2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Equivalence between Mean Square, Stochastic and Exponential Stability for Singular Jump Linear Systems

Jorge R. Chávez-Fuentes[a], Eduardo F. Costa[b] and M. H. Terra[c]

Abstract— **This paper studies Mean square stability, Stochastic stability and Exponential stability for discrete-time singular linear systems whose parameters are driven by a finite state Markov chain. It is shown the equivalence of these notions under certain conditions. New necessary conditions for mean square stability in terms of generalized Lyapunov equations for homogeneous and non-homogeneous of this class of systems are also given.**

I. Introduction

Singular systems have received considerable attention in the recent years. These kind of systems, that are also called descriptor, have many applications including, for example, economic [1], mechanical, [2], electrical [3], robotic [4] and aircraft modeling systems [5]. When the parameters that determine the system change abruptly, they are called Singular jump linear systems (SJLS) and the theory is a generalization of the very well known Markov jump linear system theory [6]. A primary and fundamental problem to address for system design is related to the stability of the system [7]. In this paper, three different concepts of stability for a SJLS are addressed: Mean square stability (MSS), Stochastic stability (SS) and Exponential stability (ES). For singular systems without jumping parameters many results regarding SS have been presented in the literature (see e.g., [8], [9], [10], [11], [12], [13]). The theory for SJLS is less developed and only a few important results have been given (see e.g., [14], [15]). To the best of our knowledge MSS and ES have not been addressed before for the considered class of systems. In this paper, it is shown that they are equivalent to SS under certain conditions.

This research was supported by FAPESP under grant 2010/12360-3 and CNPq under grant 306466/2010-4

[a] Dept. of Sciences, Pontificia Universidad Católica del Perú, Lima, Perú.

[b] Dept. of Applied Mathematics and Statistics., University of São Paulo at São carlos, Brazil,

[c] Dept. of Electrical Engineering-EESC University of São Paulo at São Carlos, Brazil

It was given in [15] a necessary and sufficient condition for SS in terms of a Generalize Lyapunov equation (GLE). For MSS, new necessary conditions in terms of GLEs are given for the homogeneous and non-homogeneous version of the SJLS.

The rest of the paper is organized as follows. In Section II, the notation used throughout the paper is presented. The MSS, the SS and the ES of a system driven by a general discrete-time finite state Markov chain (MC) are introduced in Section III. The equivalence between MSS and ES with SS are also shown in this section. New necessary conditions in terms of GLSs are given in Section IV. In Section V two numerical examples are given to show the equivalence between MSS and SS. In Section VI the conclusions are given.

II. Notation

Let $\mathbb{R}^n$ denote the n-dimensional euclidian space. The normed space of matrices of order $m \times n$ over $\mathbb{R}$ is denoted by $\mathbb{M}^{mn}$ and, for short, $\mathbb{M}^n$ denotes the normed space of square matrices of order n. For a matrix $A \in \mathbb{M}^{mn}$, A^T and $\mathrm{tr}(A)$ denote the matrix transpose and the trace of A, respectively. All random variables are defined over the probability space $(\Omega, \mathcal{F}, \Pr)$, where Ω is the sample space, $\mathcal{F}$ is the sigma algebra of events and $\Pr$ is the probability measure. A random variable is written in boldface, $\boldsymbol{x}$, and its expectation is denoted by $E(\boldsymbol{x})$. A discrete-time process is simply denoted by $\boldsymbol{x}(k)$, where k is taken in $\mathbb{Z}^+$, the set of integer non negative numbers. The indicator function with respect to the event $A \in \mathcal{F}$ is denoted by 1_A. The space of matrices in $\mathbb{M}^n$ that are (symmetric) positive semi-definite is denoted by $\mathbb{M}^{n0}$. The Kroneker product is denoted by $\otimes$ and the stacking column vector operator by $\mathrm{vec}(\cdot)$. Throughout the paper the stochastic process $\boldsymbol{\theta}(k)$ is used to drive a jump linear system. The symbol Θ is denoted to refer to the set of all initial distributions of $\boldsymbol{\theta}(k)$.

