mathbf{y}}(t)\mathbf{y}^T\right), \bar{\Gamma}_A = \bar{\Gamma}_A^T > 0$$
$$\dot{\hat{B}}_0(t) = \bar{\Gamma}_B Proj\left(\hat{B}_0(t), C^T P\tilde{\mathbf{y}}(t)\mathbf{u}^T\right), \bar{\Gamma}_B = \bar{\Gamma}_B^T > 0 \quad (21)$$

where, $\tilde{\mathbf{y}}(t) \equiv \mathbf{y}(t) - \hat{\mathbf{y}}(t)$ and P is a positive definite matrix such that $P = C\tilde{P}C^T$, which can be seen from Fact 3.4. Furthermore, the matrix P is such that $P = \tilde{P}_{1_{p\times p}}$ and from Fact 3.4 $\tilde{P}_{1_{p\times p}} > 0$. *From the property of projection, $\hat{A}_0(t)$ and $\hat{B}_0(t)$ are always confined to compact sets [14].*

E. Error Dynamics

Denote the following error signals as $\mathbf{e}(t) \equiv \mathbf{x}(t) - \hat{\mathbf{x}}(t)$, $\tilde{A}_0(t) \equiv A_0^* - \hat{A}_0(t)$ and $\tilde{B}_0(t) \equiv B_0^* - \hat{B}_0(t)$. Taking the difference between Eqs. (16) and (17), we obtain

$$\dot{\mathbf{e}}(t) = A_H\mathbf{e}(t) + \tilde{B}_0(t)\mathbf{u}(t) + \tilde{A}_0(t)\mathbf{y}(t) + \mathbf{w}(t)$$
$$+\hat{A}_{0_L}(t)\mathbf{d}(t) + \boldsymbol{\phi}(t) + \mathbf{u}_2(t), \quad \hat{A}_{0_L}(t) = \hat{A}_0(t) - L \quad (22)$$

and $\hat{A}_{0_L}(t)$ is bounded from Property 3.10. The adaptation error dynamics is $\dot{\tilde{A}}_0(t) = -\bar{\Gamma}_A Proj\left(\hat{A}_0(t), C^T P\tilde{\mathbf{y}}(t)\mathbf{y}^T\right)$, $\dot{\tilde{B}}_0(t) = -\bar{\Gamma}_B Proj\left(C^T P\tilde{\mathbf{y}}(t)\mathbf{u}^T\right)$ and $\tilde{A}_0$, $\tilde{B}_0$ respectively are $\left[\begin{array}{cc} \tilde{A}_{1_{p\times p}}^T & 0_{p\times n-p} \end{array}\right]^T$ and $\left[\begin{array}{cc} \tilde{B}_{1_{m\times p}}^T & 0_{m\times n-p} \end{array}\right]^T$.

F. Lyapunov Analysis

Consider the following candidate Lyapunov function

$$V = \mathbf{e}^T\tilde{P}\mathbf{e} + tr\left(\tilde{A}_0^T\bar{\Gamma}_A^{-1}\tilde{A}_0\right) + tr\left(\tilde{B}_0^T\bar{\Gamma}_B^{-1}\tilde{B}_0\right) \quad (23)$$

