2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Output-Feedback Controlled-Invariant Sets for Systems with Linear Parameter-Varying State Transition Matrix

Andreas B. Hempel, Andreas B. Kominek and Herbert Werner

Abstract— **The notion of output-feedback controlled-invariant sets is extended from LTI systems to systems with linear parameter-varying state transition matrix. A theorem is presented that can be used to verify whether a given polytope can be made invariant under output-feedback. The theorem also provides the constraints a control input has to fulfill to make the candidate set invariant. Predictive output-feedback controllers based on such a set can satisfy hard constraints on both the plant state and the control inputs in the presence of process disturbances and measurement noise. Simulation results demonstrate the strength of such a controller that can guarantee constraints for a subset of the state space without requiring state information or estimation.**

I. Introduction

Invariant sets are a very useful concept for stability analysis and controller synthesis. In general, invariant sets are subsets of the state space where for each initial state inside the subset, control trajectories exist that keep the state within the invariant set for all time. The strength of this tool for computation of Lyapunov functions and developing controllers that guarantee hard constraints on the states is obvious: as long as the invariant set is a subset of the constraint set, the system constraints will be fulfilled for all times. A good overview of invariant sets is given in [1] and [2]. Since we use polyhedral invariant sets, this introduction focuses on those.

Invariant sets of autonomous (considering - if any - a fixed control law) discrete time linear time invariant (LTI) systems were considered in [3]–[7]. In [8]–[11], so called controlled-invariant sets of discrete time LTI systems were constructed and corresponding state-feedback controllers were computed. In [12] Dorea extended the concept of controlled invariant sets from state-feedback to output-feedback and developed a sufficient condition for output-feedback controlled invariance. Invariant sets for observer-based output-feedback control for discrete time LTI systems are considered in [13].

A similar development has evolved for discrete time LPV systems. In a number of publications, e.g. [14]–[16], invariant sets for autonomous discrete time LPV systems (also with fixed control law) were computed and used for model predictive control in [15], [17], [18]. Controlled-invariant sets for discrete time LPV systems were considered in [19].

A. Hempel is with the Automatic Control Laboratory, Swiss Federal Institute of Technology Zurich, 8092 Zurich, Switzerland (http://control.ee.ethz.ch), A. Kominek is with Germanischer Lloyd Industrial Services GmbH, 20457 Hamburg, Germany (http://www.gl-group.com/GLRenewables), H. Werner is with the Institute of Control Systems, Hamburg University of Technology, 21073 Hamburg, Germany (http://www.tu-harburg.de/rts), email:andreas.kominek@tu-harburg.de.

Manuscript received September 7, 2011.

In [20] an efficient tool for the computation of controlled-invariant sets for constrained LPV systems was presented and applied to the computation of Lyapunov functions.

A more general notation for output-feedback controlled-invariant sets was introduced in [21]. The authors complement the standard invariant set by information sets, that incorporate information gathered by the controller during the process. This approach is very useful to determine invariant sets for observer-based output-feedback controllers. An unresolved issue with this approach are the intricate numerical computations because of the information sets even when the original system is linear. Such issues would be even worse for systems with linear parameter-varying state transition matrix from now on referred to as LPV-A systems.

This is why in this paper the more conservative definition of output feedback invariant sets used in [12] is extended to LPV-A systems. A sufficient condition for output-feedback controlled invariance of LPV-A systems is developed. For computed sets for which the sufficient condition holds LPV output-feedback controllers are computed that guarantee invariance of the set under process disturbances and measurement noise. This is demonstrated with a numerical example and the performance is compared to an explicit MPC state feedback controller from [22].

This paper is structured as follows. After this introduction, the definition of the considered LPV-A systems is provided in Section II. The main results are presented in Section III and validated with a simulation example in Section IV. The conclusion summarizes the results in Section V.

Notation: A superscript in square brackets indicates the row of a matrix or the component of a vector. $A^{[j]}$ accordingly corresponds to the j^{th} row of the matrix A and $b^{[i]}$ to the i^{th} component of vector b. Two superscripts in square brackets indicate a particular element of a matrix, i.e. $A^{[i,j]}$ would be the element in the i^{th} row and j^{th} column of matrix A. A shorthand for the convex polytope $\{x \mid Ax \leq b\}$ is given by $P[A, b]$.