III. Second Moment stability equivalence

In this section, the equivalence between MSS and ES with SS of a SJLS are shown. In Definition 1, these second moment stability concepts are introduced. Let $\boldsymbol{\theta}(k)$ be a MC with state space $S_\theta \triangleq \{1,\ldots,N\}$, transition probability matrix $\Pi \triangleq [p_{ij}]$, where $p_{ij} \triangleq \Pr(\boldsymbol{\theta}(k+1)=j|\boldsymbol{\theta}(k)=i)$ and initial state probability vector $\boldsymbol{\theta}_0 \triangleq (\Pr(\boldsymbol{\theta}(0)=1),\ldots,\Pr(\boldsymbol{\theta}(0)=N))$. Consider the following SJLS

$$S_{\theta(k+1)}\boldsymbol{x}(k+1) = A_{\theta(k)}\boldsymbol{x}(k) + B_{\theta(k)}\boldsymbol{w}(k), \qquad (1)$$
$$\boldsymbol{x}(0) = \boldsymbol{x}_0,$$

where $\boldsymbol{x}(k) \in \mathbb{R}^n$, $\boldsymbol{x}_0 \in X \subset \mathbb{R}^n$ is a random vector with finite second moment, and $\boldsymbol{w}(k) \in \mathbb{R}^q$ is a zero mean white noise process with identity covariance matrix I_q and independent of $\boldsymbol{\theta}(k)$ and $\boldsymbol{x}_0$. For all $i \in S_\theta$, the matrix S_i is a square matrix of order n with $rank(S_i) \le n$. A_i and B_i, $i \in S_\theta$ are matrices of appropriate dimension. It is assumed throughout the paper that the system is stochastically regular [16], which ensures that $\boldsymbol{x}(k)$ is a well defined random variable. This allows us to introduce second moment matrices associated with $\boldsymbol{x}(k)$, as in Definition 1. Regularity eventually implies that X is some strict subset of $\mathbb{R}^n$, see e.g. Example 1; X is sometimes referred to as the set of admissible initial conditions.

For each $k \in \mathbb{Z}^+$ define

$$Q(k) \triangleq E\left(\boldsymbol{x}(k)\boldsymbol{x}^T(k)\right) \qquad (2a)$$
$$Q_i(k) \triangleq E\left(\boldsymbol{x}(k)\boldsymbol{x}^T(k)1_{\{\theta(k)=i\}}\right), \quad i \in S_\theta. \qquad (2b)$$

Thus

$$Q(k) = \sum_{i=1}^{N} Q_i(k). \qquad (3)$$

Definition 1: Consider the JLS (1).

a) The system (1) is said to be MSS if for any $\boldsymbol{x}_0 \in X$ and any $\boldsymbol{\theta}_0 \in \Theta$ there exists a matrix $Q \in \mathbb{M}^{n0}$ such that

$$Q \triangleq \lim_{k\to\infty} Q(k). \qquad (4)$$

When the system is homogeneous, that is, when $\boldsymbol{\omega}(k) = 0$ the matrix Q is the zero matrix.

b) Let $\boldsymbol{\omega}(k) = 0$. The system (1) is said to be SS if for any $\boldsymbol{x}_0 \in X$ and any $\boldsymbol{\theta}_0 \in \Theta$

$$\sum_{k=0}^{\infty} E\left(\|x(k)\|^2\right) < \infty. \qquad (5)$$

c) Let $\boldsymbol{\omega}(k) = 0$. The system (1) is said to be ES if for any $\boldsymbol{x}_0 \in X$ and any $\boldsymbol{\theta}_0 \in \Theta$ there exist constants $0 < \alpha < 1$ and $\beta > 0$ such that for all $k \ge 0$

$$E\left(\|\boldsymbol{x}(k)\|^2\right) \le \beta\alpha^k\|\boldsymbol{x}_0\|^2. \qquad (6)$$

where α and β are independent of $\boldsymbol{x}_0$ and $\boldsymbol{\theta}_0$.
Consider the homogeneous version of the system (1), that is,

$$S_{\theta(k+1)}\boldsymbol{x}(k+1) = A_{\theta(k)}\boldsymbol{x}(k), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0. \qquad (7)$$

The equivalence between MSS, SS and ES is a very well-known result for non-singular Markov jump linear systems, that is when $S_i = I$ for all $i \in S_\theta$, where I is the identity matrix (see, e.g., [6], [17]). For SJLS, no similar result exists in the literature. For this class of systems the complication appears because of the singularity of the Matrix $S_{\theta(k)}$. Before showing this equivalence some preliminary definitions and results are given first.