The derivative of the candidate Lyapunov function yields

$$\begin{aligned} \dot{V} &= \left(\mathbf{e}^T A_H^T + \mathbf{y}^T\tilde{A}_0^T + \mathbf{u}^T\tilde{B}_0^T + \mathbf{d}^T\hat{A}_{0_L}^T + \mathbf{w}^T + \boldsymbol{\phi}^T\right. \\ &\quad \left.+\mathbf{u}_2^T\right)\tilde{P}\mathbf{e} + \mathbf{e}^T\tilde{P}\left(A_H\mathbf{e} + \tilde{A}_0\mathbf{y} + \tilde{B}_0\mathbf{u} + \hat{A}_{0_L}\mathbf{d} + \mathbf{w}\right. \\ &\quad \left.+\ \boldsymbol{\phi} + \mathbf{u}_2\right) + 2tr\left(\tilde{A}_0^T\bar{\Gamma}_A^{-1}\dot{\tilde{A}}_0\right) + 2tr\left(\tilde{B}_0^T\bar{\Gamma}_B^{-1}\dot{\tilde{B}}_0\right) \\ &= \mathbf{e}^T\left(A_H^T\tilde{P} + \tilde{P}A_H\right)\mathbf{e} + \underbrace{2\mathbf{y}^T\tilde{A}_0^T\tilde{P}\mathbf{e}}_{uncty.\ term\ 1} + \underbrace{2\mathbf{u}^T\tilde{B}_0^T\tilde{P}\mathbf{e}}_{uncty.\ term\ 2} \\ &\quad +2\mathbf{d}^T\hat{A}_{0_L}^T\tilde{P}\mathbf{e} + \mathbf{w}^T\tilde{P}\mathbf{e} + +\boldsymbol{\phi}^T\tilde{P}\mathbf{e} + \mathbf{u}_2^T\tilde{P}\mathbf{e} \\ &\quad \underbrace{-\,2tr\left(\tilde{A}_0^T Proj\left(\hat{A}_0(t), C^T P\tilde{\mathbf{y}}(t)\mathbf{y}^T\right)\right)}_{adap.\ term\ 1} \\ &\quad \underbrace{-\,2tr\left(\tilde{B}_0^T Proj\left(\hat{B}_0(t), C^T P\tilde{\mathbf{y}}(t)\mathbf{u}^T\right)\right)}_{adap.\ term\ 2} \end{aligned} \quad (24)$$

Notice that by using $P = C\tilde{P}C^T$, the term marked as *adap. term* 1 in Eq. (24) can be written as $2tr\left(\tilde{A}_0^T Proj\left(\hat{A}_0(t), C^T C\tilde{P}C^T C\mathbf{e}\mathbf{y}^T\right)\right) = 2tr\left(\left[\begin{array}{cc} \tilde{A}_{1_{p\times p}}^T & 0_{p\times n-p} \end{array}\right] Proj\left(\hat{A}_0(t), \left[\begin{array}{cc} \tilde{P}_1 & 0 \\ 0 & 0 \end{array}\right]\mathbf{e}\mathbf{y}^T\right)\right)$ which always reduces to $2tr\left(\tilde{A}_1^T\tilde{P}_1\left(\mathbf{e}\mathbf{y}^T\right)_{p\times p}\right)$ or $2tr\left(\tilde{A}_1^T\tilde{P}_1\left(\mathbf{e}\mathbf{y}^T\right)_{p\times p} - \tilde{A}_1\left(\mathbf{y}^{\mathbf{T}}\mathbf{e}\right)_{p\times p}\tilde{P}_1\left(\frac{\nabla\mathcal{G}(A_1)\nabla^T\mathcal{G}(A_1)}{\|\nabla\mathcal{G}(A_1)\|^2}\mathcal{G}\right)\right)$. Thus, from Property 3.10, the terms marked as *adap. term* 1 and *uncty. term* 1 in Eq. (24) are ≤ 0 and hence can be removed from Eq. (24). Similarly, the terms marked as *adap. term* 2 and *uncty. term* 2 in Eq. (24) can be removed. Thus, with Eq. (18), Eq. (24) reduces to

$$\dot{V} = -\mathbf{e}^T Q\mathbf{e} + 2\mathbf{d}^T\hat{A}_{0_L}^T\tilde{P}\mathbf{e} + \mathbf{w}^T\tilde{P}\mathbf{e} + \boldsymbol{\phi}^T\tilde{P}\mathbf{e} + \mathbf{u}_2^T\tilde{P}\mathbf{e} \quad (25)$$

Further, the Lyapunov derivative can be upper bounded as

$$\begin{aligned} \dot{V} \leq\ & -\lambda_{\min}(Q)\|\mathbf{e}\|^2 + 2\|\mathbf{d}\|\|\hat{A}_{0_L}\|\|\tilde{P}\|\|\mathbf{e}\| \\ & +2\|\mathbf{w}\|\|\tilde{P}\|\|\mathbf{e}\| + 2\|\boldsymbol{\phi}\|\|\tilde{P}\|\|\mathbf{e}\| + 2\mathbf{u}_2^T\tilde{P}\mathbf{e} \end{aligned} \quad (26)$$