II. Preliminaries

A. LPV-A Systems

Consider the class of polytopic linear parameter-varying systems with constant input and output matrices called LPV-A systems. All variables are subject to constraints and the system to disturbances d_k and measurement noise η_k:

$$x_{k+1} = A(\theta_k)\, x_k + Bu_k + Ed_k \tag{1a}$$

$$y_k = Cx_k + \eta_k \tag{1b}$$

$$x_k \in \mathcal{X}, \quad u_k \in \mathcal{U}, \quad d_k \in \mathcal{D}, \quad \eta_k \in \mathcal{M} \tag{1c}$$

Note that by using the Δu-formulation an LPV system with a parameter-varying input matrix B can be represented by an LPV-A system, cf. [22].

The constraint sets $\mathcal{X}$, $\mathcal{U}$, $\mathcal{D}$, and $\mathcal{M}$ are assumed to be bounded convex polytopes containing the origin in their interior:

$$\mathcal{X} = \{x \in \mathbb{R}^{n_x} \mid Gx \leq h_G\} \quad \mathcal{U} = \{u \in \mathbb{R}^{n_u} \mid Vu \leq h_V\}$$
$$\mathcal{D} = \{d \in \mathbb{R}^{n_d} \mid Sd \leq h_S\} \quad \mathcal{M} = \{\eta \in \mathbb{R}^{n_y} \mid Q\eta \leq h_Q\}$$

The scheduling parameter θ is assumed to be known at each timestep and constrained to the standard simplex Θ[1], i.e.

$$\theta \in \mathbb{R}^{n_\theta}, \quad 0 \leq \theta^{[i]} \leq 1, \quad \sum_{i=1}^{n_\theta} \theta^{[i]} = 1.$$

The state-transition matrix $A(\theta)$ is constrained to the polytope defined by the vertex matrices A_i and the components of the scheduling parameter vector θ are its barycentric coordinates:

$$A(\theta) = \sum_{i=1}^{n_\theta} A_i \theta^{[i]} \tag{2}$$

Since the controlled-invariance of a given polytope Ω should be guaranteed by the controller using only the noise corrupted measured outputs, the set $\mathcal{Y}(\Omega)$ is defined as:

Definition 1: (Set of admissible outputs) The set

$$\mathcal{Y}(\Omega) := \{y \in \mathbb{R}^{n_y} \mid y = Cx + \eta \quad \text{with} \quad x \in \Omega, \eta \in \mathcal{M}\}$$

is the set of admissible outputs corresponding to Ω.

III. Output-feedback controlled-invariant sets for LPV Systems

The problem addressed in this paper is twofold: given a dynamic system (1) and a convex polytope Ω, determine whether the polytope can be made controlled-invariant under output-feedback. If that is the case, compute an (explicit) output-feedback model-based controller ensuring the controlled-invariance of Ω. The first part is addressed in Sections III-A and III-B while the second part is handled in Section III-C.

A. Output-feedback controlled-invariant sets for LPV-A systems

The notion of an output-feedback controlled-invariant (o.f.c.i.) polytope as given in [12] is extended to LPV-A systems:

Definition 2: (Output-feedback controlled-invariant polytope for an LPV-A system) A polytope $\Omega = \{x \mid Gx \leq h_G\}$ is called o.f.c.i. with respect to LPV-A system (1) with contraction rate $\lambda \in [0, 1]$ if and only if

$$\forall y \in \mathcal{Y}(\Omega), \theta \in \Theta \quad \exists u = u(y, \theta):$$
$$G(A(\theta)x + Bu + Ed) \leq \lambda h_G \tag{3a}$$
$$Vu \leq h_V \tag{3b}$$
$$\forall d \quad \text{with} \quad Sd \leq h_S \tag{3c}$$
$$\forall \eta \quad \text{with} \quad Q\eta \leq h_Q \tag{3d}$$

[1]It should be noted that any physical scheduling parameter can be transformed into such a representation by using its barycentric coordinates.

B. Output-feedback controlled-invariance verification

Let

$$u_k = u(y_k, \theta_k) = \sum_{j=1}^{n_\theta} \theta_k^{[j]} u_j(y_k) \tag{4}$$

be the control law to be computed. Using the polytopic structure of the control law in (4) and of $A(\theta)$ in (2), the worst-case effects δ of the disturbances d and the output equation (1b) to rewrite (3) yields the following condition for a polytope Ω to be output-feedback controlled-invariant with respect to LPV-A system (1), which has to hold for all $j = 1, \ldots, n_\theta$:

$$\forall y \in \mathcal{Y}(\Omega) \quad \exists u_j:$$
$$\phi(y, j) + GBu_j \leq \lambda h_G - \delta, \tag{5a}$$
$$Vu_j \leq h_V, \tag{5b}$$

where $\phi(y, j) \in \mathbb{R}^g$ is the maximum value of the state influence GA_jx computed componentwise as:

$$\begin{aligned} \phi^{[i]}(y, j) \quad &= \quad \max_x G^{[i]} A_j x \\ \text{s.t.} \quad &Gx \leq h_G \\ &-QCx \leq h_Q - Qy \end{aligned} \tag{6}$$

Let the polyhedral cone Γ be defined as

$$\Gamma = \left\{ \begin{bmatrix} t \\ w \end{bmatrix} \in \mathbb{R}_+^{g+v} \;\middle|\; \begin{bmatrix} t^T & w^T \end{bmatrix} \begin{bmatrix} GB \\ V \end{bmatrix} = 0 \right\}. \tag{7}$$

The rows of the matrix $\begin{bmatrix} T & W \end{bmatrix}$ consist of the elements of the minimal generating set of Γ.