Lemma 1: If the system (7) is MSS then for all $i \in S_\theta$

$$\lim_{k\to\infty} Q_i(k) = 0, \qquad (8)$$

where $Q_i(k)$ is defined in (2b).

Proof: For all $i \in S_\theta$ the following inequality holds

$$\|Q_i(k)\| = \left\|E(\boldsymbol{x}(k)\boldsymbol{x}^T(k)1_{\{\theta(k)=i\}})\right\| \le \left\|E(\boldsymbol{x}(k)\boldsymbol{x}^T(k))\right\|.$$

MSS makes it possible to take limit as $k \to \infty$ on both sides of this equation resulting in (8). ■

The operators $\mathcal{A}$, $\mathcal{S}$ and $\mathcal{T}$ defined below play a fundamental role in what follows.

$$\mathcal{A} \triangleq (\Pi^T \otimes I_{n^2})diag(A_i \otimes A_i), \quad i \in S_\theta \qquad (9)$$
$$\mathcal{S} \triangleq diag(S_i \otimes S_i), \quad i \in S_\theta \qquad (10)$$
$$\mathcal{T} \triangleq (\mathcal{S} - \mathcal{A}). \qquad (11)$$

Now define

$$q_i(k) \triangleq \mathrm{vec}(Q_i(k)), \quad i \in S_\theta, \qquad (12)$$

$$\widetilde{q}(k) \triangleq \begin{bmatrix} q_1(k) \\ \vdots \\ q_N(k) \end{bmatrix}. \qquad (13)$$

A characterization of MSS in terms of $\widetilde{q}(k)$ is given in the following Lemma.

Lemma 2: The system (7) is MSS if and only if

$$\lim_{k\to\infty} \widetilde{q}(k) = 0. \qquad (14)$$

Proof: It follows from Lemma 1 and Equations (3), (12) and (13). ■

Important partial sums are introduced next. For each $k \in \mathbb{Z}^+$ define

$$M(k) \triangleq \sum_{n=0}^{k} Q(n) = \sum_{n=0}^{k} E(\boldsymbol{x}(n)\boldsymbol{x}^T(n)) \tag{15}$$

$$\begin{aligned} M_i(k) &\triangleq \sum_{n=0}^{k} Q_i(n) \quad (16)\\ &= \sum_{n=0}^{k} E(\boldsymbol{x}(n)\boldsymbol{x}^T(n))1_{\{\theta(n)=i\}}, \quad i \in S_\theta. \end{aligned}$$

Lemma 3: Let $M_i(k)$ be defined as in (16). Then for each $k \in \mathbb{Z}^+$ the following equality holds

$$\sum_{i=1}^{N} \operatorname{tr}(M_i(k)) = \sum_{n=0}^{k} E\left(\|\boldsymbol{x}(n)\|^2\right).$$

Proof:

$$\begin{aligned} &\sum_{i=1}^{N} \operatorname{tr}(M_i(k)) \\ &= \sum_{i=1}^{N} \operatorname{tr}\left(\sum_{n=0}^{k} E\left(\boldsymbol{x}(n)\boldsymbol{x}^T(n)1_{\{\theta(n)=i\}}\right)\right) \\ &= \sum_{i=1}^{N}\sum_{n=0}^{k} E\left(\operatorname{tr}\left(\boldsymbol{x}(n)\boldsymbol{x}^T(n)1_{\{\theta(n)=i\}}\right)\right) \\ &= \sum_{i=1}^{N}\sum_{n=0}^{k} E\left(\|\boldsymbol{x}(n)\|^2 1_{\{\theta(n)=i\}}\right) \\ &= \sum_{n=0}^{k} E\left(\sum_{i=1}^{N} \|\boldsymbol{x}(n)\|^2 1_{\{\theta(n)=i\}}\right) \\ &= \sum_{n=0}^{k} E\left(\|\boldsymbol{x}(n)\|^2 \sum_{i=1}^{N} 1_{\{\theta(n)=i\}}\right) \\ &= \sum_{n=0}^{k} E\left(\|\boldsymbol{x}(n)\|^2\right). \end{aligned}$$

■

Lemma 4: Let $\mathcal{A}$ and $\mathcal{S}$ be defined as in (9) and (10), respectively, and consider the system (7). Then the following equation holds

$$\mathcal{S}\widetilde{q}(k+1) = \mathcal{A}\widetilde{q}(k). \tag{17}$$

Proof: This is part (b) of Theorem 2 in [18]. ■

The main result of this section follows. It is shown that MSS and SS are equivalent.