Using Remark 3.11 and $\|\mathbf{e}\| > \max(\|\mathbf{d}\|, \|\mathbf{w}\|, \|\boldsymbol{\phi}\|)$, Eq. (26) can be further upper bounded as

$$\dot{V} \leq -\lambda_{\min}(Q)\|\mathbf{e}\|^2 + 2\tilde{\sigma}^*\|\mathbf{e}\|^2 + 2\mathbf{u}_2^T\tilde{P}\mathbf{e} \quad (27)$$

where, $\tilde{\sigma}^* = \left(\|\hat{A}_{0_L}\|\ \bar{\sigma}_{\tilde{P}} + 2\bar{\sigma}_{\tilde{P}}\right)$. Choosing $\mathbf{u}_2 = -\tilde{P}^{-1}C^T\tilde{\mathbf{y}}\tilde{\sigma}^*$ and Q such that $\lambda_{\min}(Q) > 2\tilde{\sigma}^*$, $\Rightarrow \dot{V} < 0$. Thus, $\mathbf{e}(t)$, $\tilde{A}_0(t)$ and $\tilde{B}_0(t)$ are uniformly bounded for $t \geq 0$ [1]. Further, with $\mathbf{e}$ bounded and with Assumption 2.1, $\hat{\mathbf{x}}$ is bounded. Thus, the estimates of the unknown states and parameters are bounded.

Special Case: $C = I$: We specialize the results in Section III-F when $p = n$ which $\Rightarrow \tilde{\mathbf{y}} = \mathbf{e}$. Thus the pseudo control input becomes $\mathbf{u}_2 = -\tilde{P}^{-1}\tilde{\mathbf{y}}\tilde{\sigma}^* = -\tilde{P}^{-1}\mathbf{e}\tilde{\sigma}^*$ and with this form of $\mathbf{u}_2$, Eq. (27) reduces to $\dot{V} \leq -\lambda_{\min}(Q)\|\mathbf{e}\|^2$. Thus, the equilibrium point defined by $\left(\mathbf{e}, \tilde{A}_0, \tilde{B}_0\right) = 0$ is uniformly stable. Furthermore, invoking arguments similar to Section II-G, we show that $\dot{V}\left(\mathbf{e}, \tilde{A}_0, \tilde{B}_0\right) \to 0$ as $t \to \infty$, which implies that $\mathbf{e} \to 0$ as $t \to \infty$. Further, if the input $\mathbf{u}$ is persistently exciting then $\tilde{A}_0, \tilde{B}_0 \to 0$ as $t \to \infty$ [1]. ***Thus as the number of measurements approaches the number of states, the number of parameters that can be treated as unknowns (and estimated with bounded errors) in the A matrix are n^2 and in the B matrix are nm.***

IV. SIMULATION RESULTS

We provide simulation results to illustrate the efficacy of the adaptive approach when all and some state are measured. For both cases, we sample the system at 50 Hz ($dt = 0.02$ seconds) and consider a white Gaussian measurement noise of 0.1 standard deviation. Furthermore, the known input is chosen as $u(t) = \sin(t) + \sin(5t)$.

1) All States Measured: Consider the dynamics of Eq. (1) with $A = \left[\begin{array}{cc} -1 & 1 \\ -4 & -2 \end{array}\right]$ and $B = \left[\begin{array}{cc} 1 & 2 \end{array}\right]^T$. Choose $A_H = -10I$. Let $\mathbf{x}_0 = \left[\begin{array}{cc} 1 & 5 \end{array}\right]^T$, $\hat{\mathbf{x}}_0 = 0$ and $Q = 20I$. For the adaptive laws in Eq. (4), $\bar{\Gamma}_A = 40$ and $\bar{\Gamma}_B = 1$. In Figure 1, the first two subplots show the plots of the true state (solid) and the estimated state (dotted) histories and the third subplot shows the corresponding error plots. It can be seen from these figures that the errors are bounded and the adaptive estimator in Eq. (2) behaves similar to an unbiased estimator. The estimates for A and B are $\begin{bmatrix} -1.429 & 0.98761 \\ -4.561 & -2.194 \end{bmatrix}$ and $\begin{bmatrix} 0.96722 & 2.0292 \end{bmatrix}^T$ respectively.