In order to check whether a given polytope $\Omega = P[G, h_G]$ is o.f.c.i., (5a),(5b) can be used. Together with (7) and Farka's Lemma, Theorem 1 is obtained:

Theorem 1: **Output-feedback controlled-invariant polytope for LPV-A system (1).** Given is an LPV-A sytem (1) for which the states x are constrained to the convex polytope $\Omega = P[G, h_G]$. The n_r rows of the matrix $\begin{bmatrix} T & W \end{bmatrix}$ are given by the minimal generating set of the pointed polyhedral cone Γ (7) and $\phi(y, j)$ is defined as in (6).

Then Ω is output-feedback controlled-invariant with respect to LPV-A system (1) if and only if

$$\forall i = 1, \ldots, n_r, j = 1, \ldots, n_\theta:$$
$$\sum_{l=1}^{g} T^{[i,l]} G^{[l]} A_j \xi_l \leq T^{[i]}(\lambda h_G - \delta) + W^{[i]} h_V,$$
$$\forall \xi_l, y\colon G\xi_l \leq h_G, Q(y - C\xi_l) \leq h_Q.$$

Proof: Is straightforward in analogy to the proof in [12] with the necessary modifications for the LPV-A case given above. ■

From Theorem 1 it can be deducted that output-feedback controlled-invariance of a given polytope $\Omega = P[G, h_G]$ can be verified by solving a number of linear optimization problems for calculating δ and the minimal generating set of Γ as well as solving the following $n_r n_\theta$ optimization problems. Let

$$\begin{aligned}\mu(i,j) &= \max_{y,\xi_l} \sum_{l=1}^{g} T^{[i,l]} G^{[l]} A_j \xi_l, \\ &\quad \text{s.t.} \quad G\xi^j \le h_G, \quad Qy - QC\xi^j \le h_Q.\end{aligned}$$

Then Ω is o.f.c.i. with respect to LPV-A system (1) if and only if $\forall i = 1, \ldots, n_r, j = 1, \ldots, n_\theta$

$$\mu(i,j) \le T^{[i]}(\lambda h_G - \delta) + W^{[i]} h_V.$$

C. Computation of the control input

Any set of control inputs u_j satisfying the constraints (5) will guarantee the invariance of Ω. By adding an appropriate cost function to the constraints an optimal set of vertex control laws u_j can be chosen. Combining the constraints (5) for all values of j with an appropriate cost function yields the optimization problem

$$\min_{\mathbf{U},y} \quad J(y, \mathbf{U}, \bar{x}) \tag{9a}$$

$$\text{s.t.} \quad \phi(y,j) + GBu_j \le \lambda h_G - \delta \tag{9b}$$

$$Vu_j \le h_V \tag{9c}$$

$$y \in \mathcal{Y}(\Omega)$$

$$\forall j = 1, \ldots, n_\theta$$

which has to be solved to obtain the optimal vertex control laws u_j which have been collected in the matrix $\mathbf{U}$. Note that $\bar{x}$ represents an assumed state consistent with the currently measured output given by:

$$\begin{aligned}\bar{x}(y) &= \arg\min_{x(y)} \quad x^T x \\ &\quad \text{s.t.} \quad Q(y - Cx) \le h_Q \\ &\qquad\qquad Gx \le h_G\end{aligned}$$

In general the objective function can take any form the user desires. One possibility that proved to yield good results is based on known operating points $\bar{\Theta} = \begin{bmatrix}\bar{\theta}_1 & \bar{\theta}_2 & \ldots & \bar{\theta}_{n_{\bar{\theta}}}\end{bmatrix}$ for the scheduling parameter θ. These will be incorporated into the objective function to optimize the controller for these particularly important points. Tuning matrices P_j and R_j corresponding to each operating point can be selected as appropriate weighting factors. The resulting cost function J is then

$$\begin{aligned}J(\mathbf{U}, \bar{x}(y), \bar{\Theta}) = \sum_{j=1}^{n_{\bar{\theta}}} \Big(u\left(y, \bar{\theta}_j\right)^T R_j u\left(y, \bar{\theta}_j\right) + \\ \left(C(A(\bar{\theta}_j)\bar{x} + Bu(y, \bar{\theta}_j))\right)^T P_j \left(C(A(\bar{\theta}_j)\bar{x} + Bu(y, \bar{\theta}_j))\right)\Big)\end{aligned} \tag{10}$$