Theorem 1: Assume that the operator $\mathcal{T}$ defined in (11) is full rank. Then the System (7) is MSS if and only if it SS.

Proof: (MSS $\Rightarrow$ SS). Let us assume that the system (7) is MSS. From (17) it can be shown by induction that

$$(\mathcal{S} - \mathcal{A})\,\widetilde{q}(k) = \mathcal{A}\left(\widetilde{q}(k-1) - \widetilde{q}(k)\right), \quad k \geq 1.$$

Hence

$$(\mathcal{S} - \mathcal{A})\left(\sum_{n=1}^{k} \widetilde{q}(n)\right) = \mathcal{A}\left(\widetilde{q}(0) - \widetilde{q}(k)\right)$$

and

$$\mathcal{T}\left(\sum_{n=0}^{k} \widetilde{q}(n)\right) = \mathcal{S}\widetilde{q}(0) - \mathcal{A}\widetilde{q}(k).$$

Lemma 2 implies

$$\lim_{k\to\infty} \mathcal{T}\left(\sum_{n=0}^{k} \widetilde{q}(n)\right) = \mathcal{S}\widetilde{q}(0).$$

Since $\mathcal{T}$ is full rank then it is invertible. Then

$$\sum_{k=0}^{\infty} \widetilde{q}(k) = \mathcal{T}^{-1}\mathcal{S}\widetilde{q}(0).$$

Thus by (12) and (13), there exists $T_i \in \mathbb{R}^n$ such that for all $i \in S_\theta$

$$T_i = \sum_{k=0}^{\infty} q_i(k) = \sum_{k=0}^{\infty} \operatorname{vec}(Q_i(k)).$$

Since the stacking column vector operator is continuous it follows

$$T_i = \operatorname{vec}\left(\sum_{k=0}^{\infty} Q_i(k)\right).$$

Hence by (16)

$$\lim_{k\to\infty} M_i(k) = \sum_{k=0}^{\infty} Q_i(k) = \operatorname{vec}^{-1}(T_i), \tag{18}$$

and by Lemma 3

$$\begin{aligned} \sum_{n=0}^{\infty} E\left(\|\boldsymbol{x}(n)\|^2\right) &= \sum_{i=1}^{N} \operatorname{tr}\left(\lim_{k\to\infty} M_i(k)\right) \\ &= \sum_{i=1}^{N} \operatorname{tr}\left(\operatorname{vec}^{-1}(T_i)\right) < \infty. \end{aligned}$$

Thus the condition (5) is satisfied which implies that the system is SS.

(MSS $\Leftarrow$ SS). Let us assume now that the System (7) is SS. Then $\sum_{k=0}^{\infty} E\left(\|\boldsymbol{x}(k)\|^2\right) < \infty$. Hence

$$\lim_{k\to\infty} E\left(\|\boldsymbol{x}(k)\|^2\right) = 0.$$

Since

$$\left\|\boldsymbol{x}(k)\boldsymbol{x}^T(k)\right\| = \|\boldsymbol{x}(k)\|^2 \quad (19)$$

then by Jensen's inequality and (19) it follows that

$$\begin{aligned}\left\|E\left(\boldsymbol{x}(k)\boldsymbol{x}^T(k)\right)\right\| &\leq E\left(\left\|\boldsymbol{x}(k)\boldsymbol{x}^T(k)\right\|\right)\\ &= E\left(\|\boldsymbol{x}(k)\|^2\right).\end{aligned}$$

Tacking limits as $k \to \infty$ on both sides of this inequality gives

$$\lim_{k\to\infty}\|Q(k)\| \leq \lim_{k\to\infty} E\left(\|\boldsymbol{x}(k)\|^2\right) = 0,$$

which implies that the system is MSS. ■

Now the equivalence between SS and ES is shown. The arguments largely follow the ideas developed in Sections 3.1 and IV of [16] and [15], respectively. A definition and a lemma are given first.