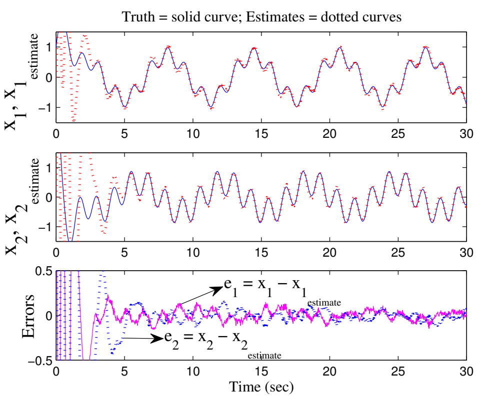

Fig. 1. True states and corresponding error plots of the state estimates.

Furthermore, in a noise free system, if we assume that the state estimate $\hat{\mathbf{x}}_f(t)$ is equal to $\mathbf{x}(t)$, i.e., $\mathbf{d}(t) = 0$ and simulate the above system we obtain $A = \begin{bmatrix} -1.0041 & 1.0005 \\ -4.0359 & -1.9959 \end{bmatrix}$ and $B = \begin{bmatrix} 1.0009 & 2.0076 \end{bmatrix}^T$.

2) Some States Measured: Consider the dynamics of Eq. (14) with $A = \begin{bmatrix} -1 & 1 \\ -4 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \end{bmatrix}^T$ and $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$, where $n = 2$, $m = 1$ and $p = 1$. Choose $A_1 = 0$ and $B_1 = 0$ (below Eq. (16)) and $L = \begin{bmatrix} -5 & 4 \end{bmatrix}^T$. Thus $\bar{A} = \begin{bmatrix} 0 & 0 \\ -4 & -2 \end{bmatrix}$ and $\bar{B} = \begin{bmatrix} 0 & 2 \end{bmatrix}^T$ and with the above choice of L, $A_H = \begin{bmatrix} -5 & 0 \\ 0 & -2 \end{bmatrix}$. Let $\mathbf{x}_0 = \begin{bmatrix} 1 & 5 \end{bmatrix}^T$, $\hat{\mathbf{x}}_0 = 0$ and $Q = I$. For the adaptive laws in Eq. (21), $\bar{\Gamma}_A = 10$ and $\bar{\Gamma}_B = 1$. In Figure 2 the first two subplots show the plots of the true state (solid) and the estimated state (dotted) histories and the third subplot shows the corresponding error plots. It can be seen from these figures that the errors are bounded and the adaptive estimator in Eq. (17) behaves similar to an unbiased estimator. The estimates for A_1^* and B_1^* are -1.0113 and 0.41629 respectively. Notice that since we do not have any guarantee on the convergence of the parameter errors, these values are at most bounded.

V. CONCLUSIONS

In this paper, a joint state and adaptive parameter identification scheme is presented for a class of unknown linear dynamical systems for the cases when all the states of the system are measured and when only some states of the system are measured. When all the states are measured, we have proven that in the presence of process and measurement noise, the state and parameter estimation errors are bounded.

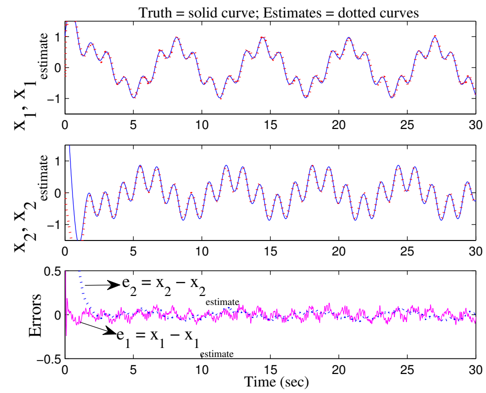

Fig. 2. True states and corresponding error plots of the state estimates.

To this end, we show that this is possible only through the appropriate design of a virtual input which ensures that the system error signals are bounded. In the more general case where only some states are measured, we have shown that via projection based adaptation, that the state and parameter errors are bounded. For each of the cases proposed and solved, we have specialized the results to situations when the bounded disturbances are absent and have shown that the state estimation error converges to 0. Simulation results illustrate the efficacy of this approach.

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