The optimization problem (9) with cost function J can be solved implicitly (online) or explicitly (offline). The former requires more computation time during plant operation but offers a relatively straightforward implementation. The latter is a little more involved as the explicit solutions for the components $\phi^{[i]}(y,j)$ from (6) have to be compiled into a vector-valued expression of $\phi(y,j)$. Both solutions lead to a controller that guarantees the invariance of Ω under the influence of disturbances and measurement noise. The computation and implementation of the explicit controller will be outlined in the following.

The solution for all components $\phi^{[i]}(y,j)$ for each vertex system matrix A_j will be continuous and piecewise affine in y over a partition of the set of admissible outputs $\mathcal{Y}(\Omega)$, cf. [23]:

$$\phi^{[i]}(y,j) = \beta_l(i,j)y + \gamma_l(i,j) \quad \text{for} \quad y \in \Psi_l(i,j) \tag{11a}$$

$$\bigcup_l \Psi_l(i,j) = \mathcal{Y}(\Omega), \quad \Psi_m(i,j) \cap \Psi_l(i,j) = \emptyset \tag{11b}$$

In order to obtain an affine expression of the whole vector $\phi(i,j)$ that can be used in (9b), the solutions obtained for the components from (6) of the form outlined in (11a) have to be combined into continuous and piecewise affine expressions for $\phi(y,j)$:

$$\phi(y,j) = \beta_l(j)y + \gamma_l(j) \quad \text{for} \quad y \in \Phi_l(j) \tag{12a}$$

$$\bigcup_l \Phi_l(j) = \mathcal{Y}(\Omega), \quad \Phi_k(j) \cap \Phi_l(j) = \emptyset \tag{12b}$$

The $x^*(i,j)$ are the maximizers from (6) and are continuous and piecewise affine functions in y. From (12a) it is clear that $\beta_l(j) \in \mathbb{R}^{g \times n_y}$ and $\gamma_l(j) \in \mathbb{R}^g$. It is important to note that the elements $\Psi_l(i,j)$ of the partition (11b) are generally not the same as the elements $\Phi_l(j)$ of partition (12b) since for two different components of $\phi(y,j)$ the convex partition of the parameter space $\mathcal{Y}(\Omega)$ will in general be different[2]. It is therefore necessary to construct the partition (12b) from the partitions in (11b).

Let $P_i(j)$ be the partition described in (11b) for the component $\phi^{[i]}(y,j)$ and $S(j)$ be the partition described in (12b) for the whole vector $\phi(y,j)$:

$$P_i(j) = \{\Psi_1(i,j), \Psi_2(i,j), \ldots, \Psi_{p_i}(i,j)\} \tag{13a}$$

$$S(j) = \{\Phi_1(j), \Phi_2(j), \ldots, \Phi_s(j)\} \tag{13b}$$

In the following an algorithm will be described that generates $S(j)$ from the g different partitions $P_i(j)$. For this, two facts have to be noted that will be important for this algorithm. First, from the constraints in (6) it can be seen that for all components $\phi^{[i]}(y,j)$ the parameter space is the same because the constraints do not depend on vertex system A_j or the row of G that is currently dealt with. Therefore all partitions P_i cover the same space, or $\bigcup_j P_i(j) = \bigcup_j P_l(j) = \mathcal{Y}(\Omega)$ for any two values of i and l. Second, all partitions contain only convex polytopes since they are generated from the solution of multiparametric linear programs.

The goal of the algorithm is to find a convex partition $S(j)$ together with the corresponding matrices $\beta_l(j)$ and $\gamma_l(j)$ that describe $\phi(y,j)$ as shown in (12a). The partitions $P_i(j)$ together with the corresponding matrices $\beta_l(i,j)$ and $\gamma_l(i,j)$ are taken by the algorithm as inputs. A shorthand description is given as Algorithm 1. The key operations of the algorithm will be described in the following.

[2]Please see [24] for two algorithms that create feasible partitions of the parameter space of a quadratic multiparametric program.