Definition 2: ([16]) Let W_i be a matrix in $\mathbb{M}^{n0}$ for all $i \in S_\theta$. The system (7) is said to be observable in $W_{\theta(k)}$ if for all $k \geq 0$ there exist $\gamma > 0$ and $T > 0$, independent of k and $\boldsymbol{\theta}(k)$ such that

$$E\left(\sum_{t=k}^{k+T}\boldsymbol{x}^T(t)W_{\theta(k)}\boldsymbol{x}(t)\middle|\mathcal{F}_k\right) \geq \gamma\|\boldsymbol{x}(k)\|^2.$$

Define the following Lyapunov function

$$V(\boldsymbol{x}(k),\boldsymbol{\theta}(k)) \triangleq \boldsymbol{x}^T(k)W_{\theta(k)}\boldsymbol{x}(k).$$

Lemma 5: Let the following Lyapunov equation be defined as

$$\begin{aligned}S_i^T X_i S_i = A_i^T\left(\sum_{j=1}^N p_{ij}X_j\right)A_i + A_i^T\left(\sum_{j=1}^N p_{ij}\overline{S}_j\right)R_i \quad (20)\\ + R_i^T\left(\sum_{j=1}^N p_{ij}\overline{S}_j^T\right)A_i + W_i,\end{aligned}$$

where $S_i, A_i \in \mathbb{R}^n$, $R_i \in \mathbb{M}^{(n-r)\times n}$ and $\overline{S}_i \in \mathbb{M}^{n\times(n-r)}$ is a full column rank matrix such that $S_i\overline{S}_i = 0$ and r =rank(S_i). If (20) is satisfied for all $i \in S_\theta$ then the following equality holds

$$\begin{aligned}V(\boldsymbol{x}(k),\boldsymbol{\theta}(k)) - E(V(\boldsymbol{x}(k+1),\boldsymbol{\theta}(k+1)))\\ = \boldsymbol{x}^T(k)W_{\theta(k)}\boldsymbol{x}(k).\end{aligned}$$

Proof: This is Equation (21) of the proof of Theorem 1 in [15]. ■

Theorem 2: SS and ES are equivalent.

Proof: a) (SS ⇒ ES) Let $W_i = I$, $i \in S_\theta$, in such a manner that the system (7) is observable in W_θ with $\gamma = 1$. According to Theorem 1 in [15], if the system is SS then for $W_i \in \mathbb{M}^{n0}$ there exist solutions $S_i^T X_i S_i \in \mathbb{M}^{n0}$ and $R_i,\ i \in S_\theta$ to the Equation (20). Now, according to Theorem 3.1 in [16] if (20) holds then the System is ES.

b) (ES ⇒ SS) Let us assume now that the system is ES. Since $\alpha < 1$ in (6), the claim follows immediately by the definition of SS.

■

IV. Necessary conditions for MSS

In this section, new necessary conditions for MSS of system (1) are given. These conditions are given in terms of GLEs for the homogeneous and non-homogeneous version of the system.

A. The homogeneous case

The following result gives an equivalent characterization of MSS.

Lemma 6: Assume that $\mathcal{T}$ is full rank. Then the system (7) is MSS if and only if there exists $M_i \in \mathbb{M}^{n0}$ such that

$$M_i = \sum_{k=0}^{\infty} Q_i(k), \quad i \in S_\theta. \quad (21)$$

Proof: (⇒) If $\mathcal{T}$ is full rank and the system is MSS then (18) holds, that is, for all $i \in S_\theta$ there exists M_i such that $M_i = \sum_{k=0}^{\infty} Q_i(k)$. Clearly $M_i \in \mathbb{M}^{n0}$.

(⇐) If (21) holds then $\lim_{k\to\infty} Q_i(k) = 0$ for all $i \in S_\theta$. Then by (3)

$$\lim_{k\to\infty} Q(k) = \lim_{k\to\infty}\left(\sum_{i=1}^N Q_i(k)\right) = 0,$$

thus the system is MSS. ■

Theorem 3: If the system (7) is MSS and $\mathcal{T}$ is full rank then the set of matrices M_i defined in (21) satisfy the generalized Lyapunov equation

$$\sum_{i=1}^N S_i M_i S_i^T = \sum_{i=1}^N A_i M_i A_i^T + X^0, \quad (22)$$

where $X^0 \triangleq E\left(S_{\theta(0)}\boldsymbol{x}(0)\boldsymbol{x}^T(0)S_{\theta(0)}^T\right)$.

Proof: Let the system (7) be MSS. By Lemma 6 it follows

$$\sum_{k=0}^{\infty} E\big(S_{\theta(k)}\boldsymbol{x}(k)\boldsymbol{x}^T(k)S_{\theta(k)}^T\big)$$
$$= X^0 + \sum_{k=1}^{\infty} E\big(A_{\theta(k-1)}\boldsymbol{x}(k-1)\boldsymbol{x}^T(k-1) A_{\theta(k-1)}^T\big)$$
$$= X^0 + \sum_{k=1}^{\infty} E\left(\sum_{i=1}^{N} A_i\boldsymbol{x}(k-1)\boldsymbol{x}^T(k-1) A_i^T 1_{\{\theta(k-1)=i\}}\right)$$
$$= X^0 + \sum_{i=1}^{N} A_i\left(\sum_{k=1}^{\infty} E\big(\boldsymbol{x}(k-1)\boldsymbol{x}^T(k-1) 1_{\{\theta(k-1)=i\}}\big)\right)A_i^T$$
$$= \sum_{i=1}^{N} A_i M_i A_i^T + X^0.$$

The left hand side of this equation can be written similarly resulting in (22). ■

B. The non-homogeneous case

In this subsection, a necessary GLE condition for MSS of the system (1) when $\boldsymbol{\omega}(k) \neq 0$ is given. Let $p_i(k) \triangleq \Pr(\boldsymbol{\theta}(k) = i)$ be the probability distribution of $\boldsymbol{\theta}(k)$.

Lemma 7: If the system (1) is MSS and the limits $\lim_{k\to\infty} p_i(k)$ exist for every $i \in S_\theta$ then there exists a matrix $Q_i \in \mathbb{M}^{n0}$ such that $Q_i = \lim_{k\to\infty} Q_i(k)$.

Proof: Since the system is MSS then there exists a matrix $Q \in \mathbb{M}^{n0}$ such that $Q = \lim_{k\to\infty} Q(k)$. The convergence of $Q_i(k)$ is proved by showing that $\{Q_i(k), k \in \mathbb{Z}^+\}$ is a Cauchy sequence. Fix $i \in S_\theta$ and take n and m in $\mathbb{Z}^+$ such that $n > m$.

$$\|Q_i(n) - Q_i(m)\| = \big\|E\left(\boldsymbol{x}(n)\boldsymbol{x}^T(n)1_{\{\theta(n)=i\}}\right) - E\left(\boldsymbol{x}(m)\boldsymbol{x}^T(m)1_{\{\theta(m)=i\}}\right)\|$$
$$= \big\|E\left(\boldsymbol{x}(n)\boldsymbol{x}^T(n)1_{\{\theta(n)=i\}}\right) - E(Q1_{\{\theta(n)=i\}}) + E(Q1_{\{\theta(n)=i\}}) - E(Q1_{\{\theta(m)=i\}}) + E(Q1_{\{\theta(m)=i\}}) - E\left(\boldsymbol{x}(m)\boldsymbol{x}^T(m)1_{\{\theta(m)=i\}}\right)\|$$

By the triangle inequality and simplifying the arguments it follows

$$\|Q_i(n) - Q_i(m)\| \leq \big\|E\left(\boldsymbol{x}(n)\boldsymbol{x}^T(n)\right) - Q\big\| + \|Q\||p_i(n) - p_i(m)| + \big\|E\left(\boldsymbol{x}(m)\boldsymbol{x}^T(m)\right) - Q\big\|.$$

Since the sequence $\{p_i(k), k \in \mathbb{Z}^+\}$ is convergent, then it is Cauchy. Thus for any $N > 0$ there exists $\epsilon > 0$ such that for $n > m > N$

$$\|Q_i(n) - Q_i(m)\| < \frac{\epsilon}{3} + \frac{\epsilon}{3\|Q\|}\|Q\| + \frac{\epsilon}{3} = \epsilon.$$