Algorithm 1 Generating a partition $S(j)$ that describes $\phi(y, j)$

Require: partitions $P_i(j)$ as described by (13a) and (11b) with corresponding matrices $\beta_l(i, j)$ and $\gamma_l(i, j)$ as described by (11a)

```
1:  S(j) = P_1(j)
2:  for l = 1 to |S(j)| do
3:      β_l(j) = β_l(1, j)
4:      γ_l(j) = γ_l(1, j)
5:  end for
6:  for i = 2 to g do
7:      S̃(j) = ∅
8:      n = 1
9:      for l = 1 to |S(j)| do
10:         for m = 1 to |P_i(j)| do
11:             if Φ_l(j) ∩ Ψ_m(i, j) ≠ ∅ then
12:                 S̃(j) = { S̃(j), Φ_l(j) ∩ Ψ_m(i, j) }
13:                 β̃_n(j) = [ β_l(j) ; β_m(i, j) ]
14:                 γ̃_n(j) = [ γ_l(j) ; γ_m(i, j) ]
15:                 n = n + 1
16:             end if
17:         end for
18:     end for
19:     S(j) = S̃(j)
20:     for l = 1 to |S(j)| do
21:         β_l(j) = β̃_l(j)
22:         γ_l(j) = γ̃_l(j)
23:     end for
24: end for
```

The target partition $S(j)$ and the corresponding matrices $\beta_l(j)$ and $\gamma_l(j)$ are initialized in lines 1 to 5 with the values from the first component partition $P_1(j)$. In the next loop each element $\Psi_m(i, j)$ of the partition $P_i(j)$ for the next component will be compared with all elements $\Phi_l(j)$ of the so far constructed partition $S(j)$. If an intersection is detected it is stored in an auxiliary partition (line 12) and the vectors $\beta_l(j)$ and $\gamma_l(j)$ corresponding to $\Phi_l(j)$ are augmented by the $\beta_m(i, j)$ and $\gamma_m(i, j)$ that correspond to $\Psi_m(i, j)$ and stored in auxiliary matrices $\tilde{\beta}_n(j)$ and $\tilde{\gamma}_n(j)$ (lines 13 and 14). Whenever another component $\phi^{[i]}(y, j)$ is fully added to the auxiliary partition $\tilde{S}(j)$ in this way, the auxiliary variables are stored back to the target variables $S(j)$, $\beta_l(j)$, and $\gamma_l(j)$ (lines 19 to 23). These then represent all the information necessary to obtain $\phi(y, j)$ as shown in (12a) and (13b).

Since all partitions $P_i(j)$ cover the same space regardless of the value for i the same space will also be covered by $S(j)$. Additionally no special cases have to be considered where parts may be included in $P_i(j)$ but not in another partition. Furthermore, since the elements $\Psi_m(i, j)$ of all partitions $P_i(j)$ are convex polytopes, the same holds for the elements $\Phi_l(j)$ of $S(j)$ because these are generated as the intersection of two convex polytopes $\Psi_m(i, j)$ and $\Psi_n(i, j)$.

Algorithm 1 has to be executed for all $j = 1, \dots, n_\theta$ to obtain piecewise continuous and affine representations of all $\phi(y, j)$ as in (12a) that can be used in the following computations of optimal control inputs u_j.

Since $\phi(y, j)$ is defined over the partition $S(j)$ of $\mathcal{Y}(\Omega)$, the calculation of the optimal control inputs u_j has to be conducted separately for all elements of $S(j)$. Using the affine representation of $\phi(j)$ from (12a) in the constraints of (5) together with a suitable objective function J leads to the multiparametric optimization problem that is used to choose an optimal control input vector u_j:

$$
\begin{aligned}
u_{j,l}(y) &= \arg\min_{u_j(y)} J(y, u_j, \bar{x}) && \text{(14a)}\\
\text{s.t.}\quad \beta_l(j)y + \gamma_l(j) + GBu_j &\le \lambda h_G - \delta && \text{(14b)}\\
Vu_j &\le h_V && \text{(14c)}\\
y &\in \Phi_l(j)
\end{aligned}
$$

The values of λ in (14b) and R_j and P_j in (10) can be considered as tuning parameters to obtain a desirable performance of the closed-loop system.

The main problem remains to find an output-feedback controlled-invariant set for system (1). One strategy would be to calculate a controlled-invariant set Ω employing an algorithm as presented in [20] . Theorem 1 is then used to verify whether it is also controlled-invariant under output-feedback. If this is successful (14) with an appropriate cost function, e.g. (10), can be used to calculate an explicit controller that guarantees the invariance under output feedback. There currently exists no algorithm to directly construct o.f.c.i. polytopes. The same problem exists for LTI systems and awaits solution.

IV. Example

First simulations with randomly generated second-order LPV-A systems indicate that the invariant output-feedback control strategy described above can provide a relatively good performance compared to other control strategies employing state- or state-estimate-feedback.