Since $\mathbb{M}^{mn}$ is a Banach space, the convergence of $Q_i(k)$ follows from this inequality. In addition, since $Q_i(k) \in \mathbb{M}^{n0}$ for every $k \in \mathbb{Z}^+$ then $Q_i \in \mathbb{M}^{n0}$. ■

Theorem 4: If the system (7) is MSS and $p_i = \lim_{k\to\infty} p_i(k)$ then the set of matrices Q_i given in Lemma 7 satisfy the GLE

$$\sum_{i=1}^{N} S_i Q_i S_i^T = \sum_{i=1}^{N} A_i Q_i A_i^T + \sum_{i=1}^{N} B_i B_i^T p_i. \quad (23)$$

Proof: It is similar to the one in Theorem 3. ■

V. EXAMPLES

Example 1: Consider the system (7) with $S_\theta = \{1, 2\}$, $S_1 = 1$, $A_1 = 0.5$, $S_2 = 0$, $A_2 = 1$ and

$$\Pi = \begin{bmatrix} 1 & 0 \\ p & 1-p \end{bmatrix},$$

with $0 \leq p \leq 1$. This is a quite simple, illustrative example of a stochastically regular SJLS with one dimensional $\boldsymbol{x}$-state, and set of admissible initial conditions given by $X = \{0\}$ (the origin, only) because the initial condition $\boldsymbol{\theta}_0 = 2$ imposes $0\boldsymbol{x}(1) = 1\boldsymbol{x}(0)$, that is, $\boldsymbol{x}(0) = 0$. Hence the system is trivially MSS, SS and ES irrespectively of p. This is in accordance with Theorem 2. Theorem 3 holds trivially with $M_i = X^0 = 0$. Regarding Theorem 1, from (11) we have that

$$\mathcal{T} = \begin{bmatrix} 0.75 & -p \\ 0 & -(1-p) \end{bmatrix},$$

which is of full rank except for $p = 1$, indicating that the result is conservative only for $p = 1$.

Example 2: Consider the system (7) with $S_\theta = \{1, 2\}$, $S_1 = S_2 = I$, $A_1 = 1.2I$,

$$A_2 = \begin{bmatrix} 0.5 & 0.2 \\ -0.1 & 0.7 \end{bmatrix}, \text{ and } \Pi = \begin{bmatrix} 0.6 & 0.4 \\ p & 1-p \end{bmatrix},$$

with $0 \leq p \leq 1$. This is a standard nonsingular Markov jump linear system, which is MSS and SS if and only

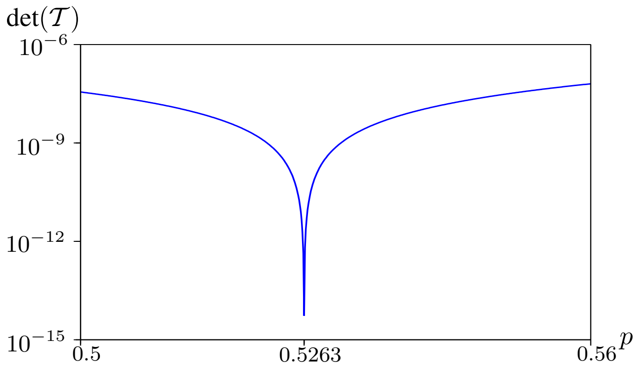

Fig. 1. Determinant of $\mathcal{T}$ for different values of p in Example 2.

if the eigenvalues of $\mathcal{A}$ are in the unit disk [6]; this allows us to check that the system is stable for $p < \bar{p}$ with $\bar{p} \approx 0.5263$. Figure 1 indicates that $\mathcal{T}$ is of full rank except for $\bar{p}$, therefore the hypothesis of Theorem 1 holds whenever the system is stable.

Remark. For nonsingular Markov jump linear systems, Theorem 1 retrieves the well known fact that MSS and SS are equivalent (see, e.g., [6], [17]). Actually, in this context one has that MSS implies that all eigenvalues of $\mathcal{A}$ are inside the unit disk, yielding that $\mathcal{T} = I - \mathcal{A}$ is of full rank (as illustrated in Example 2) and Theorem 1 implies that the system is SS. The same argument is valid to show that SS implies MSS.

VI. Conclusions

Three concepts of second moment stability for SJLS have been introduced and shown to be equivalent under certain conditions. Up to the best of our knowledge this is a new result and represents an extension of the very well know result for Markov jump linear system that establishes the same property for the same three stability concept of stability. It was also given necessary conditions for MSS in terms of a generalized Lyapunov equation for SJLS. These necessary conditions were given for the homogeneous and non-homogeneous versions of the system.

References

[1] D. G. Luemberger and A. Arbel, "Singular dynamic Leontief systems," *Econometrica*, vol. 45, no. 4, pp. 991–995, May 1977.

[2] H. Hemami and B. F. Wyman, "Modeling and control of constrained dynamic systems with application to biped locomotion in the frontal plane," *IEEE Trans. Autom. Control*, vol. AC-24, no. 4, pp. 526–535, 1979.

[3] R. W. Newcomb, "The Semistate description of non linear time-variable circuits," *IEEE Trans. Autom. Control*, vol. CAS-28, no. 1, pp. 62–71, January 1981.

[4] J. Mills and A. Goldemberg, "Force and position pontrol of manipulator during constrained motion tasks," *IEEE Trans. Autom. Control*, vol. 5, no. 1, pp. 30–46, February 1989.

[5] B. L. Stevens and F. L. Lewis, *Aircraft Modeling, Dynamics and Control*. New York, NY: Wiley, 1991.

[6] O. L. V. Costa, M. D. Fragoso, and R. P. Marques, *Discrete-Time Markov jump linear systems*, ser. Springer Series in Probability and its applications. Springer, 2004.

[7] E. K. Boukas, *Control of singular systems with random abrupt changes*, ser. Springer Series in communication and control engineering. Springer, 2004.

[8] V. Bajic, "Algebraic conditions for stability of linear singular systems," *Int. Symp. Circuits and Systems*, vol. 2, pp. 1089–1092, 1991.

[9] E. Fridman, "A Lyapunov-based approach to stability of descriptor systems with delay," in *Proc. 40th IEEE Conf. on Control and Decision*, Orlando, FL, 2001, pp. 2850–2855.

[10] Z. Gao, X. Wang, and J. Wang, "Internal properness and stability in singular decentralized control systems," in *Proc. 1997 American Control Conf.*, vol. 4, 2008, pp. 2520–2521.

[11] C. Chen and Y. Liu, "Lyapunov stability analysis of linear singular dynamical systems," in *IEEE Int. Conf. Intelligent-Processing Systems, ICIPS'97*, vol. 1, 2004, pp. 635–639.

[12] S. Xu and J. Lam, "Robust stability and stabilization of discrete singular systems: An equivalent characterization," *IEEE Trans. Autom. Control*, vol. 49, no. 4, pp. 568–574, April 2004.

[13] J. Y. Ishihara and M. H. Terra, "On the Lyapunov theorem for singular systems," *IEEE Trans. Autom. Control*, vol. 47, no. 11, pp. 1926–1930, November 2002.

[14] E. K. Boukas and Z. K. Liu, "Delay-dependent stabilization of singular perturbed jump linear systems," *Int. J. Control*, vol. 77, no. 3, pp. 310–319, 2001.

[15] A. L. P. Manfrim, M. H. Terra, E. F. Costa, and J. Y. Ishihara, "Stochastic stability for discrete-time singular systems with Markov jump parameters," in *Proc. 2008 American Control Conf.*, Seattle, W, 2008, pp. 1650–1655.

[16] A. L. P. Manfrim, "Sistemas Lineares Singulares Sujeitos a Saltos Markovianos," Ph.D. dissertation, Engineering School of São Carlos, University of São Paulo, 2010.

[17] Y. Ji, J. Chizeck, X. Feng, and K. A. Loparo, "Stability and control of discrete-time jump linear systems," *Contr. Theory Adv. Tech.*, vol. 7, no. 2, pp. 247–270, 1991.

[18] A. L. P. Manfrim, M. H. Terra, E. F. Costa, and J. Y. Ishihara, "On the regularity for singular linear systems with markov jump parameters," in *Proc. 2008 International Federation of Automatic Control*, Seoul, Korea, 2008, pp. 9987–9992.