In order to visualize some of the unique properties of an explicit output-feedback invariant controller (EOFIC), its performance for an exemplary second order LPV system (1) is compared to an explicit model-predictive controller (EMPC) obtained from [22]. The necessary computations are done with the Multiparametric Toolbox [23], [25] and YALMIP [26] for MATLAB.

The two control schemes are applied to the LPV-A system (1) with the system matrices[3]

$$
A_1 = \begin{bmatrix} 1.9851 & 1.9188 \\ -1.5255 & -1.1930 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2.0382 & 2.1189 \\ -1.2427 & -1.1107 \end{bmatrix}
$$
$$
B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad E = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}, \quad C = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}.
$$

The constraints are chosen such that $\|x\|_\infty \le 10$, $|u| \le 3$, $|d| \le 1$, and $|\eta| \le 0.3$. The control objective is to guarantee

[3]Note that only the first four digits are shown. For details please contact the authors.

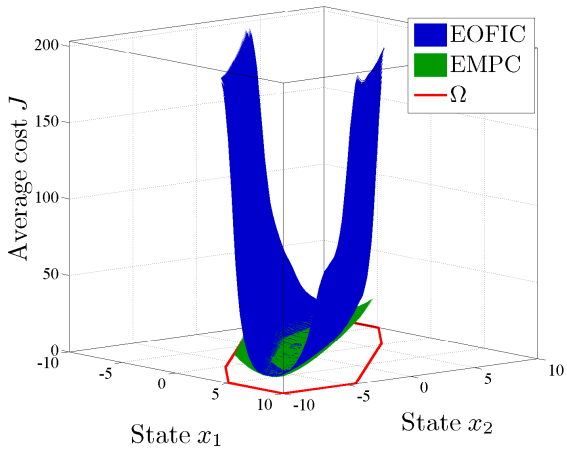

Fig. 1. Average cost J

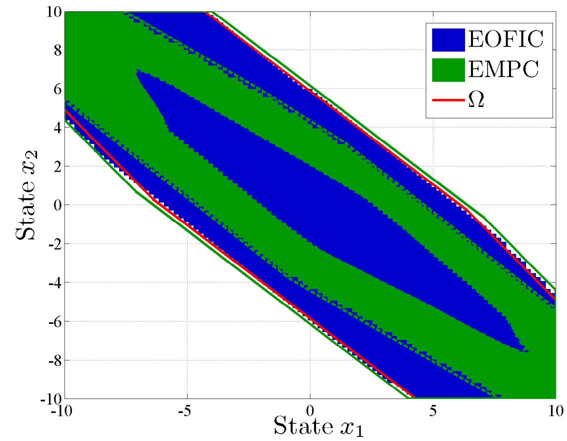

Fig. 2. Areas of lowest cost J

the satisfaction of these constraints for a maximal subset of the state space.

For the design of the EOFIC the weighting matrices are chosen as $P_1 = P_2 = 0.1I_2$ and $R_1 = R_2 = 1$ while the vertices e_1 and e_2 of the standard simplex Θ are selected as the operating points for (10). The same tuning parameters are used for the EMPC which uses a prediction horizon of 5 steps and a control horizon of 2 steps. For feedback purposes an Extended Kalman Filter (EKF) is used to estimate the current state. The EKF is initialized correctly with the actual initial state and disturbances and measurement noise introduce uncertainty.

The total state space is gridded into 100 points per direction and each of these states is used as the initial state for 20 simulations with different random trajectories for disturbances d, noise η, and parameter θ.

Fig. 1 shows the average cost of control as given by (10) on the z-axis resulting from the 20 simulations from each initial state in the x, y-plane. The invariant polytope Ω is shown in red. In case one of the simulations leads to a constraint violation the corresponding initial state is discarded from the plot. Constraint violations can occur in the state (the plant state is controlled outside the allowable state space $\mathcal{X}$) or in the control input (the use of a state outside the region of operation of the controller leads to a nominal u outside $\mathcal{U}$).

The blue surface shows the average cost J for the output-feedback controller being relatively low along the center diagonal of the invariant polytope and rising significantly towards its edges. The costs for the EMPC (green surface) are comparable along the diagonal but rise more slowly towards the edges.

Although the performance of the EMPC appears better in terms of cost, a different aspect becomes apparent when Fig. 1 is viewed at from below. Fig. 2 provides this view in which colored areas indicate regions where one controller performs better than the other. The first thing to note is that the output-feedback controller actually performs a little better for most states along the diagonal. The second is that in spite of the steep increase in cost towards the edges of Ω the EMPC cost surface is not seen in this area. This is due to the significant number of constraint violations for the closed-loop system.

The solid green line shows the nominal area of operation for the EMPC which is slightly larger than Ω. The dashed green line marks the convex hull of all states that do not lead to constraint violations in the simulations. Not only exists a significant gap between the two areas but also between the feasible region of operation for the EMPC and Ω on which the output-feedback controller is defined. Furthermore, even some states outside Ω lead to feasible simulations with the invariant controller. The described effects are also present when the actual state is used for feedback.

The peculiar behavior of the EMPC can of course be attributed to the influences of disturbances and measurement noise. The invariant output-feedback controller recognizes the existence of these influences and accounts for them while the used EMPC scheme does not. The choice of values for the tuning matrices P_j and R_j leads to significantly higher cost for the EOFIC as compared to the EMPC as shown in Fig. 1. By choosing different values for P_j and R_j the gap between the nominal and actual control area of the EMPC can be lessened but this also leads to better overall performance for the EOFIC. The designer of the EMPC faces a trade-off between superior performance over the EOFIC and a larger area of control.

Other simulations confirm these impressions. Furthermore, the general performance of the EOFIC for most systems where it can be applied is comparable to or even better than that of the EMPC or other control schemes from [18], [27] while at the same time delivering guarantees with regard to hard constraints even under external influences.

V. Conclusion

The notion of robust controlled-invariant polytopes is very useful for designing state-feedback controllers for linear parameter-varying (LPV) systems. Computation algorithms can take process disturbances into account in the polytope computation making the resulting control method robust with respect to unknown but bounded disturbances.

As an extension of this, a theorem is presented that allows to verify whether a given polytope is output-feedback controlled-invariant (o.f.c.i.) with respect to a given LPV-A system. The invariance property of the polytope can then be guaranteed by calculating the control input from the measured outputs only. The outputs can be subject to bounded measurement noise. The calculation of a suitable control input requires some sort of robustification of the constraints with respect to the unknown plant state x. The robust constraints are complemented by a suitable cost function to calculate an output-feedback controller that can guarantee hard constraints on states and control inputs in the presence of process disturbances and measurement noise.

The output-feedback control method (EOFIC) is compared to an explicit model-predictive controller (EMPC) in an exemplary simulation. The EOFIC guarantees the observance of system constraints in the presence of both process disturbances and measurement noise for all states within the invariant polytope. On the other hand the EMPC has major problems dealing with these external influences. Even when using state-feedback the nominal area of operation and the actual set of states for which it does not lead to constraint violations are significantly different from each other. This effect is strengthened by using state estimation which is affected by the presence of measurement noise. The EOFIC scheme guarantees this robustness for any choice of weighting matrices while the EMPC can be tuned to work on a larger area than shown here at the expense of giving up its performance advantage.

There are several possible routes for future research. The most obvious is indicated by the fact that finding an o.f.c.i. polytope for a given system can be difficult. It will be useful to find conditions LPV (or even LTI) systems have to fulfill to indicate the (non-)existence of an o.f.c.i. polytope for the respective system. The next step will then be to develop algorithms that allow for the calculation of these polytopes directly. Other paths involve the extension to other model structures, e.g. full LPV systems or input-output LPV systems.

References

[1] F. Blanchini, “Set invariance in control,” *Automatica*, vol. 35, pp. 1747–1767, 1999.

[2] F. Blanchini and S. Miani, *Set-Theoretic Methods in Control*. Boston: Birkhauser, 2008.

[3] E. G. Gilbert and K. T. Tan, “Linear systems with state and control constraints: The theory and application of maximal output admissible sets,” *IEEE Transactions on Automatic Control*, vol. 36, no. 9, pp. 1008–1020, 1991.

[4] V. M. Kuntsevich and B. N. Pshenichnyi, “Minimal invariant sets of dynamic systems with bounded disturbances,” *Cybernetics and Systems Analysis*, vol. 32, no. 1, pp. 58–64, 1996.

[5] I. Kolmanovsky and E. G. Gilbert, “Theory and computation of disturbance invariant sets for discrete time linear systems,” *Mathematical Problems in Engineering: Theory, Methods and Applications*, vol. 4, pp. 317–367, 1998.

[6] S. V. Rakovic, E. C. Kerrigan, K. I. Kouramas, and D. Q. Mayne, “Invariant approximations of the minimal robust positively invariant set,” *IEEE Transactions on Automatic Control*, vol. 50, no. 3, pp. 406–410, March 2005.

[7] D. Sui, L. Feng, and M. Hovd, “Robust output feedback model predictive control for linear systems via moving horizon estimation,” in *Proceedings of the American Control Conference*, Seattle, Washington, USA, June 2008, pp. 453–458.

[8] F. Blanchini, “Minimum-time control for uncertain discrete-time linear systems,” in *Proceedings of the 31st IEEE Conference on Decision and Control*, Tucson, Arizona, Dec. 1992, pp. 2629–2634.

[9] C. E. T. Dórea and J. C. Hennet, “(A,B)-invariant polyhedral sets of linear discrete-time systems,” *Journal of Optimization Theory and Applications*, vol. 103, no. 3, pp. 521–542, Dec. 1999.

[10] Z. Artstein and S. V. Rakovic, “Feedback and invariance under uncertainty via set-iterates,” *Automatica*, vol. 44, no. 2, pp. 520–525, feb 2008.

[11] S. V. Rakovic and M. Baric, “Parameterized robust control invariant sets for linear systems: Theoretical advances and computational remarks,” *IEEE Transactions on Automatic Control*, vol. 55, no. 7, pp. 1599–1613, July 2010.

[12] C. E. T. Dórea, “Output-feedback controlled-invariant polyhedra for constrained linear systems,” in *Proceedings of the 48th IEEE Conference on Decision and Control, 2009 held jointly with the 2009 28th Chinese Control Conference. CDC/CCC 2009.*, Dec. 2009, pp. 5317–5322.

[13] S. V. Rakovic and K. I. Kouramas, “The minimal robust positively invariant set for linear discrete time systems: Approximation methods and control applications,” in *Proceedings of the 45th IEEE Conference on Decision and Control*, San Diego, CA, USA, Dec. 2006, pp. 4562–4567.

[14] R. K. Brayton and C. H. Tong, “Constructive stability and asymptotic stability of dynamical systems,” *IEEE Trans. Circuits and Systems*, vol. 27, no. 11, pp. 1121–1130, April 1980.

[15] E. Kerrigan, “Robust constraint satisfaction: Invariant sets and predictive control,” Ph.D. dissertation, Camebridge, 2000.

[16] B. Plyumers, J. A. Rossiter, J. Suykens, and B. D. Moor, “Reduced complexity polyhedral invariant sets for LPV systems,” Katholieke Universiteit Leuven, Leuven, B, Internal Report 04-230, Mar. 2005.

[17] ——, “Interpolation based MPC for LPV systems using polyhedral invariant sets,” in *Proceedings of the American Control Conference*, Portland, OR, USA, Jun. 2005, pp. 810–815.

[18] T. Besselmann, J. Löfberg, and M. Morari, “Constrained Time Optimal Control of Linear Parameter-Varying Systems,” in *IEEE Conference on Decision and Control*, Shanghai, China, Dec. 2009.

[19] F. Blanchini, “Ultimate boundedness control for uncertain discrete-time systems via set-induced lyapunov functions,” *IEEE Transactions on Automatic Control*, vol. 39, no. 2, pp. 428–433, FEB 1994.

[20] S. Miani and C. Savorgnan, “MAXIS-G: a software package for computing polyhedral invariant sets for constrained LPV systems,” in *Proceedings of the 44th IEEE Conference on Decision and Control and the European Control Conference*, Seville, ESP, Dec. 2005.

[21] Z. Artstein and S. V. Raković, “Set invarianc under output feedback: a set-dynamics approach,” *International Journal of Systems Science*, pp. 1–17, 2010.

[22] T. Besselmann, J. Löfberg, and M. Morari, “Explicit Model Predictive Control for Systems with Linear Parameter-Varying State Transition Matrix,” in *Proceedings of the 17th World Congress of the International Federation of Automation and Control*, Seoul, Korea, Jul. 2008.

[23] M. Kvasnica, *Real-Time Model Predictive Control via Multi-Parametric Programming - Theory and Tools*. Saarbrücken, D: VDM Verlag Dr. Müller, 2009.

[24] M. Baotić, “An efficient algorithm for multi-parametric quadratic programming,” ETH Zürich, Institut fr Automatik, Internal Report, 2002.

[25] M. Kvasnica, P. Grieder, and M. Baotić, “Multi-Parametric Toolbox (MPT),” 2004. [Online]. Available: http://control.ee.ethz.ch/ mpt/

[26] J. Löfberg, “YALMIP : A toolbox for modeling and optimization in MATLAB,” in *Proceedings of the CACSD Conference*, Taipei, Taiwan, 2004. [Online]. Available: http://control.ee.ethz.ch/ joloef/yalmip.php

[27] P. Apkarian, P. Gahinet, and G. Becker, “Self-scheduled H_∞ control of linear parameter-varying systems: a design example,” *Automatica*, vol. 31, no. 9, pp. 1251–1261, 1995.